Quest Journals
Journal of Research in Applied Mathematics
Volume 8 ~ Issue 7 (2022) pp: 06-11
ISSN(Online) : 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Research Paper

A Note on Completely Monotone Hyper Functions

DeepthiA.N.[1],MangalambalN.R[2]
1(Department of Mathematics, K K T M Government College,Pullut,Thrissur,Kerala,India)
2(CentreforResearchinMathematicalSciences,St.Joseph'sCollege,
Irinjalakuda,Thrissur,Kerala,India)
Corresponding Author: Deepthi A N

ABSTRACT: *Hyperfunctions are the generalisation of the concept of functionsbytheJapaneseMathematicianMikioSato.Inthispapertheideaof Completely Monotone Hyperfunction is introduced for real valued hyperfunctions.Some properties of completely Monotone Hyperfunctionsareprovedforaclassofhyperfunctionswithboundedexponentialgrowth.*

Mathematics SubjectClassification:*32A45,46F15,58J15*

KEYWORDS: *Hyperfunctions,Completelymonotonefunctions*

***Received** 25 June, 2022; **Revised** 05 July, 2022; **Accepted** 07 July, 2022 © The author(s) 2022.*
Published** with open access at **www.questjournals.org

I. INTRODUCTION

Mikio Sato introduce the idea of hyperfunctions[4],[5] to mention his generalization ofthe concept of functions.Urs Graf use sato's idea which uses the classical complex functiontheorytogeneralizethenotionoffunctionofarealvariableandappliedvarioustransformslikeLaplacetransform ,Fouriertransform,Hilberttransform,Mellintransforms,Hankeltransform to a class of hyperfunctions in his book.[1].A H Zemanian[2] introduces differenttransforms for generalized function for solving differential and partial differential equationsoccurringinsolvingphysicalproblemsinvariousengineeringfields.

Inthispapertheideaofcompletelymonotonehyperfunctionsisdefinedforreal valued hyperfunctions.Some properties of this were proved for a class of real valuedhyperfunctionshavingboundedexponentialgrowth..

II. PRELIMINARIES

Theupperandlowerhalf-planeofthecomplexplane$\mathbb{C}$ aredenotedby$\mathbb{C}_+ = \{z \in \mathbb{C}: Iz > 0\}, \mathbb{C}_- = \{z: Iz < 0\}$respectively.

Definition 1.1[1]: "For an open interval $I$of the real line, the open subset $N(I) \subset \mathbb{C}$ iscalleda*complexneighborhood*of$I$,if$I$isaclosedsubsetof$N(I)$.Let$N_+(I) = N(I) \cap \mathbb{C}_+andN_-(I) = N(I) \cap \mathbb{C}_-$.

$\mathfrak{O}(N(I)\backslash I)$denotestheringofholomorphicfunctionsin$N(I)\backslash I$.Foragiveninterval$I$afunction$F(z) \in \mathfrak{O}(N(I)\backslash I)$canbewrittenas

$F_+(z)$ for $z \in N_+(I)$,
$F(Z) = F_-(z)$ for $z \in N_-(I)$

where $F_+(z) \in \mathfrak{O}(N_+(I))$ and $F_-(z) \in \mathfrak{O}(N_-(I))$ are called upper and lower component of $F(z)$ respectively. In general the upper and lower component of $F(z)$ need not be related to each other. If they are analytic continuations from each other we call $F(z)$ a global analytic function on $N(I)$ and we write $F_+(z) = F_-(z) = F(z)$."

Definition1.2[1]:" Two functions $F(z)$ and $G(z)$ in $\mathfrak{O}(N(I) \setminus I)$ are equivalent if for $z \in N_1(I) \cap N_2(I)$, $G(z) = F(z) + \varphi(z)$, with $\varphi(z) \in \mathfrak{O}(N(I))$ where $N_1(I)$ and $N_2(I)$ are complex neighborhoods of I of $F(z)$ and $G(z)$ respectively."

