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Research Paper

Variants of $\mathcal{R}$ −Weakly Commuting and Reciprocal Continuous Mappings in Fuzzy Metric space

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Abstract
In this paper, we prove common fixed-point theorems for variants of $\mathcal{R}$ −weakly commuting and reciprocal mappings in fuzzy metric space that contains cubic and quadratic terms of $\mathcal{M}(x, y, t)$.
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I. INTRODUCTION

Since the 16th century, probability theory has been studying a sort of uncertainty in the occurrence of an event. The concept of a fuzzy metric space appears to be more relevant when the uncertainty is related to fuzziness rather than randomness, as it is occasionally in the measurement of an ordinary length.

The concept of fuzzy sets was developed by Zadeh [22], and it provides a precise natural framework for mathematical modelling of real-world situations marked by ambiguity and uncertainty due to non-probabilistic elements. A major developmental path for fuzzy set theory has been the fuzzification of those branches of mathematics that are founded on set theory, which resulted in a new branch of mathematics known as "Fuzzy Mathematics.

The fact that the distance between two points is frequently inexact rather than a single integer prompted the creation of the probabilistic metric space. In their own unique approaches, Erceg [2], Kaleva and Seikkala [9], and Kramosils and Michalek [8] have all created the concept of fuzzy metric space. Grabiec [3] established the fuzzy form of the Banach contraction principle, which was followed by Kramosil and Michalek [8]."

Definition 1.1[17]"Let f and g be mappings from a fuzzy metric space $(\mathfrak{B}, \mathcal{M}, *)$ into itself. The mappings f and g are said to be compatible if $lim_{n\to\infty}\mathcal{M}(fgx_n, gfx_n, t) = 1$, for all $t > 0$, whenever $\{x_n\}$ is a sequence in $\mathfrak{B}$ such that $lim_{n\to\infty} fx_n = lim_{n\to\infty} gx_n = u$ for some $u \in \mathfrak{B}$."

In 1996, Jungck [4] introduced the notion of weakly compatible mappings and showed that compatible maps are weakly compatible, but converse may not be true.

Two self-mappings f and g on a fuzzy metric space $(\mathfrak{B}, \mathcal{M}, *)$ are called weakly compatible if they commute at their coincidence point.

Definition 1.2[11] "Let f and g be mappings from a fuzzy metric space $(\mathfrak{B}, \mathcal{M}, *)$ into itself. Then f and g are said to be $\mathcal{R}$ −weakly commuting if there exists $\mathcal{R} > 0$ such that $\mathcal{M}(fgx, gfx, t) \geq \mathcal{M}(fx, gx, \frac{t}{\mathcal{R}})$ for all $x \in \mathfrak{B}$."

Obviously weak commutativity implies $\mathcal{R}$ −weak commutativity. However, $\mathcal{R}$-weak commutativity implies weak commutativity only when $\mathcal{R} \leq 1$.

In 1999, Pant [15] introduced a new continuity condition, known as reciprocal continuity as follows:

Definition 1.3 [18]"Two self-mappings f and g of a fuzzy metric space $(\mathfrak{B}, \mathcal{M}, *)$ are called reciprocally continuous if $lim_n \mathcal{M}(fgx_n, fu, t) = 1$ and $lim_n \mathcal{M}(gfx_n, gu, t) = 1$, whenever $\{x_n\}$ is a sequence in $\mathfrak{B}$ such that $lim_n fx_n = lim_n gx_n = u$, for some u in $\mathfrak{B}$ and for all $t > 0$."

Remark 1.1 Continuous mappings are reciprocally continuous on $(\mathfrak{B}, \mathcal{M}, *)$ but the converse is not true.

II. PRELIMINARIES

Definition 2.1 [3]"The triplet $(\mathfrak{B},\mathcal{M},*)$ is a fuzzy metric space if $\mathfrak{B}$ is arbitrary set, $*$ is a continuous t-norm, $\mathcal{M}$ is a fuzzy set in $\mathfrak{B}^2\times[0,\infty)$ satisfying the following conditions:

(i) $\mathcal{M}(x,y,0) = 0$,

(ii) $\mathcal{M}(x,y,t) = 1$ for all $t > 0$ if and only if $x = y$,

(iii) $\mathcal{M}(x,y,t) = \mathcal{M}(y,x,t)$,

(iv) $\left(\mathcal{M}(x, y,t) * \mathcal{M}(y,z,s)\right) \leq \mathcal{M}(x,z,t+s)$,

(v) $\mathcal{M}(x,y,.) : [0,\infty) \to [0,1]$ is left continuous for all $x, y,z \in \mathfrak{B}$ and $s, t > 0$,

(vi) $lim_{t\to\infty}\mathcal{M}(x,y,t) = 1$, for all x,y, in $\mathfrak{B}$.

$\mathcal{M}(x,y,t)$ can be thought of as the degree of nearness between x and $y$with respect to t."

Definition 2.2[13] "Let $(\mathfrak{B},\mathcal{M},*)$ be a fuzzy metric space. A sequence $\{x_n\}$ in $\mathfrak{B}$ is said to be:

(i) Converge to $x \in \mathfrak{B}$ if $lim_{n\to\infty}\mathcal{M}(x_n,x,t) = 1$ fo each $t > 0$.

(ii) Cauchy sequence if $lim_{n\to\infty}\mathcal{M}(x_{n+p},x_n,t) = 1$ for all $t > 0$ and $p > 0$.

(iii) Complete if every Cauchy sequence in $\mathfrak{B}$ is convergent in $\mathfrak{B}$. "

Proposition 2.1[7] Let $\mathcal{S}$ and $\mathcal{T}$ be compatible mappings of a fuzzy metric space $(\mathfrak{B},\mathcal{M},*)$ into itself, If $\mathcal{S}t = \mathcal{T}t$ for some$t$ in$\mathfrak{B}$, then$\mathcal{S}\mathcal{T}t = \mathcal{S}\mathcal{S}t = \mathcal{T}\mathcal{T}t = \mathcal{T}\mathcal{S}t$.

Proposition 2.2[7] Let $\mathcal{P}$ and $\mathcal{Q}$ be compatible mappings of a fuzzy metric space $(\mathfrak{B},\mathcal{M},*)$ into itself. Suppose that $lim_n\mathcal{P}x_n= lim_n\mathcal{Q}x_n= t$ for some t in$\mathfrak{B}$. Then the following holds:

(i) $lim_n\mathcal{Q}\mathcal{P}x_n= \mathcal{P}t$ if $\mathcal{P}$ is continuous at t;

(ii) $lim_n\mathcal{P}\mathcal{Q}x_n= \mathcal{Q}t$ if $\mathcal{Q}$ is continuous at t;

(iii) $\mathcal{P}\mathcal{Q}t = \mathcal{Q}\mathcal{P}t$ and $\mathcal{P}t = \mathcal{Q}t$ if $\mathcal{P}$ and $\mathcal{Q}$ are continuous at t.

Lemma 2.1[17]Let $(\mathfrak{B},\mathcal{M},*)$ be a fuzzy metric space. If there exists $q \in (0,1)$ such that $\mathcal{M}(x,y,qt) \geq \mathcal{M}(x,y,t)$ for all $x,y \in \mathfrak{B}$, and $t > 0$, then $x = y$.

Lemma 2.2[17]Let $\{y_n\}$ be a sequence in a fuzzy metric space $(\mathfrak{B},\mathcal{M},*)$. If there exists $q \in (0,1)$ such that $\mathcal{M}(y_{n+2},y_{n+1},qt) \geq \mathcal{M}(y_{n+1},y_n,t), t > 0, n \in N$, then$y_n$is a Cauchy sequence in $\mathfrak{B}$.

Lemma 2.3[20] Let $(\mathfrak{B},\mathcal{M},*)$ be a fuzzy metric space. If there is a sequence $\{x_n\}$in X, such that for every $n\in N$.

$$\mathcal{M}(x_n,x_{n+1},t) \geq \mathcal{M}(x_0,x_1,k^n t)$$

for every k>1, then the sequence is a Cauchy sequence.

Definition 2.3[11]"Two self-mappings f and g of a fuzzy metric space $(\mathfrak{B},\mathcal{M},*)$ are said to be weakly commuting if $\mathcal{M}(fgx,gfx,t) \geq \mathcal{M}(fx,gx,t)$ for each $x \in \mathfrak{B}$and for each $t > 0$."

Definition 2.4[14]"Let $f$and g are self-mappings on a fuzzy metric space $(\mathfrak{B},\mathcal{M},*)$. Then $f$and g are said to be:

(i) $\mathcal{R}$ −weakly commuting of type (A_g) if there exists $\mathcal{R} > 0$ such that $\mathcal{M}(ffx,gfx,t) \geq \mathcal{M}(fx,gx,\frac{t}{\mathcal{R}})$;

(ii) $\mathcal{R}$− weakly commuting of type (A_f) if there exists $\mathcal{R} > 0$ such that $\mathcal{M}(fgx,ggx,t) \geq \mathcal{M}(fx,gx,\frac{t}{\mathcal{R}})$;

(iii) $\mathcal{R}$ − weakly commuting of type (P) if there exists $\mathcal{R} > 0$ such that$\mathcal{M}(ffx,ggx,t) \geq \mathcal{M}(fx,gx,\frac{t}{\mathcal{R}})$ for all $x,y \in \mathfrak{B}$ and $t > 0$."

