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Research Paper

Existence Theory for Quadratic Random Differential Equation

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***Abstract**: In this paper, we investigate the first order quadratic random functional differential equation on unbounded intervals. We prove the existence and attractivity results of the solution using hybrid fixed point theory.*
***Keywords:** Random differential equation, Random solution, Fixed point theory, global attractivity.*
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I. Introduction

Consider the following quadratic random functional differential equation on unbounded intervals,

$$\left[\frac{p(t,\omega)u(t,\omega)}{f(t,u(t,\omega),\omega)}\right]' = g(t,u(t,\omega),u_t,\omega)+h(t,u(t,\omega),u_t,\omega)+k(t,u(t,\omega),u_t,\omega)\text{ , } a.e.\, t\in R_+ \quad (1.1)$$

Where $p\in CRB(R_+),\ f:R_+\times R\to R\setminus\{0\}$, $g:R_+\times R\times C\to R\, and\, h:R_+\times R\times C\to R.$

The problem(1.1)have been studied on closed and bounded intervals by many authors..The above problem (1.1) is not discussed on unbounded intervals .Here, we have discussed on unbounded intervals and prove the existence and attractivity results by application of hybrid fixed point theory.

II. Auxiliary Results

Let $I_0=[-\delta,0]$ be a closed ,bounded interval in real line R for some real number $\delta>0$ and let $J=I_0\cup R_+$.
We have use the following result for proving the main existence result.

Theorem2.1 (Dhage[10]). Let S be a non-empty, closed convex and bounded subset of the Banach algebra U and Let $A:U\to U\ and\ B:S\to U$ be two operators such that

(i) $A(\omega)$ $A(\omega)$ is D-Lipschitz with D-function ψ,

(ii) $B(\omega)$ $B(\omega)$ is completely continuous,

(iii) $u=Au\,Bv\Rightarrow u\in S\ for\ all\ v\in S, and$

(iv) $M\,\psi(t)<r, where\ M=\|B(S)\|=\sup\{\|Bu\|:u\in S\}$

Then the operator equation $Au\,Bu=u$ has a solution in S.

We have needed following definitions.

Definition 2.1. The solutions of the operator equation $Qu(t)=u(t)$ are locally attractive if there exists a closed ball $\bar{B}_r(u_0)$ in $BC(I_0\cup R_+,R)$ for some $u_0\in BC(I_0\cup R_+,R)$ such that for arbitrary solutions $u=u(t)\, and\, v=v(t)$ of equation $Qu(t)=u(t)$ belonging to $\bar{B}_r(u_0)$.

In the case when the limit is uniform with respect to the set $\overline{B}_r(u_0)$, then say that solutions of equation $Qu(t)=u(t)$ are uniformly locally attractive on $I_0 \cup R_+$.

Definition 2.2. A solution $u = u(t,)$ of equation $Qu(t)=u(t)$ is said to be globally attractive if $\lim_{t\to\infty}(u(t)-v(t))=0$ holds for each solution $v = v(t)$ of $Qu(t)=u(t)$ in $BC(I_0 \cup R_+, R)$.

III. Main Result

Consider the following.

(A_1). There is a continuous function h : $R_+ \to R_+$ such that $|g(t,u,v)| \le h(t) \quad a.e.\ t\in R_+$

for all $u \in R$ and $v \in C$. Also, let $\lim_{t\to\infty} |\overline{p}(t)| \int_0^t h(s)ds = 0$

$(A_2)\ \phi(0) \ge 0$.

(A_3). The function $t \to f(t,0,0)$ is bounded on R_+ with $F_0 = \sup\{|f(t,0,0)| \ : \ t \in R_+\}$.

(A_4) .The function $f : R_+ \times R \to R$ is continuous and there exists a function $\ell \in BC(R_+, R)$ and a real number K > 0 such that

$$|f(t,u) - f(t,v)| \ell(t) \frac{|u-v|}{K+|u-v|} \quad \text{for all } t \in R_+ \ and \ u,v \in R.$$ also

suppose $\sup_{t\ge 0} \ell(t) = L.$

(A_5). $\lim_{t\to\infty}\left[|f(t,u)| - f(t,v)\right] = 0 \ for\ all\ u \in R.$

(A_6) . $f(0,\phi(0)) = 1$

(A_7). Suppose $u \to \dfrac{u}{f(0,u)}$ *is injective*.

Theorem 3.1. Suppose that (A_1), (A_3), (A_4), (A_6) and(A_7)) holds. Further, assume that

$$L \max\{\|\phi\|, |\phi(0)| \|\overline{a}\| + W\} \le K. \tag{3.1}$$

Then problem (1.1) admits a solution and solution is uniformly globally attractive.

