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Research Paper

K-EQUIDISTANT SETS OF HYPERCUBES

Eric Choi and Anthony Delgado

ABSTRACT

A *k-equidistant set*, S_k, of a graph is a subset of three or more vertices such that if x and y are distinct vertices in S_k, then the distance, $d(x,y) = k$. The hypercube, Q_n, is defined by $Q_1 = K_2$, and $Q_{n+1} = Q_n \times K_2$. Results on k-equidistant sets of Q_n are presented. In particular, given a hypercube, Q_n, where $n = 2^m$, we develop a matrix algorithm for determining k-equidistant sets, where $k = 2^{m-1}$.

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We follow the notation of [1]. A ***k-equidistant set***, S_k, of a graph is a subset of three or more vertices such that if x and y are distinct vertices in S_k, then the distance, $d(x,y) = k$. The hypercube, Q_n, is defined by $Q_1 = K_1$, and $Q_{n+1} = Q_n \times K_2$. Hypercubes are important for massively parallel computing, error-correcting codes, and other topics in computer science. They are the subject of much research. See [2, 3, …, 6]. Throughout this paper, $S_k \subset V(Q_n)$.

Theorem 1: Given $S_k \subset V(Q_n)$, then k must be even.

Proof: Proceeding by contradiction, assume k is odd. Given $v_1, v_2, v_3 \in S_k$, since k is odd and hypercubes are bipartite, v_1 and v_2 must have opposite color. Assume, WLOG, that v_1 is black and v_2 is white. This implies that v_3 , which is a distance of k away from both v_1 and v_2 , cannot be either black or white, which is impossible. (Note that Theorem 1 applies to a k-equidistant set in any bipartite graph.) ☐

Theorem 2: Given $S_k \subset V(Q_n)$, then $n \geq \frac{3}{2}k$.

Proof: We will use binary strings of length n for vertex labels in Q_n. Let the label of $v_1 \in S_k$ consist of n ‘0’s. Clearly, for $i \geq 2$, v_i has k ‘1’s. For $r, s \geq 2$, place disagreements between v_r and v_s are of two types. **(1)** v_r has a 0 in the same place in which v_s has a 1, and **(2)** v_r has a 1 in the same place in which v_s has a 0. Let D be the number of disagreements of type (1). Since v_r and v_s have the same number of 1’s, the number of disagreements of type (2) must also be D. Hence we have $d(v_r, v_s) = k = 2D \Rightarrow D = \frac{k}{2} \in \mathbb{N}$ (since k is even). Then each of v_r and v_s have k ‘1’s and $n-k$ ‘0’s, and we need $n-k \geq D = \frac{k}{2}$ which implies that $n \geq \frac{3}{2}k$. ☐

Theorem 3: Let $f(n,k)$ be the *maximum* size of S_k in Q_n. Then $\frac{2n}{k} \leq f(n,k) < \binom{n}{k} + 1$.

Proof: We will concatenate two kinds of "horizontal" vectors to yield the binary labels of the vertices of S_k in Q_n. [0] represents $\frac{k}{2}$ '0's, and [1] represents $\frac{k}{2}$ '1's. It is clear by inspection that the rows in matrix A below are binary labels of vertices in S_k:

$$A = \begin{pmatrix} [0] & [0] & [0] & \cdots [0] \\ [1] & [1] & [0] & \cdots [0] \\ [1] & [0] & [1] & \cdots [0] \\ \vdots & \vdots & \vdots & \ddots [0] \\ [1] & [0] & [0] & \cdots [1] \end{pmatrix}$$

For $i \geq 2$, the i-th row starts with a [1] vector, and the i-th vector is [1]. (The remaining vectors are [0]'s.) Since the number of columns (of vectors) in A is $\left\lfloor \frac{n}{\left(\frac{k}{2}\right)} \right\rfloor = \left\lfloor \frac{2n}{k} \right\rfloor$, and the vectors in A as depicted above using bracketed entries, give it the "appearance" of a square matrix, $\left\lfloor \frac{2n}{k} \right\rfloor$ equals the number of rows (and therefore, vertex addresses, of S_k) in A. Hence $\left\lfloor \frac{2n}{k} \right\rfloor \leq$ $f(n, k)$. On the other hand, $f(n,k) < \binom{n}{k} + 1$, which is the total number of addresses with k '1's plus the one address with n '0's. □

Lemma: Let $\overline{v_i}$ denote the *antipode* of v_i (also called the *binary complement* of v_i). Suppose for $v_i, v_j \in Q_n$ we have $d(v_i, v_j) = k$. Then $d(\overline{v_i}, v_j) = n - k$.