It is obvious that point wise $\mathcal{R}$ −weakly commuting mappings commute at their coincidence points and $\mathcal{R}$ −weak commutativity is equivalent to commutativity at coincidence points. It may be noted that both compatible and non-compatible mappings can be $\mathcal{R}$ −weakly commuting of the type (A_g) or of type (A_f), but converse need not be true.

Remark 2.1 If $\mathcal{R}\leq 1$, then $\mathcal{R}$ −weakly commuting mappings are weakly commuting.

III. MAIN RESULTS

A class of Implicit Relation

Let **Ψ** be set of all continuous functions ψ: $[0,1] \to [0,1]$ increasing in any coordinate and $\psi(t) > t$.

Theorem 3.1 Let $(\mathfrak{B},\mathcal{M},*)$ be a complete fuzzy metric space. Let $\mathcal{N},\mathcal{P},\mathcal{Q}$ and $\mathcal{S}$ are four mappings of a complete fuzzy metric space $(\mathfrak{B},\mathcal{M},*)$ into itself satisfying

$(C_1)\mathcal{N}(\mathfrak{B}) \subseteq \mathcal{S}(\mathfrak{B}), \mathcal{P}(\mathfrak{B}) \subseteq \mathcal{Q}(\mathfrak{B})$;

$(C_2)(\mathcal{Q},\mathcal{N})$ and $(\mathcal{S},\mathcal{P})$ are point wise $\mathcal{R}$ −weakly commuting pairs;

$(C_3)(\mathcal{Q},\mathcal{N})$ and $(\mathcal{S},\mathcal{P})$are compatible pairs of reciprocally continuous mappings;

$$(C_4)\mathcal{M}^3(\mathcal{N}u,\mathcal{P}v,kt) \geq \psi\left\{min\begin{pmatrix}\mathcal{M}^2(\mathcal{Q}u,\mathcal{N}u,t)\mathcal{M}(\mathcal{S}v,\mathcal{P}v,t)\\ \mathcal{M}(\mathcal{Q}u,\mathcal{N}u,t)\mathcal{M}^2(\mathcal{S}v,\mathcal{P}v,t),\\ \mathcal{M}(\mathcal{Q}u,\mathcal{N}u,t)\mathcal{M}(\mathcal{N}u,\mathcal{P}v,t)\mathcal{M}(\mathcal{S}v,\mathcal{P}v,t),\\ \mathcal{M}(\mathcal{S}v,\mathcal{P}v,t)\mathcal{M}(\mathcal{Q}u,\mathcal{S}v,t)\mathcal{M}(\mathcal{Q}u,\mathcal{N}u,t)\end{pmatrix}\right\}$$

for all $u, v \in \mathfrak{V}$, $k > 1$and $\psi \in \mathbf{\Psi}$

Then $\mathcal{N}, \mathcal{P}, \mathcal{Q}$ and $\mathcal{S}$ have a unique common fixed point in $\mathfrak{V}$.

Proof. Let $x_0 \in \mathfrak{V}$ be an arbitrary point. From (C_1) we can find a point x_1 such that $\mathcal{N}(x_0) = \mathcal{S}(x_1) = y_0$. For this point x_1 one can find a point $x_2 \in \mathfrak{V}$ such that $\mathcal{P}(x_1) = \mathcal{Q}(x_2) = y_1$.Continuing in this way, one can construct a sequence $\{x_n\}$ such that $y_{2n} = \mathcal{N}(x_{2n}) = \mathcal{S}(x_{2n+1})$, $y_{2n+1} = \mathcal{P}(x_{2n+1}) = \mathcal{Q}(x_{2n+2})$, for each $n \geq 0$. (3.1)

For brevity, we write $\alpha_m(t) = \mathcal{M}(y_m, y_{m+1}, t)$

First, we prove that $\{y_n\}$ is a Cauchy sequence

Case I If *n* is even, taking $u = x_{2n}$ and $v = x_{2n+1}$ in (C_4), we get

$$\mathcal{M}^3(\mathcal{N}x_{2n}, \mathcal{P}x_{2n+1}, kt) \geq \psi\left\{min\begin{pmatrix} \mathcal{M}^2(\mathcal{Q}x_{2n}, \mathcal{N}x_{2n}, t)\mathcal{M}(\mathcal{S}x_{2n+1}, \mathcal{P}x_{2n+1}, t), \\ \mathcal{M}(\mathcal{Q}x_{2n}, \mathcal{N}x_{2n}, t)\mathcal{M}^2(\mathcal{S}x_{2n+1}, \mathcal{P}x_{2n+1}, t), \\ \mathcal{M}(\mathcal{Q}x_{2n}, \mathcal{N}x_{2n}, t)\mathcal{M}(\mathcal{N}x_{2n}, \mathcal{P}x_{2n+1}, t)\mathcal{M}(\mathcal{S}x_{2n+1}, \mathcal{P}x_{2n+1}, t), \\ \mathcal{M}(\mathcal{S}x_{2n+1}, \mathcal{P}x_{2n+1}, t)\mathcal{M}(\mathcal{Q}x_{2n}, \mathcal{S}x_{2n+1}, t)\mathcal{M}(\mathcal{Q}x_{2n}, \mathcal{N}x_{2n}, t) \end{pmatrix}\right\}$$

Using (3.1), we have

$$\mathcal{M}^3(y_{2n}, y_{2n+1}, kt) \geq \psi\left\{min\begin{pmatrix} \mathcal{M}^2(y_{2n-1}, y_{2n}, t)\mathcal{M}(y_{2n}, y_{2n+1}, t), \\ \mathcal{M}(y_{2n-1}, y_{2n}, t)\mathcal{M}^2(y_{2n}, y_{2n+1}, t), \\ \mathcal{M}(y_{2n-1}, y_{2n}, t)\mathcal{M}(y_{2n}, y_{2n+1}, t)\mathcal{M}(y_{2n}, y_{2n+1}, t), \\ \mathcal{M}(y_{2n}, y_{2n+1}, t)\mathcal{M}(y_{2n-1}, y_{2n}, t)\mathcal{M}(y_{2n-1}, y_{2n}, t) \end{pmatrix}\right\}$$

On using $\alpha_{2n}(t) = \mathcal{M}(y_{2n}, y_{2n+1}, t)$ in the above inequality, we have

$$\alpha_{2n}^3(kt) \geq \psi\left\{min\begin{pmatrix} \alpha_{2n-1}^2(t)\alpha_{2n}(t), \alpha_{2n-1}(t)\alpha_{2n}^2(t), \\ \alpha_{2n-1}(t)\alpha_{2n}^2(t), \alpha_{2n}(t)\alpha_{2n-1}^2(t) \end{pmatrix}\right\}, \quad (3.2)$$

We claim that $\alpha_{2n}(kt) \geq \alpha_{2n-1}(kt)$

If $\alpha_{2n}(kt) < \alpha_{2n-1}(kt)$, then(3.2) reduces to

$$\alpha_{2n}^3(kt) \geq \psi\{min(\alpha_{2n}^3(t), \alpha_{2n}^3(t), \alpha_{2n}^3(t), \alpha_{2n}^3(t))\}$$

. Using property of ψ we get

$$\alpha_{2n}^3(kt) > \alpha_{2n}^3(t) \Rightarrow \alpha_{2n}(kt) > \alpha_{2n}(t), \text{a contradiction}$$

Therefore $\alpha_{2n}(kt) \geq \alpha_{2n-1}(kt)$

In a similar way, if n is odd, then we can obtain $\alpha_{2n+1}(kt) \geq \alpha_{2n}(kt)$.

It follows that the sequence $\{\alpha_n(t)\}$ is increasing in $[0,1]$, thus (3.2) reduces to

$$\alpha_{2n}^3(kt) \geq \psi\{min(\alpha_{2n-1}^3(t), \alpha_{2n-1}^3(t), \alpha_{2n-1}^3(t), \alpha_{2n-1}^3(t))\}$$

Using property of ψ we get

$$\alpha_{2n}^3(kt) > \alpha_{2n-1}^3(t)$$

Thus we get $\alpha_{2n}(kt) \geq \alpha_{2n-1}(t)$

Similarly for an odd integer m=2n+1, we have $\alpha_{2n+1}(kt) \geq \alpha_{2n}(t)$,

Hence $\alpha_n(kt) \geq \alpha_{n-1}(t)$, That is,

$$\mathcal{M}(y_n, y_{n+1}, kt) \geq \mathcal{M}(y_{n-1}, y_n, t)$$

Hence by Lemma(2.2) $\{y_n\}$ is a Cauchy sequence in $\mathfrak{V}$. Since $(\mathfrak{V}, \mathcal{M}, *)$ is a complete fuzzy metric space, therefore, the sequence $\{y_n\}$ converges to a point z in $\mathfrak{V}$ as $n \to \infty$. Consequently, the subsequence's $\{\mathcal{N}x_{2n}\}, \{\mathcal{Q}x_{2n}\}, \{\mathcal{P}x_{2n+1}\}$ and $\{\mathcal{S}x_{2n+1}\}$ also converges to the same point z.

