Quest Journals
Journal of Research in Applied Mathematics
Volume 8 ~ Issue 7 (2022) pp: 21-27
ISSN(Online) : 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Research Paper

Rishi Transform for Solving Second Kind Linear Volterra Integral Equations

[1]Dinesh Thakur, [2]Prakash Chand Thakur
[1,2]School of Basic Sciences, Department of Mathematics
Bahra University, Solan, Himachal Pradesh, India

ABSTRACT - *Many scientific and technical problems, such as neutron diffusion, radiation transfer, heat transfer and electric circuit problems can be theoretically represented in terms of the Volterra integral equation. We employed the Rishi transform in this work to solve second kind Volterra integral equations. Some applications are given in the application section to demonstrate the utility of the Rishi transform in solving second kind Volterra integral equations.*
KEYWORDS- *Second kind Volterra integral equations, Rishi transform, Rishi Inverse transform, Convolution theorem.*

***R**eceived 06 July, 2022; **R**evised 18 July, 2022; **A**ccepted 20 July, 2022 © The author(s) 2022.*
P**ublished with open access at **www.questjournals.org

I. INTRODUCTION

The following is a second kind of linear Volterra integral equation [1-3]:

$$\phi(x) = f(x) + \lambda \int_0^x k(x,t)\phi(t)\,dt$$

(1)

Here the function $f(x)$ and kernel $k(x,t)$ are known real-valued functions. The unknown function is denoted by $\phi(x)$ and λ is a non-zero real parameter.

The Rishi transform of the function $\omega(t)$ is piecewise continuous exponential order. The function $\omega(t)$ defined in the interval $[0,\infty)$ is given by (16):

$$R\{\omega(t)\} = \left(\frac{\sigma}{\varepsilon}\right)\int_0^\infty \omega(t)\,e^{-\left(\frac{\varepsilon}{\sigma}\right)t}\,dt = T(\varepsilon,\sigma),\quad \varepsilon,\sigma > 0.$$ Here R denotes the Rishi transform operator.

In more advanced eras, researchers developed a novel type of integral transform to solve advanced challenges in science, space, engineering, and real-world scenarios. The most typical application of integral transformations is to solve initial value problems**.** Song and Kim (4) used the Elzaki transform to find numerical solutions to Volterra integral equations of the second kind. The convolution for Kamal and Mahgoub transforms was provided by Fadhil [5]. For the first kind of linear Volterra integral equations, Kumar *et al.* [6] provided applications of the Mohand transform. Aggarwal *et al.* [7] presented a method for solving linear Volterra integral equations of the first kind using the Kamal transform. Aggarwal *et al.* [8] solved linear Volterra integral equations of the second kind with the help of Mohand transform. Applications discussed by Aggarwal *et al.* [9] for solving improper integrals with error function as integrands with the aid of the Laplace transform. Chauhan and Aggarwal [10] used the Laplace transformation to solve convolution type linear V.I.E. of the second kind. Aggarwal *et al.* [11] employed the Shehu transform to solve first kind of Volterra integral equations. Higazy *et al.* [12] solved the Volterra integral equation using the Sawi decomposition method. Aggarwal *et al.* [13] used the Shehu transform to identify the primitive of the second kind linear Volterra integral equation. This newly developed method was given the name Sawi decomposition method. Aggarwal and Kumar [14] solved a system of second kind linear Volterra integro-differential equations using the Laplace transform. Recently, Kumar *et al*. [15] proposed a new integral transform called the Rishi Transform and used it to solve first kind linear Volterra integral equations.

The primary goal of this paper is to use the Rishi Transform to solve second kind of linear Volterra integral equations.

II. LINEARITY PROPERTY OF RISHI TRANSFOM

If $R\{\omega_i(t)\} = T_i(\varepsilon,\sigma)$, then $R\{\sum_{i=1}^{n}\alpha_i\omega_i(t)\} = \alpha_i\sum_{i=1}^{n}\{R\,\omega_i(t)\} = \alpha_i\sum_{i=1}^{n}T_i(\varepsilon,\sigma),$ where α_i are arbitrary constant [15].

III. SOME USEFUL FUNCTIONS WITH RISHI TRANSFORM [15]

TABLE.1

Sr. No.	$\omega(t)$	$R\{\omega(t)\} = T(\varepsilon,\sigma)$
i)	1	$\left(\frac{\sigma}{\varepsilon}\right)^2$
ii)	t	$\left(\frac{\sigma}{\varepsilon}\right)^3$
iii)	t^2	$2\left(\frac{\sigma}{\varepsilon}\right)^4$
iv)	$t^n, n \in N$	$n!\left(\frac{\sigma}{\varepsilon}\right)^{n+2}$
v)	$t^n, n \in -1$	$\Gamma(n+1)\left(\frac{\sigma}{\varepsilon}\right)^{n+2}$
vi)	e^{lt}	$\frac{\sigma^2}{\varepsilon(\varepsilon - l\sigma)}$
vii)	$\sin lt$	$\frac{l\sigma^3}{\varepsilon(\varepsilon^2 + \sigma^2 l^2)}$
viii)	$\cos lt$	$\frac{\sigma^2}{(\varepsilon^2 + \sigma^2 l^2)}$
ix)	$\sinh lt$	$\frac{l\sigma^3}{\varepsilon(\varepsilon^2 - \sigma^2 l^2)}$
x)	$\cosh lt$	$\frac{\sigma^2}{(\varepsilon^2 - \sigma^2 l^2)}$