Definition1.3[1]:" An equivalence class of functions $F(z) \in \mathfrak{O}(N(I) \setminus I)$ defines a hyperfunction $f(x)$ on I. Which is denoted by $f(x) = [F(z)] = [F_+(z), F_-(z)]$. $F(z)$ is called defining or generating function of the hyperfunction. The set of all hyperfunctions defined on the interval I is denoted by $\mathfrak{B}(I)$. Then $\mathfrak{B}(I) = \mathfrak{O}(N(I)\setminus I)\setminus\mathfrak{O}(N(I))$. A real analytic function $\varphi(x)$ on I is defined by the fact that $\varphi(x)$ can analytically be continued to a full neighborhood $\mathfrak{U}$ containing I i.e. we then have $\varphi(z) \in \mathfrak{O}(\mathfrak{U})$. For any complex neighborhood $N(I)$ containing $\mathfrak{U}$ we may then write $\mathfrak{B}(I) = \mathfrak{O}(N(I) \setminus I) \setminus \mathcal{A}(I)$, where $\mathcal{A}(I)$ is the ring of all real analytic functions on I. Thus a hyperfunction $f(x) \in \mathfrak{B}(I)$ is determined by a defining function $F(z)$ which is holomorphic in an adjacent neighborhood above and below the interval I, but is only determined up to a real analytic function on I. The value of a hyperfunction at a point $x \in I$ is

$$f(x) = F(x + i0) - F(x - i0) = \lim_{\epsilon\to 0^+}\{F + (x + i\epsilon) - F - (x - i\epsilon)\}$$

provided the limit exists."

Example[1]:" Dirac delta function at $x = 0$ is represented in terms of hyperfunction as $\delta(x) = [-\frac{1}{2\pi iz}]$. Here the defining function is $(z) = -\frac{1}{2\pi iz}$. $F(z)$ is defined except at $z = 0$. At $z = 0$, $F(z)$ has an isolated singularity, which is a pole of order 1. For every real number $x \neq 0$ the limit $\lim_{\epsilon\to 0^+}\{F + (x + i\epsilon) - F_-(x - i\epsilon)\}$ exists and equal to 0."

Definition1.4[1]: "A hyperfunction $f(x)$ is called holomorphic at $x = a$, if the lower and upper component of the defining function can analytically be continued to a full (two-dimensional) neighborhood of the real point a i.e. the upper/lower component can analytically be continued across a into the lower/upper half-plane."

Definition1.5[1]:" Let $f(x) = [F_+(z), F_-(z)]$ be a hyperfunction, holomorphic at both endpoints of the finite interval $[a, b]$, then the (definite) integral of $f(x)$ over $[a, b]$ is defined and denoted by

$$\int_a^b f(x)dx = \int_{\gamma_{a,b}^+} F_+(z)dz - \int_{\gamma_{a,b}^-} F_-(z)dz = -\oint_{(a,b)} F(z)dz$$

where the contour $\gamma_{a,b}^+$ runs in N_+ from a to b above the real axis, and the contour $\gamma_{a,b}^-$ is in N_- from a to b below the real axis."

Example[1]:" $\int_{-\infty}^{\infty} \delta(x)dx = -\oint \frac{-1}{2\pi iz} dz = 1$"

Definition 1.6[1]:"Let Σ_0 be the largest open subset of the real line where the hyper-function $\boldsymbol{f(x)} = [\boldsymbol{F(z)}]$ is vanishing. Its complement $\boldsymbol{K_0} = \mathbb{R} \setminus \boldsymbol{\Sigma_0}$ is said to be the support of the hyperfunction $\boldsymbol{f(x)}$ denoted by $\boldsymbol{suppf(x)}$. Let $\boldsymbol{\Sigma_1}$ be the largest open subset of the real line where the hyperfunction $\boldsymbol{f(x)} = [\boldsymbol{F(z)}]$ is holomorphic. Its complement $\boldsymbol{K_1} = \mathbb{R} \setminus \boldsymbol{\Sigma_1}$ is said to be the singular support of the hyperfunction $\boldsymbol{f(x)}$ denoted by $\boldsymbol{singsuppf(x)}$.