If $\mathcal{S}$ and $\mathcal{P}$ are compatible, then

$$\lim_{n\to\infty}(\mathcal{SP}x_n, \mathcal{PS}x_n, t) = 1 \text{ whenever} \lim_{n\to\infty} \mathcal{S}x_n = \lim_{n\to\infty} \mathcal{P}x_n = z$$

that is, $\mathcal{S}z = \mathcal{P}z$

Also by the reciprocal continuity of $\mathcal{S}$ and $\mathcal{P}$ we have

$$\lim_{n\to\infty} \mathcal{SP}x_n = \mathcal{S}z \text{ and} \lim_{n\to\infty} \mathcal{PS}x_n = \mathcal{P}z$$

Since $\mathcal{P}(\mathfrak{V}) \subseteq \mathcal{Q}(\mathfrak{V})$, there exists a point$w$ in $\mathfrak{V}$ such that $\mathcal{P}z = \mathcal{Q}w$

Setting u=w and v=z in (C_4), we get

$$\mathcal{M}^3(\mathcal{N}w, \mathcal{P}z, kt) \geq \psi\left\{min\begin{pmatrix} \mathcal{M}^2(\mathcal{Q}w, \mathcal{N}w, t)\mathcal{M}(\mathcal{S}z, \mathcal{P}z, t), \\ \mathcal{M}(\mathcal{Q}w, \mathcal{N}w, t)\mathcal{M}^2(\mathcal{S}z, \mathcal{P}z, t), \\ \mathcal{M}(\mathcal{Q}w, \mathcal{N}w, t)\mathcal{M}(\mathcal{N}w, \mathcal{P}z, t)\mathcal{M}(\mathcal{S}z, \mathcal{P}z, t), \\ \mathcal{M}(\mathcal{S}z, \mathcal{P}z, t)\mathcal{M}(\mathcal{Q}w, \mathcal{S}z, t)\mathcal{M}(\mathcal{Q}w, \mathcal{N}w, t) \end{pmatrix}\right\}$$

Using $\mathcal{P}z = \mathcal{Q}w$ and $\mathcal{P}z = \mathcal{S}z$, we get

$$\mathcal{M}^3(\mathcal{N}w,\mathcal{P}z,kt) \geq \psi\left\{min\begin{pmatrix}\mathcal{M}^2(\mathcal{P}z,\mathcal{N}w,t)\mathcal{M}(\mathcal{P}z,\mathcal{P}z,t),\\ \mathcal{M}(\mathcal{P}z,\mathcal{N}w,t)\mathcal{M}^2(\mathcal{P}z,\mathcal{P}z,t),\\ \mathcal{M}(\mathcal{P}z,\mathcal{N}w,t)\mathcal{M}(\mathcal{N}w,\mathcal{P}z,t)\mathcal{M}(\mathcal{P}z,\mathcal{P}z,t),\\ \mathcal{M}(\mathcal{P}z,\mathcal{P}z,t)\mathcal{M}(\mathcal{P}z,\mathcal{P}z,t)\mathcal{M}(\mathcal{P}z,\mathcal{N}w,t)\end{pmatrix}\right\}$$

$$\mathcal{M}^3(\mathcal{N}w,\mathcal{P}z,kt) \geq \psi\left\{min\begin{pmatrix}\mathcal{M}^2(\mathcal{P}z,\mathcal{N}w,t).1,\\ \mathcal{M}(\mathcal{P}z,\mathcal{N}w,t).1.1,\\ \mathcal{M}(\mathcal{P}z,\mathcal{N}w,t)\mathcal{M}(\mathcal{N}w,\mathcal{P}z,t).1,\\ 1.1.\mathcal{M}(\mathcal{P}z,\mathcal{N}w,t)\end{pmatrix}\right\}$$

Suppose $\mathcal{P}z \neq \mathcal{N}w$, then $\mathcal{M}(\mathcal{N}w,\mathcal{P}z,t) < 1$, using this in above inequality we get

$$\mathcal{M}^3(\mathcal{N}w,\mathcal{P}z,kt) \geq \psi\{min(\mathcal{M}^3(\mathcal{N}w,\mathcal{P}z,t),\mathcal{M}^3(\mathcal{N}w,\mathcal{P}z,t),\mathcal{M}^3(\mathcal{N}w,\mathcal{P}z,t),\mathcal{M}^3(\mathcal{N}w,\mathcal{P}z,t))\}$$

Using property of ψ we get

$$\mathcal{M}^3(\mathcal{N}w,\mathcal{P}z,kt) > \mathcal{M}^3(\mathcal{N}w,\mathcal{P}z,t) \Rightarrow \mathcal{M}(\mathcal{N}w,\mathcal{P}z,kt) > M(\mathcal{N}w,\mathcal{P}z,t)$$

This implies $\mathcal{N}w = \mathcal{P}z$ (using lemma 2.1)

Thus $\mathcal{N}w = \mathcal{Q}w = \mathcal{P}z = \mathcal{S}z$ (3.3)

The point wise $\mathcal{R}-$ weak commutativity of $\mathcal{S}$ and $\mathcal{P}$ implies that there exists an $\mathcal{R} > 0$ such that $\mathcal{M}(\mathcal{S}\mathcal{P}z,\mathcal{P}\mathcal{S}z,t) \geq \mathcal{M}(\mathcal{S}z,\mathcal{P}z,\frac{t}{\mathcal{R}})$;

which implies that $\mathcal{S}\mathcal{P}z = \mathcal{P}\mathcal{S}z$ and $\mathcal{S}\mathcal{S}z = \mathcal{P}\mathcal{S}z = \mathcal{S}\mathcal{P}z = \mathcal{P}\mathcal{P}z$ for all $z \in \mathfrak{V}$. (3.4)

Similarly, the point wise $\mathcal{R}$ −weak commutativity of $\mathcal{Q}$ and $\mathcal{N}$ implies that there exists an $\mathcal{R} > 0$ such that $\mathcal{M}(\mathcal{Q}\mathcal{N}w,\mathcal{N}\mathcal{Q}w,t) \geq \mathcal{M}(\mathcal{Q}w,\mathcal{N}w,\frac{t}{\mathcal{R}})$;

which implies that $\mathcal{Q}\mathcal{N}w = \mathcal{N}\mathcal{Q}w$ and $\mathcal{Q}\mathcal{Q}w = \mathcal{Q}\mathcal{N}w = \mathcal{N}\mathcal{Q}w = \mathcal{N}\mathcal{N}w$. (3.5)

Again substituting $u = w$ and $v = \mathcal{P}z$ in (C_4), we get

$$\mathcal{M}^3(\mathcal{N}w,\mathcal{P}\mathcal{P}z,kt) \geq \psi\left\{min\begin{pmatrix}\mathcal{M}^2(\mathcal{Q}w,\mathcal{N}w,t)\mathcal{M}(\mathcal{S}\mathcal{P}z,\mathcal{P}\mathcal{P}z,t),\\ \mathcal{M}(\mathcal{Q}w,\mathcal{N}w,t)\mathcal{M}^2(\mathcal{S}\mathcal{P}z,\mathcal{P}\mathcal{P}z,t),\\ \mathcal{M}(\mathcal{Q}w,\mathcal{N}w,t)\mathcal{M}(\mathcal{N}w,\mathcal{P}\mathcal{P}z,t)\mathcal{M}(\mathcal{S}\mathcal{P}z,\mathcal{P}\mathcal{P}z,t),\\ \mathcal{M}(\mathcal{S}\mathcal{P}z,\mathcal{P}\mathcal{P}z,t)\mathcal{M}(\mathcal{Q}w,\mathcal{S}\mathcal{P}z,t)\mathcal{M}(\mathcal{Q}w,\mathcal{N}w,t)\end{pmatrix}\right\}$$

Using (3.3) and (3.4) in above equation we get,

$$\mathcal{M}^3(\mathcal{P}z,\mathcal{P}\mathcal{P}z,kt) \geq \psi\left\{min\begin{pmatrix}\mathcal{M}^2(\mathcal{N}w,\mathcal{N}w,t)\mathcal{M}(\mathcal{P}\mathcal{P}z,\mathcal{P}\mathcal{P}z,t),\\ \mathcal{M}(\mathcal{N}w,\mathcal{N}w,t)\mathcal{M}^2(\mathcal{P}\mathcal{P}z,\mathcal{P}\mathcal{P}z,t),\\ \mathcal{M}(\mathcal{N}w,\mathcal{N}w,t)\mathcal{M}(\mathcal{P}z,\mathcal{P}\mathcal{P}z,t)\mathcal{M}(\mathcal{P}\mathcal{P}z,\mathcal{P}\mathcal{P}z,t),\\ \mathcal{M}(\mathcal{P}\mathcal{P}z,\mathcal{P}\mathcal{P}z,t)\mathcal{M}(\mathcal{P}z,\mathcal{P}\mathcal{P}z,t)\mathcal{M}(\mathcal{N}w,\mathcal{N}w,t)\end{pmatrix}\right\}$$