Proof: If $d(v_i, v_j) = k$, then v_i and v_j must disagree in k places in their addresses. Since $\overline{v_i}$ disagrees with v_i in all places, $\overline{v_i}$ agrees with v_j in $n - k$ places, that is, $d(\overline{v_i}, v_j) = n - k$. □

Corollary: When $d(v_i, v_j) = k = \frac{n}{2}$, we have $n - k = k$, implying $d(v_i, v_j) = d(\overline{v_i}, v_j)$. □

Theorem 4: Given Q_n, where $n = 2^m$ and $m \geq 2$, let $k = 2^{m-1}$. Then there exists a k-equidistant set, S_k, such that $|S_k| = n$.

Proof: By construction. $S_{2^{m-1}}$ can be constructed recursively using square submatrices.

Initial Case: $n = 4$. Then $m = \log_2 n = 2$ and $k = 2^{m-1} = 2$. Let $B_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$. The rows of B_2 yield a 2-equidistant set with $n = 4$ elements.

Generating Scheme: Let $B_{m+1} = \begin{pmatrix} B_m & B_m \\ B_m & \overline{B_m} \end{pmatrix}$, where $\overline{B_m}$ is the binary complement of B_m.

Example: $n = 8$. Then $m = \log_2 n = 3$ and $k = 2^{m-1} = 4$. Let $B_3 = \left(\begin{array}{cccc|cccc} 0&0&0&0&0&0&0&0\\ 1&1&0&0&1&1&0&0\\ 1&0&1&0&1&0&1&0\\ 1&0&0&1&1&0&0&1\\ \hline 0&0&0&0&1&1&1&1\\ 1&1&0&0&0&0&1&1\\ 1&0&1&0&0&1&0&1\\ 1&0&0&1&0&1&1&0 \end{array}\right)$

Verify that the rows of B_3 yield a 4-equidistant set in Q_8 with $n = 8$ elements. Note that the lower right 4×4 submatrix guarantees that the eight rows are distinct and that the place disagreements between any two distinct rows is 4. Continuing in this manner, we obtain k-equidistant sets, S_k, in Q_n, such that $n = 2^m$, $k = 2^{m-1}$, and $|S_k| = n$.

We use induction to show that the above recursion generates $S_{2^{m-1}}$ in Q_n. The base case is the Initial Case above for Q_4.

Inductive Leap: Given rows v_i, v_j in B_m, $d(v_i, v_j) = k = 2^{m-1}$ by the inductive assumption, let $B_{m+1} = \begin{pmatrix} B_m & B_m \\ B_m & \overline{B_m} \end{pmatrix}$. Since $\overline{B_m}$ is the binary complement of B_m, every row in B_{m+1} is obtained by *concatenating* a vertex in B_m with itself or with its antipode, thereby doubling its length. For $1 \le j \le n$, designate these concatenations $v_j * v_j$ and $v_j * \overline{v_j}$, respectively. Note that any *pair* of rows in B_{m+1} is of one of three cases:

(1) $v_i * v_i$ and $v_j * v_j$, $i \ne j$; both concatenated vectors are rows of the upper half of B_{m+1}.

(2) $v_i * \overline{v_i}$ and $v_j * \overline{v_j}$, $i \ne j$; both concatenated vectors are rows of the lower half of B_{m+1}.

(3) $v_i * v_i$ and $v_j * \overline{v_j}$; the concatenated vectors are from different halves of B_{m+1}.

Then we have

(1) $d(v_i * v_i, v_j * v_j) = d(v_i, v_j) + d(v_i, v_j) = 2d(v_i, v_j) = 2k$.

(2) $d\left(v_i * \overline{v_i}, v_j * \overline{v_j}\right) = d\left(v_i, v_j\right) + d\left(\overline{v_i}, \overline{v_j}\right) = 2d\left(v_i, v_j\right) = 2k$, since $d\left(\overline{v_i}, \overline{v_j}\right) = d\left(v_i, v_j\right)$.

(3) $d\left(v_i * v_i, v_j * \overline{v_j}\right) = d\left(v_i, v_j\right) + d\left(v_i, \overline{v_j}\right) = 2d\left(v_i, v_j\right) = 2k$, using the Corollary. □

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*Corresponding Author: Eric Choi