To define hyperfunctions of bounded exponential growth, consider open sets $\boldsymbol{J} = (a, 0) \cup (0, b)$ with some $a < 0$ and some $b > 0$ and compact subsets $K = [a', a''] \cup [b', b'']$ with $a < a' \leq a'' < 0$ and $0 < b' \leq b'' < b$. Also consider the following open neighborhoods $[-\delta, \infty) + iJ$ and $(-\infty, \delta] + iJ$ of $\mathbb{R}_+$ and $\mathbb{R}_-$ respectively for some $\delta > 0$.

Consider the subclass $\mathfrak{O}(\mathbb{R}_+)$ of hyperfunctions $f(x) = [F(z)]$ on $\mathbb{R}$ satisfying

(i) The support $suppf(x)$ is contained in $[0, \infty)$

(ii) Either the support $suppf(x)$ is bounded on the right by a finite number $\beta > 0$ or we demand that among all equivalent defining functions, there is one, $F(z)$ defined in $[-\delta, \infty) + iJ$ such that for any compact set $K \subset J$ there exist some real constant $M' > 0$ and σ' such that $|F(z)| \leq M' e^{\sigma' \mathcal{R}z}$ holds uniformly for all $z \in [0, \infty) + \boldsymbol{iK}$.

Because $suppf(x) \subset \mathbb{R}_+$ and since the singular support $singsuppf$ is a subset of the support, we have $singsuppf \subset \mathbb{R}_+$. Therefore $f(x)$ is a holomorphic hyperfunction for all $x < 0$. Moreover, the fact that $F_+(x + i0) - F_-(x - i0) = 0$ for all $x < 0$ shows that $F(z)$ is real analytic on the negative part of the real axis. Hence $f(x) \in \mathfrak{O}(\mathbb{R}_+)$ implies that $\chi_{(-\epsilon,\infty)} f(x) = f(x)$ for any $\epsilon > 0$."

Definition 1.7[1]:"We call the subclass of hyperfunctions $\mathfrak{O}(\mathbb{R}_+)$ the class of *right sided originals*. In the case of an unbounded support $suppf(x)$, let $\sigma_- = inf\sigma'$ be the greatest lower bound of all σ' where the infimum is taken over all σ' and all equivalent defining functions satisfying(ii). This number $\sigma_- = \sigma_-(f)$ is called the growth index of $f(x) \in \mathbb{R}_+$. It has the properties

(i) $\sigma_- \leq \sigma'$

(ii) For every $\epsilon > 0$ there is a σ' with $\sigma_- \leq \sigma' \leq \sigma_- + \epsilon$ and an equivalent defining function $F(z)$ such that $|F(z)| \leq M' e^{\sigma' \mathcal{R}z}$ uniformly for all $z \in [0, \infty) + \boldsymbol{iK}$.

In the case of a bounded support $suppf(x)$, we set $\sigma_-(f) = -\infty$."

Definition 1.8[1]:"The Laplace transform of a right-sided original $f(x) = [F(z)] \in \mathfrak{O}(\mathbb{R}_+)$ is now defined by $\hat{f}(s) = \mathcal{L}[f](s) = -\int_{\infty}^{(0+)} e^{-sz} F(z)dz$. The image function $\hat{f}(s)$ of $f(x) \in \mathfrak{O}(\mathbb{R}_+)$ is holomorphic in the right half-plane $\mathcal{R}s > \sigma_-(f)$

Similarly, we introduce the class $\mathfrak{O}(\mathbb{R}_+)$ of hyperfunctions specified by

(i) The support $suppf(x)$ is contained in $\mathbb{R}_- = (-\infty, 0]$

(ii) Either the support $suppf(x)$ is bounded on the left by a finite number $\alpha < 0$, or we demand that among all equivalent defining functions there is one, denoted by $F(z)$ and defined in $(-\infty, \delta] +$

iJsuchthatforanycompactsubset$K \subset J$therearesomerealconstants$M'' > 0$ andσ'' suchthat$|F(z)| \leq M'' e^{\sigma'' \mathcal{R}z}$holdsuniformlyforz∈$(-\infty,0]+\boldsymbol{iK}$.“

Definition1.9[1]:“ Theset$\mathfrak{O}(\mathbb{R}_-)$issaidtobetheclassofleft-sidedoriginals.