Suppose $\mathcal{P}z \neq \mathcal{P}\mathcal{P}z$, then $\mathcal{M}(\mathcal{P}z,\mathcal{P}\mathcal{P}z,t) < 1$, using this in above inequality we get

$$\mathcal{M}^3(\mathcal{P}z,\mathcal{P}\mathcal{P}z,kt) \geq \psi\{min(\mathcal{M}^3(\mathcal{P}z,\mathcal{P}\mathcal{P}z,t),\mathcal{M}^3(\mathcal{P}z,\mathcal{P}\mathcal{P}z,t),\mathcal{M}^3(\mathcal{P}z,\mathcal{P}\mathcal{P}z,t),\mathcal{M}^3(\mathcal{P}z,\mathcal{P}\mathcal{P}z,t))\}$$

Using property of ψ we get

$$\mathcal{M}^3(\mathcal{P}z,\mathcal{P}\mathcal{P}z,kt) > \mathcal{M}^3(\mathcal{P}z,\mathcal{P}\mathcal{P}z,t) \Rightarrow \mathcal{M}(\mathcal{P}z,\mathcal{P}\mathcal{P}z,kt) > M(\mathcal{P}z,\mathcal{P}\mathcal{P}z,t)$$
$$\Longrightarrow \mathcal{P}z = \mathcal{P}\mathcal{P}z$$

Thus $\mathcal{P}z = \mathcal{P}\mathcal{P}z = \mathcal{S}\mathcal{P}z$

Therefore, $\mathcal{P}z$ is a common fixed point of $\mathcal{P}$ and $\mathcal{S}$.

Taking $u = \mathcal{N}w$ and $v = z$ in (C_4), we get

$$\mathcal{M}^3(\mathcal{N}\mathcal{N}w,\mathcal{P}z,kt) \geq \psi\left\{min\begin{pmatrix}\mathcal{M}^2(\mathcal{Q}\mathcal{N}w,\mathcal{N}\mathcal{N}w,t)\mathcal{M}(\mathcal{S}z,\mathcal{P}z,t),\\ \mathcal{M}(\mathcal{Q}\mathcal{N}w,\mathcal{N}\mathcal{N}w,t)\mathcal{M}^2(\mathcal{S}z,\mathcal{P}z,t),\\ \mathcal{M}(\mathcal{Q}\mathcal{N}w,\mathcal{N}\mathcal{N}w,t)\mathcal{M}(\mathcal{N}\mathcal{N}w,\mathcal{P}z,t)\mathcal{M}(\mathcal{S}z,\mathcal{P}z,t),\\ \mathcal{M}(\mathcal{S}z,\mathcal{P}z,t)\mathcal{M}(\mathcal{Q}\mathcal{N}w,\mathcal{S}z,t)\mathcal{M}(\mathcal{Q}\mathcal{N}w,\mathcal{N}\mathcal{N}w,t)\end{pmatrix}\right\}$$

Using (3.3) and (3.5) $, we\ get$

$$\mathcal{M}^3(\mathcal{N}\mathcal{N}w,\mathcal{N}w,kt) \geq \psi\left\{min\begin{pmatrix}\mathcal{M}^2(\mathcal{N}\mathcal{N}w,\mathcal{N}\mathcal{N}w,t)\mathcal{M}(\mathcal{P}z,\mathcal{P}z,t),\\ \mathcal{M}(\mathcal{N}\mathcal{N}w,\mathcal{N}\mathcal{N}w,t)\mathcal{M}^2(\mathcal{P}z,\mathcal{P}z,t),\\ \mathcal{M}(\mathcal{N}\mathcal{N}w,\mathcal{N}\mathcal{N}w,t)\mathcal{M}(\mathcal{N}\mathcal{N}w,\mathcal{N}w,t)\mathcal{M}(\mathcal{P}z,\mathcal{P}z,t),\\ \mathcal{M}(\mathcal{P}z,\mathcal{P}z,t)\mathcal{M}(\mathcal{N}\mathcal{N}w,\mathcal{N}w,t)\mathcal{M}(\mathcal{N}\mathcal{N}w,\mathcal{N}\mathcal{N}w,t)\end{pmatrix}\right\}$$

Suppose $\mathcal{N}\mathcal{N}w \neq \mathcal{N}w$, then $\mathcal{M}(\mathcal{N}\mathcal{N}w,\mathcal{N}w,t) < 1$, using this in above inequality we get

$$\mathcal{M}^3(\mathcal{N}\mathcal{N}w,\mathcal{N}w,kt) \geq \psi\left\{min\begin{pmatrix}\mathcal{M}^3(\mathcal{N}\mathcal{N}w,\mathcal{N}w,t),\mathcal{M}^3(\mathcal{N}\mathcal{N}w,\mathcal{N}w,t),\\ \mathcal{M}^3(\mathcal{N}\mathcal{N}w,\mathcal{N}w,t),\mathcal{M}^3(\mathcal{N}\mathcal{N}w,\mathcal{N}w,t)\end{pmatrix}\right\}$$

Using property of ψ we get

$$\mathcal{M}^3(\mathcal{N}\mathcal{N}w,\mathcal{N}w,kt) > \mathcal{M}^3(\mathcal{N}\mathcal{N}w,\mathcal{N}w,t)$$
$$\Longrightarrow \mathcal{M}(\mathcal{N}\mathcal{N}w,\mathcal{N}w,t) > M(\mathcal{N}\mathcal{N}w,\mathcal{N}w,kt),$$

Hence $\mathcal{N}w = \mathcal{N}\mathcal{N}w$.

Thus $\mathcal{N}w = \mathcal{N}\mathcal{N}w = \mathcal{Q}\mathcal{N}w$

Thus $\mathcal{N}w$ is a common fixed point of $\mathcal{N}$ and $\mathcal{Q}$.

If $\mathcal{N}w = \mathcal{P}z = r$, then$\mathcal{N}r$=$\mathcal{P}r$=$\mathcal{Q}r$= $\mathcal{S}r$. Hence r is a common fixed point of$\mathcal{N}$,$\mathcal{P}$,$\mathcal{Q}$ and $\mathcal{S}$.

Uniqueness: Suppose that $z \neq w$ are two common fixed points of $\mathcal{N}$,$\mathcal{P}$,$\mathcal{Q}$ and $\mathcal{S}$. On putting u=w, v=z in (C_4), we have

$$\mathcal{M}^3(\mathcal{N}w,\mathcal{P}z,kt) \geq \psi\left\{min\begin{pmatrix} \mathcal{M}^2(\mathcal{Q}w,\mathcal{N}w,t)\mathcal{M}(\mathcal{S}z,\mathcal{P}z,t), \\ \mathcal{M}(\mathcal{Q}w,\mathcal{N}w,t)\mathcal{M}^2(\mathcal{S}z,\mathcal{P}z,t), \\ \mathcal{M}(\mathcal{Q}w,\mathcal{N}w,t)\mathcal{M}(\mathcal{N}w,\mathcal{P}z,t)\mathcal{M}(\mathcal{S}z,\mathcal{P}z,t), \\ \mathcal{M}(\mathcal{S}z,\mathcal{P}z,t)\mathcal{M}(\mathcal{Q}w,\mathcal{S}z,t)\mathcal{M}(\mathcal{Q}w,\mathcal{N}w,t) \end{pmatrix}\right\}$$

If $z \neq w$ thSen $\mathcal{M}(z,w,t) < 1$,using in above inequality and on simplification we get

$$\mathcal{M}(z,w,kt) > M(z,w,t),$$

Hence z= w

This completes the proof.

IV. $\mathcal{R}$ − WEAKLY COMMUTING MAPPINGS OF TYPE (P).

Imdad and Ali [5] created the notion of R - weakly commuting mappings of type (P) in fuzzy metric spaces in 2006, have used these mappings to prove a common fixed-point theorem.

We now prove a common fixed-point theorem for pairs of $\mathcal{R}$ − weakly commuting mappings of type (P) that satisfy a weak contraction condition covering numerous combinations of metric functions in fuzzy metric space.

Theorem 4.1 Let $(\mathfrak{V},\mathcal{M},*)$ be a complete fuzzy metric space. Let $\mathcal{N},\mathcal{P},\mathcal{Q}$ and $\mathcal{S}$ are four mappings of a complete fuzzy metric space $(\mathfrak{V},\mathcal{M},*)$ into itself satisfying

(C_1), (C_4) and the following condition:

$(\mathcal{Q},\mathcal{N})$ and $(\mathcal{S},\mathcal{P})are\mathcal{R}$ −weakly commuting of type (P), Then $\mathcal{Q}$,$\mathcal{N}$,$\mathcal{S}$and$\mathcal{P}$ have a unique common fixed point.