Proof. Now, using hypotheses (A_6) and (A_7) it can be shown that the problem (1.1) is equivalent to the functional integral equation

$$u(t) = \begin{cases} [f(t,u(t))]\left(\phi(0)\overline{a}(t) + \overline{a}(t)\int_0^t [g(s,u(s),u_s) + h(s,u(s),u_s) + k(s,u(s),u_s)]ds\right), & if\ t \in R_+ \\ \phi(t), \quad if\ t \in I_0 \end{cases} \tag{3.2}$$

Set $U = BC(I_0 \cup R_+, R)$ and define a closed ball $\overline{B}_r(0)$ in U centered at origin of radius r given by

$$r = \max\{1, L + F_0\} \max\{\|\phi\|, |\phi(0)| \|\overline{a}\| + W\}$$

Define the operators A, B on X , $\overline{B}_r(0)$ respectively by

$$Au(t) = \begin{cases} f(t,u(t)), & if\ t \in R_+ \\ 1, & if\ t \in I_0 \end{cases} \tag{3.3}$$

And $$Bu(t) = \begin{cases} \phi(0)\overline{p}(t) + \overline{p}\int_0^t \left[g(s,u(s),u_s) + h(s,u(s),u_s) + k(s,u(s),u_s)\right]ds, & if\ t \in R_+ \\ \phi(t), \quad if\ t \in I_0. \end{cases}$$

Then the equation(3.2) is transformed into the operator equation as

$$Au(t)\,Bu(t) = u(t), \ t \in I_0 \cup R_+. \tag{3.4}$$

We have to Show that A and B satisfy all the conditions of Theorem 2.1 on $BC(I_0 \cup R_+, R)$ First we show that the operators A and B define the mappings $A : U \to U \; and \; B : \bar{B}_r(0) \to U.$ be arbitrary. Obviously, Au is a continuous function on $I_0 \cup R_+$. We show that Au is bounded on $I_0 \cup R_+$. Thus, if $t \in R_+$, then we obtain:

$$|Au(t)| = |f(t,u(t))| \le |f(t,u(t)) - f(t,0)| + |f(t,0)|$$

$$\le \ell(t) \frac{|u(t)|}{K + |u(t)|} + F_0 \le L + F_0$$

Similarly, $|Au(t)| \le 1 \; for\, all\, t \in I_0$. Therefore, as supremum ,

$$|Au\| \le \max\{1, L + F_0\} = N \; \|Au\| \le \max\{1, L + F_0\} = N$$

Thus Au is continuous and bounded on $I_0 \cup R_+$. As a result $Au \in U$. It can be shown that $Bu \in U$ and in particular, $A : U \to U \;\; and \;\; B : \bar{B}_r(0) \to U.$ We show that A is a Lipschitz on U. Let $u, v \in U$ be arbitrary. Then, by hypothesis (A_3),

$$\|Au - Av\| = \sup_{t \in I_0 \cup \mathbb{R}_+} |Au(t) - Av(t)|$$

$$\le \max\left\{ \sup_{t \in I_0} |Au(t) - Av(t)|, \sup_{t \in \mathbb{R}_+} |Au(t) - Av(t)| \right\}$$

$$\le \max \left\{ 0, \sup_{t \in \mathbb{R}_+} \ell(t) \frac{|u(t) - v(t)|}{K + |u(t) - v(t)|} \right\}$$

$$\le \frac{L\|u - v\|}{K + \|u - v\|}$$

for all u, $v \in U$. This shows that A is a D-Lipschitz on U with D-function $\psi \; (r) = \dfrac{Lr}{K + r}$ next, it can be shown that B is a compact and continuous operator on U and in particular on $\bar{B}_r(0)$ Next , we estimate the value of the constant M.By definition of M, as

$$|B(\bar{B}_r(0))\| = \sup\{\|Bu\| : u \in \bar{B}_r(0)\}$$

$$= \sup\left\{ \sup_{t \in I_0 \cup \mathbb{R}_+} |Bu(t)| : u \in \bar{B}_r(0) \right\}$$

$$\le \sup\left\{ \max\left\{ \sup_{t \in I_0} |Bu(t)|, \sup_{t \in \mathbb{R}_+} |Bu(t)| \right\} : \; u \in \bar{B}_r(0) \right\}$$

$$\le \sup_{u \in \bar{B}_r(0)} \Big\{ \max\Big\{ \|\phi\|, |\phi(0)\| \bar{u}(t)|$$