Inthecaseofanunboundedsupportlet$\sigma_+ = \sup \sigma''$betheleastupperboundofallσ'',where the supremum is taken over all σ'' and all equivalent defining functions satisfying (ii).The number $\sigma_+ = \sigma_+(f)$ is called the growth index of $f(x) \in \mathfrak{O}(\mathbb{R}_-)$.It has the properties

(i) $\sigma'' \leq \sigma_+$

(ii)Forevery$\epsilon > 0$thereisaσ''suchthat$\sigma_+ - \epsilon \leq \sigma'' \leq \sigma_+$andadefiningfunction$F(z)$suchthat$|F(z)| \leq M'' e^{\sigma'' \mathcal{R}z}$uniformlyfor$z \in (-\infty, 0] + \boldsymbol{iK}$.

Ifthesupport$supp f(x)$isbounded,weset$\sigma_+(f) = +\infty$”

Definition1.10[1]: “TheLaplace transform of a left-sided original $f(x) = [F(z)] \in \mathfrak{O}(\mathbb{R}_-)$isdefinedby$\hat{f}(s) = \mathcal{L}[f](s) = -\int_{-\infty}^{(0+)} e^{-sz} F(z) dz$. Theimagefunction $\hat{f}(s)$of$f(x) \in \mathfrak{O}(\mathbb{R}_-)$isholomorphicinthelefthalf-plane$\mathcal{R}s < \sigma_+(f)$.

Withaleft-sidedoriginal$g(x) \in \mathfrak{O}(\mathbb{R}_-)$withgrowthindex$\sigma_+(g)$andaright-sided original$f(x) \in \mathfrak{O}(\mathbb{R}_+)$with growth index $\sigma_-(f)$ form the hyperfunction $h(x) = g(x) + h(x)$whosesupportisnowtheentirereal axis.If$\hat{g} = \mathcal{L}[g(t)](s), \mathcal{R}s < \sigma_+(g)and\hat{f} = \mathcal{L}[f(t)](s), \mathcal{R}s > \sigma_-(f)$we can add the two image functions provided they have a commonstripofconvergence,i.e.$\sigma_-(f) < \sigma_+(g)$holds.”

Definition1.11[1]: “With$g(x) \in \mathfrak{O}(\mathbb{R}_-) f(x) \in \mathfrak{O}(\mathbb{R}_+)$,$h(x) = g(x) + h(x)$,

$\mathcal{L}[h(t)](s) = \widehat{g(x)}(s) + \widehat{f(x)}(s), \sigma_-(f) < \mathcal{R}s < \sigma_+(g)$, provided $\sigma_-(f) < \sigma_+(g)$.”

Definition 1.12[1]:“Hyperfunctions of the subclass $\mathfrak{O}(\mathbb{R}_+)$ are said to be of bounded ex-ponential growth as $x \to \infty$ and hyperfunctions of the subclass $\mathfrak{O}(\mathbb{R}_-)$ are said to be ofboundedexponentialgrowthas$x \to -\infty$.

An ordinary function $f(x)$ is called of bounded exponential growth as$x \to \infty$,if therearesomerealconstants$M' > 0$and$\sigma' > 0$suchthat$|f(x)| \leq M' e^{\sigma' x}$forsufficientlylarge$x$.Itiscalledofboundedexponentialgrowthas$x \to -\infty$,iftherearesomerealconstants$M'' > 0$and$\sigma'' > 0$suchthat$|f(x)| \leq M'' e^{\sigma'' x}$,forsufficientlynegativelarge$x$.Afunctionorahyperfunctionisofboundedexponentialgrowth,ifitisofboundedexponential growth for $x \to -\infty$ as well as for $x \to \infty$ .Thus a hyperfunction or ordi-naryfunction$f(x)$hasaLaplacetransform,ifitisofbondedexponentialgrowth,andif$\sigma_-(f) < \sigma_+(f)$.”