Proof. Let $x_0 \in \mathfrak{V}$ be an arbitrary point. From (C_1) we can find a point x_1 such that $\mathcal{N}(x_0) = \mathcal{S}(x_1) = y_0$. For this point x_1 one can find a point $x_2 \in \mathfrak{V}$ such that $\mathcal{P}(x_1) = \mathcal{Q}(x_2) = y_1$.Continuing in this way, one can construct a sequence $\{x_n\}$ such that $y_{2n} = \mathcal{N}(x_{2n}) = \mathcal{S}(x_{2n+1})$, $y_{2n+1} = \mathcal{P}(x_{2n+1}) = \mathcal{Q}(x_{2n+2})$, for each $n \geq 0$.

From Theorem 3.1,$\{y_n\}$ is a Cauchy sequence in $\mathfrak{V}$. From the completeness of $\mathfrak{V}$,the sequence $\{y_n\}$ converges to a point$z$ in $\mathfrak{V}$ as $n \to \infty$. Consequently, the subsequence's $\{\mathcal{N}x_{2n}\}$, $\{\mathcal{Q}x_{2n}\}$, $\{\mathcal{P}x_{2n+1}\}$ and $\{\mathcal{S}x_{2n+1}\}$ also converges to the same point z.

Case 1: Suppose that $\mathcal{Q}$ is continuous. Then $\{\mathcal{Q}\mathcal{Q}x_{2n}\}$and $\{\mathcal{Q}\mathcal{N}x_{2n}\}$ converges to $\mathcal{Q}z$ as $n \to \infty$. Since the mappings $\mathcal{Q}$ and $\mathcal{N}$ are $\mathcal{R}$ −weakly commuting of type (P), we have

$$.\ \mathcal{M}(\mathcal{Q}\mathcal{Q}x,\mathcal{N}\mathcal{N}x,t) \geq \mathcal{M}(\mathcal{Q}x,\mathcal{N}x,\tfrac{t}{\mathcal{R}})$$

Letting$n \to \infty$, we get $\lim_{n\to\infty} \mathcal{N}\mathcal{Q}\,x_{2n} = \mathcal{Q}z$

Now we claim that $z = \mathcal{Q}z$. For this put $u = \mathcal{Q}x_{2n}$ and $v = x_{2n+1}$ in (C_4), we get

$$\mathcal{M}^3(\mathcal{N}\mathcal{Q}x_{2n},\mathcal{P}x_{2n+1},kt)$$
$$\geq \psi\left\{min\begin{pmatrix} \mathcal{M}^2(\mathcal{Q}\mathcal{Q}x_{2n},\mathcal{N}\mathcal{Q}x_{2n},t)\mathcal{M}(\mathcal{S}x_{2n+1},\mathcal{P}x_{2n+1},t), \\ \mathcal{M}(\mathcal{Q}\mathcal{Q}x_{2n},\mathcal{N}\mathcal{Q}x_{2n},t)\mathcal{M}^2(\mathcal{S}x_{2n+1},\mathcal{P}x_{2n+1},t), \\ \mathcal{M}(\mathcal{Q}\mathcal{Q}x_{2n},\mathcal{N}\mathcal{Q}x_{2n},t)\mathcal{M}(\mathcal{N}\mathcal{Q}x_{2n},\mathcal{P}x_{2n+1},t)\mathcal{M}(\mathcal{S}x_{2n+1},\mathcal{P}x_{2n+1},t), \\ \mathcal{M}(\mathcal{S}x_{2n+1},\mathcal{P}x_{2n+1},t)\mathcal{M}(\mathcal{Q}\mathcal{Q}x_{2n},\mathcal{S}x_{2n+1},t)\mathcal{M}(\mathcal{Q}\mathcal{Q}x_{2n},\mathcal{N}\mathcal{Q}x_{2n},t) \end{pmatrix}\right\}$$

Taking as $n \to \infty$, the above inequality reduces to,

$$\mathcal{M}^3(\mathcal{Q}z,z,kt) \geq \psi\left\{min\begin{pmatrix} \mathcal{M}^2(\mathcal{Q}z,\mathcal{Q}z,t)\mathcal{M}(z,z,t), \\ \mathcal{M}(\mathcal{Q}z,\mathcal{Q}z,t)\mathcal{M}^2(z,z,t), \\ \mathcal{M}(\mathcal{Q}z,\mathcal{Q}z,t)\mathcal{M}(\mathcal{Q}z,z,t)\mathcal{M}(z,z,t), \\ \mathcal{M}(z,z,t)\mathcal{M}(\mathcal{Q}z,z,t)\mathcal{M}(\mathcal{Q}z,\mathcal{Q}z,t) \end{pmatrix}\right\}$$

$$\mathcal{M}^3(\mathcal{Q}z,z,kt) \geq \psi\{min(1.1.1\ ,1.1.1\ ,1.\mathcal{M}(\mathcal{Q}z,z,t).1\ ,1.\mathcal{M}(\mathcal{Q}z,z,t).1)\}$$

Suppose $\mathcal{Q}z \neq z$ then $\mathcal{M}(\mathcal{Q}z,z,t) < 1$,

Using this in the above inequality we get

$$\mathcal{M}^3(\mathcal{Q}z,z,kt) \geq \psi\left\{min\begin{pmatrix} \mathcal{M}(\mathcal{Q}z,z,t)\mathcal{M}(\mathcal{Q}z,z,t)\mathcal{M}(\mathcal{Q}z,z,t), \\ \mathcal{M}(\mathcal{Q}z,z,t)\mathcal{M}(\mathcal{Q}z,z,t)\mathcal{M}(\mathcal{Q}z,z,t), \\ \mathcal{M}(\mathcal{Q}z,z,t)\mathcal{M}(\mathcal{Q}z,z,t)\mathcal{M}(\mathcal{Q}z,z,t), \\ \mathcal{M}(\mathcal{Q}z,z,t)\mathcal{M}(\mathcal{Q}z,z,t)\mathcal{M}(\mathcal{Q}z,z,t) \end{pmatrix}\right\}$$

$$\mathcal{M}^3(\mathcal{Q}z,z,kt) \geq \psi\{min(\mathcal{M}^3(\mathcal{Q}z,z,t),\mathcal{M}^3(\mathcal{Q}z,z,t),\mathcal{M}^3(\mathcal{Q}z,z,t),\mathcal{M}^3(\mathcal{Q}z,z,t))\}$$

Using property of ψ we get,

$$\mathcal{M}^3(\mathcal{Q}z,z,kt) > \mathcal{M}^3(\mathcal{Q}z,z,t) \implies \mathcal{M}(\mathcal{Q}z,z,kt) > M(\mathcal{Q}z,z,t)\,,$$

Hence $\mathcal{Q}z = z$

Next, we shall show that $\mathcal{N}z=z$

For this, putting $u=z$ and $v= x_{2n+1}$ in (C_4), we get

$$\mathcal{M}^3(\mathcal{N}z,\mathcal{P}x_{2n+1},kt) \geq \psi\left\{min\begin{pmatrix} \mathcal{M}^2(\mathcal{Q}z,\mathcal{N}z,t)\mathcal{M}(\mathcal{S}x_{2n+1},\mathcal{P}x_{2n+1},t), \\ \mathcal{M}(\mathcal{Q}z,\mathcal{N}z,t)\mathcal{M}^2(\mathcal{S}x_{2n+1},\mathcal{P}x_{2n+1},t), \\ \mathcal{M}(\mathcal{Q}z,\mathcal{N}z,t)\mathcal{M}(\mathcal{N}z,\mathcal{P}x_{2n+1},t)\mathcal{M}(\mathcal{S}x_{2n+1},\mathcal{P}x_{2n+1},t), \\ \mathcal{M}(\mathcal{S}x_{2n+1},\mathcal{P}x_{2n+1},t)\mathcal{M}(\mathcal{Q}z,\mathcal{S}x_{2n+1},t)\mathcal{M}(\mathcal{Q}z,\mathcal{N}z,t) \end{pmatrix}\right\}$$

Taking as $n \to \infty$ and using $\mathcal{Q}z = z$, the above inequality reduces to,

$$\mathcal{M}^3(\mathcal{N}z,z,kt) \geq \psi\left\{min\begin{pmatrix} \mathcal{M}^2(z,\mathcal{N}z,t)\mathcal{M}(z,z,t), \\ \mathcal{M}(z,\mathcal{N}z,t)\mathcal{M}^2(z,z,t), \\ \mathcal{M}(z,\mathcal{N}z,t)\mathcal{M}(\mathcal{N}z,z,t)\mathcal{M}(z,z,t), \\ \mathcal{M}(z,z,t)\mathcal{M}(z,z,t)\mathcal{M}(z,\mathcal{N}z,t) \end{pmatrix}\right\}$$

Suppose $\mathcal{N}z \neq z$ then $\mathcal{M}(\mathcal{N}z,z,t) < 1$,

Using this in the above inequality we get

$$\mathcal{M}^3(\mathcal{N}z,z,kt) \geq \psi\{min(\mathcal{M}^3(\mathcal{N}z,z,t),\mathcal{M}^3(\mathcal{N}z,z,t),\mathcal{M}^3(\mathcal{N}z,z,t),\mathcal{M}^3(\mathcal{N}z,z,t))\}$$

Using property of ψ we get,

$$\mathcal{M}^3(\mathcal{N}z,z,kt) > \mathcal{M}^3(\mathcal{N}z,z,t) \implies \mathcal{M}(\mathcal{N}z,z,kt) > M(\mathcal{N}z,z,t),$$

Hence $\mathcal{N}z = z$

Since $\mathcal{N}(\mathfrak{V}) \subseteq \mathcal{S}(\mathfrak{V})$ and hence there exists a point $w \in \mathfrak{V}$ such that $z = \mathcal{N}z = \mathcal{S}w$.