$$+ \sup_{t \in R_+} |\bar{p}(t)| \int_0^t |g(s,u(s),u_s) + h(s,u(s),u_s) + k(s,u(s),u_s)| \, ds\}\}$$

$$\le \max \{\|\phi\|, |\phi(0))\| \bar{p}\| + W\}$$

Thus,

$$\|Bu\| \le \max\{\|\phi\|, |\phi(0)| \, \| \bar{p}| + W\} = M$$

for all $u \in \bar{B}_r(0)$. Next , let $u, v \in U$ be arbitrary. Then,

$$\begin{aligned}
|u(t)| &\leq |Au(t)|\ |Bv(t)| \\
&\leq \| Au \|\, \| Bv \| \\
&\leq \| A(U) \|\, \| B(\bar{B}_r(0)) \| \\
&\leq \max\{1, L+F_0\} M \\
&\leq \max\{1, L+F_0\} \max\{\|\phi\|, |\phi(0)|\, \|\bar{p}\| + W\} \\
&= r
\end{aligned}$$

For all $t \in I_0 \cup R_+$. Therefore, we have:

$$\| u \| \leq \max\{1, L+F_0\} \max\{\|\phi\|, |\phi(0)|\ \|\bar{p}\| + W\} = r$$

This shows that $u \in \bar{B}_r(0)$ and hypothesis (iii) of Theorem 2.1 is satisfied. Again,

$$M\phi(r) \leq \frac{L \max\{\|\phi\|\, |\phi(0)|\, \|\bar{p}\| + W\} r}{K + r} < r$$

For r>0, because $L \max\{\|\phi\|\, |\phi(0)|\, \|\bar{p}\| + W\} \leq K.$

Therefore, hypothesis (iv) of Theorem 2.1 is satisfied. Now we apply Theorem 2.1 to the operator equation $Au\, Bu = u$ to yield that the problem (1.1) has a solution on $I_0 \cup R_+$ Moreover, the solutions of the problem(1.1) are in $\bar{B}_r(0)$ Hence, solutions are global in nature.

Finally, let $u, v \in \bar{B}_r(0)$ be any two solutions of the problem(1.1) on $I_0 \cup R_+$. Then

$$\begin{aligned}
|u(t)-v(t)| \leq\ & |[f(t,u(t))]\left(\phi(0)\bar{p}(t) + \bar{p}(t)\int_0^t [g(s,u(s),u_s) + h(s,u(s),u_s)] + k(s,u(s),u_s)ds\right) \\
& -[f(t,v(t))]\left(\phi(0)\bar{p}(t) + \bar{p}(t)\int_0^t [g(s,v(s),v_s) + h(s,v(s),v_s)] + k(s,u(s),v_s)ds\right) \\
\leq\ & |f(t,u(t)) - f(t,v(t))|\left(\phi(0)\bar{p}(t) + \bar{p}(t)\int_0^t [g(s,u(s),u_s) + h(s,u(s),u_s)] + k(s,u(s),u_s)ds\right) \\
& +|\ f(t,v(t))\left(\bar{p}(t)\int_0^t \left\{\begin{matrix}[g(s,u(s),u_s) - g(s,u(s),u_s)] - \\ [h(s,u(s),u_s) - h(s,v(s),v_s)] - [k(s,u(s),u_s) - k(s,v(s),v_s)]\end{matrix}\right\} ds\right) \\
\leq\ & |\ f(t,u(t)) - f(t,v(t))|\left(|\phi(0)\|\bar{p}(t)| + |\bar{p}(t)|\int_0^t h(s)ds\right) \\
& +2[|\ f(t,u(t)) - f(t,0)| + |\ f(t,0)|]r(t) \\
\leq\ & \frown(t)\frac{|u(t)-v(t)|}{K+|u(t)-v(t)|}(|\phi(0)|\,\|\bar{p}\| + R) \qquad (3.5) \\
& +2\left[\frac{\frown(t)|v(t)|}{K+|v(t)|} + F_0\right]r(t) \\
\leq\ & \frac{L(|\phi(0)|\,\|\bar{p}\| + R)\,|u(t)-v(t)|}{K+|u(t)-vy(t)|} + 2(L+F_0)r(t)
\end{aligned}$$

Taking the limit superior as $t \to \infty$ in the above, we get

$$\lim_{t\to\infty} |u(t) - v(t)| = 0$$

Hence, there is a real number T > 0 such that $|u(t) - v(t)| < \in for\ all\ t \geq T$. Obviously, the solutions of problem(1.1) are uniformly globally attractive on $I_0 \cup R_+$.

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