Definition 1.13[1]:”For any given hyperfunction $f(x) = [F+(z), F-(z)]$ its *derivative* inthesenseofhyperfunctionisdefinedanddenotedas

$$Df(x) = f'(x) = \left[\frac{dF_+}{dz}, \frac{dF_-}{dz}\right]$$

$D^n f(x) = f^{(n)}(x) = \left[\frac{d^n F_+}{dz^n}, \frac{d^n F_-}{dz^n}\right]$"

III. COMPLETELYMONOTONEHYPERFUNCTIONS

Definition 2.1: A positive real valued hyperfunction $f(x) = [F(Z)]$ defined on $(0,\infty)$ iscalledacompletelymonotonehyperfunctionifitsatisfies$(-1)^n f^{(n)}(x) \geq 0$forall$x > 0, n = 0,1,2,\ldots$

Theorem2.2: Apositiverealvaluedhyperfunction$f(x) = [F(Z)]$definedon$(0,\infty)$isa completely monotone hyperfunction if there exists a positive real valued hyperfunction $g(x) = [G(z)]$ on $(0,\infty)$ with bounded exponential growth such that $f(s) = \mathcal{L}[g(x)](s)$, for all $s > 0$.

Proof:

Suppose there exists a positive real valued hyperfunction $g(x) = [G(z)]$ on $(0,\infty)$ withboundedexponentialgrowthsuchthat$f(s) = \mathcal{L}[g(x)](s)$,forall$s > 0$.

Then$(-1)^n f^{(n)}(x) = (-1)^n \frac{d^n}{ds^n}\mathcal{L}[g(x)](s)$

$$= \int_0^\infty x^n e^{-sx} g(x)dx \geq 0 \text{for all } s > 0$$

Hence$f(x)$isacompletelymonotonehyperfunction.

Theorem 2.3: Let $f(x) = [F(x)]$ and $g(x) = [G(x)]$ be two completely monotone hyperfunctions.Then $f(x)g(x)$ is a completely monotone hyperfunction whenever the product is defined and $f(s) = \mathcal{L}[h(x)](s)$ and $(x) = \mathcal{L}[j(x)](s)$, where $h(x) = [H(z)]$ and $j(x) = [J(z)]$aretwohyperfunctions, where $s > 0$.

Proof:

Productof$f(x)$and$g(x)$isdefinedwhen$singsuppf(x) \cap singsuppg(x) = \varphi$

If$s \notin singsuppf(x) \cap singsuppg(x)$then

$$f(s)g(s) = \mathcal{L}[h(x)](s)\mathcal{L}[j(x)](s)$$
$$= \mathcal{L}[h(x) * j(x)](s)$$

Thenbytheorem2.2theresultfollows

Theorem2.4: Let $f(x)$be a completely monotone hyperfunction and $g(x)$ be a positivereal valuedhyperfunctiondefinedon$(0,\infty)$suchthat$g'(x)$isacompletelymonotonehyperfunction, then$f \circ g$isalsoacompletelymonotonehyperfunction.

Proof:

Wearegoingtoprovetheresultusingmathematicalinduction

Clearly$(f \circ g)(x) \geq 0$for$x > 0$.

$(f o g)' = (f' o g)g' \leq 0$since$(-1)f' \geq 0andg' \geq 0$

Hencetheresulttruefor$k$=1.

Supposethattheresultistruefor$k = n$i.e.$(-1)^k (f \circ g)^k \geq 0$forall$k = 0,1,2,\ldots,n$

Since$-f'$andg'arecompletelymonotone $(-1)^{n+1}(f \circ g)^{n+1}$

$$= (-1)^n \left[\left((-f') \circ g\right)g'\right]^{(n)}$$

$$= (-1)^n \sum_{k=0}^{n} nC_k \big((-f') \circ g\big)^{(k)} (g')^{(n-k)}$$

$$= \sum_{k=0}^{n} nC_k [(-1)]^k \big((-f') \circ g\big)^k][(-1)^{n-k} (g')^{(n-k)}]$$

≥ 0

Hencetheresultistruefor$k$=n+1also.

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