We claim that $z = \mathcal{P}w$. To prove this we put $u = z$ and $v = w$ in (C_4), we get

$$\mathcal{M}^3(\mathcal{N}z,\mathcal{P}w,kt) \geq \psi\left\{min\begin{pmatrix} \mathcal{M}^2(\mathcal{Q}z,\mathcal{N}z,t)\mathcal{M}(\mathcal{S}w,\mathcal{P}w,t) \\ \mathcal{M}(\mathcal{Q}z,\mathcal{N}z,t)\mathcal{M}^2(\mathcal{S}w,\mathcal{P}w,t), \\ \mathcal{M}(\mathcal{Q}z,\mathcal{N}z,t)\mathcal{M}(\mathcal{N}z,\mathcal{P}w,t)\mathcal{M}(\mathcal{S}w,\mathcal{P}w,t), \\ \mathcal{M}(\mathcal{S}w,\mathcal{P}w,t)\mathcal{M}(\mathcal{Q}z,\mathcal{S}w,t)\mathcal{M}(\mathcal{Q}z,\mathcal{N}z,t) \end{pmatrix}\right\},$$

Using $\mathcal{N}z = z = \mathcal{S}w$ and $\mathcal{Q}z = z$ in above inequality we get

$$\mathcal{M}^3(z,\mathcal{P}w,kt) \geq \psi\left\{min\begin{pmatrix} \mathcal{M}^2(z,z,t)\mathcal{M}(z,\mathcal{P}w,t) \\ \mathcal{M}(z,z,t)\mathcal{M}^2(z,\mathcal{P}w,t), \\ \mathcal{M}(z,z,t)\mathcal{M}(z,\mathcal{P}w,t)\mathcal{M}(z,\mathcal{P}w,t), \\ \mathcal{M}(z,\mathcal{P}w,t)\mathcal{M}(z,z,t)\mathcal{M}(z,z,t) \end{pmatrix}\right\}$$

Suppose $z \neq \mathcal{P}w$, then $\mathcal{M}(z,\mathcal{P}w,t) < 1$, using this in the above inequality we get

$$\mathcal{M}^3(z,\mathcal{P}w,kt) \geq \psi\left\{min\begin{pmatrix} \mathcal{M}^2(z,\mathcal{P}w,t)\mathcal{M}(z,\mathcal{P}w,t) \\ \mathcal{M}(z,\mathcal{P}w,t)\mathcal{M}^2(z,\mathcal{P}w,t), \\ \mathcal{M}(z,\mathcal{P}w,t)\mathcal{M}(z,\mathcal{P}w,t)\mathcal{M}(z,\mathcal{P}w,t), \\ \mathcal{M}(z,\mathcal{P}w,t)\mathcal{M}(z,\mathcal{P}w,t)\mathcal{M}(z,\mathcal{P}w,t) \end{pmatrix}\right\}$$

On simplification, and using property of ψ we have

$$\mathcal{M}^3(z,\mathcal{P}w,kt) > \mathcal{M}^3(z,\mathcal{P}w,t) \Rightarrow \mathcal{M}(z,\mathcal{P}w,kt) > M(z,\mathcal{P}w,t),$$

Hence $\mathcal{P}w = z$.

Since $(\mathcal{S},\mathcal{P})$ *is* $\mathcal{R} -$ weakly commuting of type (P),we have

$\mathcal{M}(\mathcal{S}z,\mathcal{P}z,t) = \mathcal{M}(\mathcal{S}\mathcal{S}w,\mathcal{P}\mathcal{P}w,t) \geq \mathcal{M}(\mathcal{S}w,\mathcal{P}w,\frac{t}{\mathcal{R}}) = \mathcal{M}(z,z,\frac{t}{\mathcal{R}})$;

Hence $\mathcal{S}z=\mathcal{P}z$

Finally, we have

$$\mathcal{M}^3(\mathcal{N}z,\mathcal{P}z,kt) \geq \psi\left\{min\begin{pmatrix} \mathcal{M}^2(\mathcal{Q}z,\mathcal{N}z,t)\mathcal{M}(\mathcal{S}z,\mathcal{P}z,t) \\ \mathcal{M}(\mathcal{Q}z,\mathcal{N}z,t)\mathcal{M}^2(\mathcal{S}z,\mathcal{P}z,t), \\ \mathcal{M}(\mathcal{Q}z,\mathcal{N}z,t)\mathcal{M}(\mathcal{N}z,\mathcal{P}z,t)\mathcal{M}(\mathcal{S}z,\mathcal{P}z,t), \\ \mathcal{M}(\mathcal{S}z,\mathcal{P}z,t)\mathcal{M}(\mathcal{Q}z,\mathcal{S}z,t)\mathcal{M}(\mathcal{Q}z,\mathcal{N}z,t) \end{pmatrix}\right\}$$

Using $\mathcal{N}z = z, \mathcal{S}z = \mathcal{P}z$ and $\mathcal{Q}z = z$ in above inequality we get

$$\mathcal{M}^3(z,\mathcal{P}z,kt) \geq \psi\left\{min\begin{pmatrix} \mathcal{M}^2(z,z,t)\mathcal{M}(\mathcal{P}z,\mathcal{P}z,t) \\ \mathcal{M}(z,z,t)\mathcal{M}^2(\mathcal{P}z,\mathcal{P}z,t), \\ \mathcal{M}(z,z,t)\mathcal{M}(z,\mathcal{P}z,t)\mathcal{M}(\mathcal{P}z,\mathcal{P}z,t), \\ \mathcal{M}(\mathcal{P}z,\mathcal{P}z,t)\mathcal{M}(z,\mathcal{P}z,t)\mathcal{M}(z,z,t) \end{pmatrix}\right\}$$

Suppose $z \neq \mathcal{P}z$,then $\mathcal{M}(z,\mathcal{P}z,t) <1$,using this in above inequality we get

$$\mathcal{M}^3(z,\mathcal{P}z,kt) \geq \psi\{min(\mathcal{M}^3(z,\mathcal{P}z,t),\mathcal{M}^3(z,\mathcal{P}z,t),\mathcal{M}^3(z,\mathcal{P}z,t),\mathcal{M}^3(z,\mathcal{P}z,t))\}$$

Using property of ψ we get

$$\mathcal{M}^3(z,\mathcal{P}z,kt) > \mathcal{M}^3(z,\mathcal{P}z,t) \implies \mathcal{M}(z,\mathcal{P}z,kt) > M(z,\mathcal{P}z,t),$$

This implies that $\mathcal{P}z = z$. Hence $z = \mathcal{Q}z = \mathcal{P}z = \mathcal{N}z = \mathcal{S}z$. Therefore, z is a common fixed point of $\mathcal{P},\mathcal{Q},\mathcal{N}$ and $\mathcal{S}$.

Case 2: Suppose that $\mathcal{S}$is continuous. Then we can obtain the same result by using Case 1.

Case 3: Suppose that $\mathcal{N}$ is continuous. Then $\{\mathcal{N}\mathcal{N}x_{2n}\}$and $\{\mathcal{N}\mathcal{Q}x_{2n}\}$ converges to $\mathcal{N}z$ as $n \to \infty$. Since the mappings $\mathcal{Q}$ and $\mathcal{N}$ are $\mathcal{R}$ −weakly commuting of type (P), we have

$$. \mathcal{M}(\mathcal{Q}\mathcal{Q}x, \mathcal{N}\mathcal{N}x, t) \geq \mathcal{M}(\mathcal{Q}x, \mathcal{N}x, \tfrac{t}{\mathcal{R}})$$

Letting $n\to \infty$, we get $\lim_{n\to\infty} \mathcal{Q}\,\mathcal{N}x_{2n} = \mathcal{N}z$

Now,we claim that $z = \mathcal{N}z$. For this put $u = \mathcal{N}x_{2n}$ and $v = x_{2n+1}$ in (C_4), we get

$$\mathcal{M}^3(\mathcal{N}\mathcal{N}x_{2n}, \mathcal{P}x_{2n+1}, kt) \geq \psi\left\{min\begin{pmatrix}\mathcal{M}^2(\mathcal{Q}\mathcal{N}x_{2n}, \mathcal{N}\mathcal{N}x_{2n}, t)\mathcal{M}(\mathcal{S}x_{2n+1}, \mathcal{P}x_{2n+1}, t),\\ \mathcal{M}(\mathcal{Q}\mathcal{N}x_{2n}, \mathcal{N}\mathcal{N}x_{2n}, t)\mathcal{M}^2(\mathcal{S}x_{2n+1}, \mathcal{P}x_{2n+1}, t),\\ \mathcal{M}(\mathcal{Q}\mathcal{N}x_{2n}, \mathcal{N}\mathcal{N}x_{2n}, t)\mathcal{M}(\mathcal{N}\mathcal{N}x_{2n}, \mathcal{P}x_{2n+1}, t)\mathcal{M}(\mathcal{S}x_{2n+1}, \mathcal{P}x_{2n+1}, t),\\ \mathcal{M}(\mathcal{S}x_{2n+1}, \mathcal{P}x_{2n+1}, t)\mathcal{M}(\mathcal{Q}\mathcal{N}x_{2n}, \mathcal{S}x_{2n+1}, t)\mathcal{M}(\mathcal{Q}\mathcal{N}x_{2n}, \mathcal{N}\mathcal{N}x_{2n}, t)\end{pmatrix}\right\}$$

Now proceeding limit as $n \to \infty$, we have

$$\mathcal{M}^3(\mathcal{N}z, z, kt) \geq \psi\left\{min\begin{pmatrix}\mathcal{M}^2(\mathcal{N}z, \mathcal{N}z, t)\mathcal{M}(z, z, t),\\ \mathcal{M}(\mathcal{N}z, \mathcal{N}z, t)\mathcal{M}^2(z, z, t),\\ \mathcal{M}(\mathcal{N}z, \mathcal{N}z, t)\mathcal{M}(\mathcal{N}z, z, t)\mathcal{M}(z, z, t),\\ \mathcal{M}(z, z, t)\mathcal{M}(\mathcal{N}z, z, t)\mathcal{M}(\mathcal{N}z, \mathcal{N}z, t)\end{pmatrix}\right\}$$

Suppose $z \neq \mathcal{N}z$, then $\mathcal{M}(\mathcal{N}z, z, t) < 1$, using this in above inequality we get

$$\mathcal{M}^3(\mathcal{N}z, z, kt) \geq \psi\{min(\mathcal{M}^3(\mathcal{N}z, z, t), \mathcal{M}^3(\mathcal{N}z, z, t), \mathcal{M}^3(\mathcal{N}z, z, t), \mathcal{M}^3(\mathcal{N}z, z, t))\}$$

Using property of ψ we get

$$\mathcal{M}^3(\mathcal{N}z, z, kt) > \mathcal{M}^3(\mathcal{N}z, z, t) \Rightarrow \mathcal{M}(\mathcal{N}z, z, kt) > M(\mathcal{N}z, z, t),$$

Hence $z = \mathcal{N}z$.

Since $\mathcal{N}(\mathfrak{V}) \subseteq \mathcal{S}(\mathfrak{V})$ and hence there exists a point $q \in \mathfrak{V}$ such that $z = \mathcal{N}z = \mathcal{S}q$.

We claim that $z = \mathcal{P}q$.

To prove this, we put $u = \mathcal{N}x_{2n}$ and $v = q$ in (C_4) we get

$$\mathcal{M}^3(\mathcal{N}\mathcal{N}x_{2n}, \mathcal{P}q, kt) \geq \psi\left\{min\begin{pmatrix}\mathcal{M}^2(\mathcal{Q}\mathcal{N}x_{2n}, \mathcal{N}\mathcal{N}x_{2n}, t)\mathcal{M}(\mathcal{S}q, \mathcal{P}q, t),\\ \mathcal{M}(\mathcal{Q}\mathcal{N}x_{2n}, \mathcal{N}\mathcal{N}x_{2n}, t)\mathcal{M}^2(\mathcal{S}q, \mathcal{P}q, t),\\ \mathcal{M}(\mathcal{Q}\mathcal{N}x_{2n}, \mathcal{N}\mathcal{N}x_{2n}, t)\mathcal{M}(\mathcal{N}\mathcal{N}x_{2n}, \mathcal{P}q, t)\mathcal{M}(\mathcal{S}q, \mathcal{P}q, t),\\ \mathcal{M}(\mathcal{S}q, \mathcal{P}q, t)\mathcal{M}(\mathcal{Q}\mathcal{N}x_{2n}, \mathcal{S}q, t)\mathcal{M}(\mathcal{Q}\mathcal{N}x_{2n}, \mathcal{N}\mathcal{N}x_{2n}, t)\end{pmatrix}\right\}$$

Now proceeding limit as $n \to \infty$ and using $z = \mathcal{N}z = \mathcal{S}q$, we have

$$\mathcal{M}^3(z, \mathcal{P}q, kt) \geq \psi\left\{min\begin{pmatrix}\mathcal{M}^2(z, z, t)\mathcal{M}(z, \mathcal{P}q, t),\\ \mathcal{M}(zs, z, t)\mathcal{M}^2(z, \mathcal{P}q, t),\\ \mathcal{M}(z, z, t)\mathcal{M}(z, \mathcal{P}q, t)\mathcal{M}(z, \mathcal{P}q, t),\\ \mathcal{M}(z, \mathcal{P}q, t)\mathcal{M}(z, z, t)\mathcal{M}(z, z, t)\end{pmatrix}\right\}$$

Suppose $z \neq \mathcal{P}q$, then $\mathcal{M}(z, \mathcal{P}q, t) < 1$, using this in above inequality we get

$$\mathcal{M}^3(z, \mathcal{P}q, kt) \geq \psi\{min(\mathcal{M}^3(z, \mathcal{P}q, t), \mathcal{M}^3(z, \mathcal{P}q, t), \mathcal{M}^3(z, \mathcal{P}q, t), \mathcal{M}^3(z, \mathcal{P}q, t))\}$$

Using property of ψ we get

$$\mathcal{M}^3(z, \mathcal{P}q, kt) > \mathcal{M}^3(z, \mathcal{P}q, t) \Rightarrow \mathcal{M}(z, \mathcal{P}q, kt) > M(z, \mathcal{P}q, t),$$

Hence $\mathcal{P}q = z$

Since $(\mathcal{S}, \mathcal{P})$is $\mathcal{R}$ − weakly commuting of type (P),we have

$\mathcal{M}(\mathcal{S}z, \mathcal{P}z, t) = \mathcal{M}(\mathcal{S}\mathcal{S}q, \mathcal{P}\mathcal{P}q, t) \geq \mathcal{M}(\mathcal{S}q, \mathcal{P}q, \tfrac{t}{\mathcal{R}}) = \mathcal{M}(z, z, \tfrac{t}{\mathcal{R}})$;

Hence $\mathcal{S}z$=$\mathcal{P}z$.

Finally from (C_4) we have

$$\mathcal{M}^3(\mathcal{N}x_{2n}, \mathcal{P}z, kt) \geq \psi\left\{min\begin{pmatrix}\mathcal{M}^2(\mathcal{Q}x_{2n}, \mathcal{N}x_{2n}, t)\mathcal{M}(\mathcal{S}z, \mathcal{P}z, t),\\ \mathcal{M}(\mathcal{Q}x_{2n}, \mathcal{N}x_{2n}, t)\mathcal{M}^2(\mathcal{S}z, \mathcal{P}z, t),\\ \mathcal{M}(\mathcal{Q}x_{2n}, \mathcal{N}x_{2n}, t)\mathcal{M}(\mathcal{N}x_{2n}, \mathcal{P}z, t)\mathcal{M}(\mathcal{S}z, \mathcal{P}z, t),\\ \mathcal{M}(\mathcal{S}z, \mathcal{P}z, t)\mathcal{M}(\mathcal{Q}x_{2n}, \mathcal{S}z, t)\mathcal{M}(\mathcal{Q}x_{2n}, \mathcal{N}x_{2n}, t)\end{pmatrix}\right\}$$

Now proceeding limit as $n \to \infty$ and using $\mathcal{S}z = \mathcal{P}z.$, we have

$$\mathcal{M}^3(z, \mathcal{P}z, kt) \geq \psi\left\{min\begin{pmatrix}\mathcal{M}^2(z, z, t)\mathcal{M}(\mathcal{P}z, \mathcal{P}z, t),\\ \mathcal{M}(z, z, t)\mathcal{M}^2(\mathcal{P}z, \mathcal{P}z, t),\\ \mathcal{M}(z, z, t)\mathcal{M}(z, \mathcal{P}z, t)\mathcal{M}(\mathcal{P}z, \mathcal{P}z, t),\\ \mathcal{M}(\mathcal{P}z, \mathcal{P}z, t)\mathcal{M}(z, \mathcal{P}z, t)\mathcal{M}(z, z, t)\end{pmatrix}\right\}$$

Suppose $z \neq \mathcal{P}z$, then $\mathcal{M}(z, \mathcal{P}z, t) < 1$, using this in above inequality we get

$$\mathcal{M}^3(z,\mathcal{P}z,kt) \geq \psi\{min(\mathcal{M}^3(z,\mathcal{P}z,t),\mathcal{M}^3(z,\mathcal{P}z,t),\mathcal{M}^3(z,\mathcal{P}z,t),\mathcal{M}^3(z,\mathcal{P}z,t))\}$$

Using property of ψ we get

$$\mathcal{M}^3(z,\mathcal{P}z,kt) > \mathcal{M}^3(z,\mathcal{P}z,t) \Longrightarrow \mathcal{M}(z,\mathcal{P}z,kt) > M(z,\mathcal{P}z,t),$$

Hence $\mathcal{P}z = z$

Since $\mathcal{P}(\mathfrak{V}) \subseteq \mathcal{Q}(\mathfrak{V})$, therefore there exists a point $w \in \mathfrak{V}$ such that $z = \mathcal{P}z = \mathcal{Q}w$.

We claim that $z = \mathcal{N}w$.

For this we put $u = w$ and $v = z$ in (C_4) we get

$$\mathcal{M}^3(\mathcal{N}w,\mathcal{P}z,kt) \geq \psi\left\{min\begin{pmatrix}\mathcal{M}^2(\mathcal{Q}w,\mathcal{N}w,t)\mathcal{M}(\mathcal{S}z,\mathcal{P}z,t),\\ \mathcal{M}(\mathcal{Q}w,\mathcal{N}w,t)\mathcal{M}^2(\mathcal{S}z,\mathcal{P}z,t),\\ \mathcal{M}(\mathcal{Q}w,\mathcal{N}w,t)\mathcal{M}(\mathcal{N}w,\mathcal{P}z,t)\mathcal{M}(\mathcal{S}z,\mathcal{P}z,t),\\ \mathcal{M}(\mathcal{S}z,\mathcal{P}z,t)\mathcal{M}(\mathcal{Q}w,\mathcal{S}z,t)\mathcal{M}(\mathcal{Q}w,\mathcal{N}w,t)\end{pmatrix}\right\}$$

Using $z = \mathcal{P}z = \mathcal{Q}w$, *we get*

$$\mathcal{M}^3(\mathcal{N}w,z,kt) \geq \psi\left\{min\begin{pmatrix}\mathcal{M}^2(z,\mathcal{N}w,t)\mathcal{M}(z,z,t),\\ \mathcal{M}(z,\mathcal{N}w,t)\mathcal{M}^2(z,z,t),\\ \mathcal{M}(z,\mathcal{N}w,t)\mathcal{M}(\mathcal{N}w,z,t)\mathcal{M}(z,z,t),\\ \mathcal{M}(z,z,t)\mathcal{M}(z,z,t)\mathcal{M}(z,\mathcal{N}w,t)\end{pmatrix}\right\}$$

Suppose $z \neq \mathcal{N}w$, then $\mathcal{M}(\mathcal{N}w,z,t) < 1$, using this in above inequality we get

$$\mathcal{M}^3(\mathcal{N}w,z,kt) \geq \psi\{min(\mathcal{M}^3(\mathcal{N}w,z,t),\mathcal{M}^3(\mathcal{N}w,z,t),\mathcal{M}^3(\mathcal{N}w,z,t),\mathcal{M}^3(\mathcal{N}w,z,t))\}$$

Using property of ψ we get

$$\mathcal{M}^3(\mathcal{N}w,z,kt) > \mathcal{M}^3(\mathcal{N}w,z,t) \Longrightarrow \mathcal{M}(\mathcal{N}w,z,kt) > M(\mathcal{N}w,z,t),$$

Hence $\mathcal{N}w = z$

Since $(\mathcal{N},\mathcal{Q})$ *is* $\mathcal{R}$ − weakly commuting of type (P),we have

$\mathcal{M}(\mathcal{Q}z,\mathcal{N}z,t) = \mathcal{M}(\mathcal{Q}\mathcal{Q}w,\mathcal{N}\mathcal{N}w,t) \geq \mathcal{M}(\mathcal{Q}w,\mathcal{N}w,\frac{t}{\mathcal{R}}) = \mathcal{M}(z,z,\frac{t}{\mathcal{R}})$;

Hence $\mathcal{Q}z$=$\mathcal{N}z$

Hence z=$\mathcal{N}z$= $\mathcal{Q}z$= $\mathcal{S}z$ =$\mathcal{P}z$, and z is a common fixed point of $\mathcal{Q}$, $\mathcal{N}$, $\mathcal{S}$ and $\mathcal{P}$.

Case 4: Suppose that $\mathcal{P}$ is continuous. We can obtain the same result by using Case 3.

Uniqueness: Suppose that $z \neq w$ are two common fixed points of$\mathcal{Q}$, $\mathcal{N}$, $\mathcal{S}$ and $\mathcal{P}$.

Put $u = z$ and $v = w$ in (C_4), we get

$$\mathcal{M}^3(\mathcal{N}z,\mathcal{P}w,kt) \geq \psi\left\{min\begin{pmatrix}\mathcal{M}^2(\mathcal{Q}z,\mathcal{N}z,t)\mathcal{M}(\mathcal{S}w,\mathcal{P}w,t),\\ \mathcal{M}(\mathcal{Q}z,\mathcal{N}z,t)\mathcal{M}^2(\mathcal{S}w,\mathcal{P}w,t),\\ \mathcal{M}(\mathcal{Q}z,\mathcal{N}z,t)\mathcal{M}(\mathcal{N}z,\mathcal{P}w,t)\mathcal{M}(\mathcal{S}w,\mathcal{P}w,t),\\ \mathcal{M}(\mathcal{S}w,\mathcal{P}w,t)\mathcal{M}(\mathcal{Q}z,\mathcal{S}w,t)\mathcal{M}(\mathcal{Q}z,\mathcal{N}z,t)\end{pmatrix}\right\}$$

If $z \neq w$ then $\mathcal{M}(z,w,t) < 1$,using in above inequality and on simplification we get

$$\mathcal{M}(z,w,kt) > M(z,w,t),$$

Hence z= w

This completes the proof.

V. APPLICATION

In 2002 Branciari obtained a fixed point theorem for a single mapping satisfying an analogue of a Banach contraction principle for integral type inequality .

Now we prove the following theorem as an application of Theorem 3.1

Theorem 5.1 Let $\mathcal{N}$, $\mathcal{P}$, $\mathcal{Q}$and$\mathcal{S}$ be four self-mappings of a complete fuzzy metric space $(\mathfrak{V},\mathcal{M},*)$ satisfying the conditions(C_1),(C_2),(C_3) and the following condition:

$$\int_0^{\mathcal{M}^3(x,y,t)} \psi(w)dw \geq \int_0^{\sigma(u,v)} \psi(w)dw$$

$$\sigma(u,v) = \psi\left\{min\begin{pmatrix}\mathcal{M}^2(\mathcal{Q}u,\mathcal{N}u,t)\mathcal{M}(\mathcal{S}v,\mathcal{P}v,t)\\ \mathcal{M}(\mathcal{Q}u,\mathcal{N}u,t)\mathcal{M}^2(\mathcal{S}v,\mathcal{P}v,t),\\ \mathcal{M}(\mathcal{Q}u,\mathcal{N}u,t)\mathcal{M}(\mathcal{N}u,\mathcal{P}v,t)\mathcal{M}(\mathcal{S}v,\mathcal{P}v,t),\\ \mathcal{M}(\mathcal{S}v,\mathcal{P}v,t)\mathcal{M}(\mathcal{Q}u,\mathcal{S}v,t)\mathcal{M}(\mathcal{Q}u,\mathcal{N}u,t)\end{pmatrix}\right\}$$

for all u, $v \in \mathfrak{V}$, where $\psi: [0,1] \to [0,1]$ is increasing in any coordinate and $\psi(t,t,t,t) > t$ for every $t \in [0,1)$,where $\psi: [0,1]^4 \to [0,1]$is a "Lebesgue-integrable function" which is summable, nonnegative, and such that, for each$\epsilon > 0$, $\int_0^{\epsilon} \psi(w)dw > 0$. Then$\mathcal{N}$, $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{S}$have a unique common fixed point in $\mathfrak{V}$.

Proof: The proof of the theorem follows on the same lines of the proof of the theorem 3.1

Conclusion We show common fixed-point theorems for $\mathcal{R}$ −weakly commuting and reciprocal mappings in fuzzy metric space that contains cubic and quadratic terms of the distance function $\mathcal{M}$ (x,y,t) in this study.

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