IV. CONVOLUTION OF RISHI TRANSFORM

If $R\{\omega_1(t)\} = T_1(\varepsilon,\sigma)$ and $R\{\omega_2(t)\} = T_2(\varepsilon,\sigma)$ then [15]

$$R\{\omega_1(t) * \omega_2(t)\} = \left(\frac{\varepsilon}{\sigma}\right) R\{\omega_1(t)\} R\{\omega_2(t)\} = \left(\frac{\varepsilon}{\sigma}\right) T_1(\varepsilon,\sigma) T_2(\varepsilon,\sigma),$$

where $\omega_1(t)$ and $\omega_2(t)$ is denoted by $\omega_1(t) * \omega_2(t) = \int_0^t \omega_1(t-u)\omega_2(u)\,du = \int_0^t \omega_1(u)\omega_2(t-u)\,du.$

V. INVERSE OF RISHI TRANSFORM

The function $\omega(t)$ is known as inverse Rishi transform of $T(\varepsilon,\sigma)$, if $R\{\omega(t)\} = T(\varepsilon,\sigma)$ and it is expressed as [15]:

$\omega(t) = R^{-1}\{T(\varepsilon,\sigma)\}$. The operator R^{-1} is known as inverse of the Rishi transform.

VI. INVERSE RISHI TRANSFORM LINEARITY PROPERTY [15]

If $R^{-1}\{T_i(\varepsilon,\sigma)\} = \omega_i(t)$ then $R^{-1}\{\sum_{i=1}^{n}\alpha_i T_i(\varepsilon,\sigma)\} = \sum_{i=1}^{n}\alpha_i\, R^{-1}\{T_i(\varepsilon,\sigma)\}$, where α_i are arbitrary constants.

VII. SOME USEFUL FUNCTIONS WITH THE INVERSE RISHI TRANSFORM

TABLE.2

Sr. No.	$T(\varepsilon,\sigma)$	$R^{-1}\{T(\varepsilon,\sigma)\}$
i)	$\left(\frac{\sigma}{\varepsilon}\right)^2$	1
ii)	$\left(\frac{\sigma}{\varepsilon}\right)^3$	t
iii)	$\left(\frac{\sigma}{\varepsilon}\right)^4$	$\frac{t^2}{2}$
iv)	$\left(\frac{\sigma}{\varepsilon}\right)^{n+2}$	$\frac{t^n}{n!}, n \in N$
v)	$\left(\frac{\sigma}{\varepsilon}\right)^{n+2}$	$\frac{t^n}{\Gamma(n+1)}, n \in -1$
vi)	$\frac{\sigma^2}{\varepsilon(\varepsilon - l\sigma)}$	e^{lt}
vii)	$\frac{\sigma^3}{\varepsilon(\varepsilon^2 + \sigma^2 l^2)}$	$\frac{\sin lt}{l}$
viii)	$\frac{\sigma^2}{(\varepsilon^2 + \sigma^2 l^2)}$	$\cos lt$
ix)	$\frac{\sigma^3}{\varepsilon(\varepsilon^2 - \sigma^2 l^2)}$	$\frac{\sinh lt}{l}$
x)	$\frac{\sigma^2}{(\varepsilon^2 - \sigma^2 l^2)}$	$\cosh lt$

VIII. RISHI TRANSFORM FOR CONVOLUTION TYPE SECOND KIND LINEAR VOLTERRA INTEGRAL EQUATIONS

In this study, the kernel $k(x,t)$ (1) will be assumed to be a difference kernel, as described by the difference $(x-t)$. The second kind of Volterra integral equation (1) can thus expressed as:

$$\phi(x) = f(x) + \lambda\int_0^x k(x-t)\,\phi(t)\,dt \qquad (2)$$

The Rishi transform is applied to both sides of (2), yielding

$$R\{\phi(x)\} = R\{f(x)\} + \lambda\, R\{\int_0^x k(x-t)\,\phi(t)\,dt\}$$

$$\Rightarrow R\{\phi(x)\} = R\{f(x)\} + \lambda\, R\{k(x) * \phi(x)\} \qquad (3)$$

Employing the Rishi transform's convolution theorem to (3), we have

$$R\{\phi(x)\} = R\{f(x)\} + \lambda\left(\frac{\varepsilon}{\sigma}\right)R\{k(x)\}R\{\phi(x)\}$$

$$R\{\phi(x)\} - \lambda\left(\frac{\varepsilon}{\sigma}\right)R\{k(x)\}R\{\phi(x)\} = R\{f(x)\}$$

$$R\{\phi(x)\} = \left[\frac{R\{f(x)\}}{1-\lambda(\varepsilon/\sigma)R\{k(x)\}}\right] \quad (4)$$

Inverting the Rishi transform on both sides of the preceding equation (4), we obtain:

$$\phi(x) = R^{-1}\left[\frac{R\{f(x)\}}{1-\lambda(\varepsilon/\sigma)R\{k(x)\}}\right] \quad (5)$$

Equation (5) is the required solution of (2).

4.0 Applications

Some applications are presented in this part to show the utility of Rishi transform for solving Volterra integral equations of Second kind.

(A) Consider the following second type: Volterra integral equation

$$\phi(x) = \cos x + \int_0^x \sin(x-t)\,\phi(t)\,dt \quad (6)$$

The Rishi transform is applied to both sides of (6), yielding

$$R\{\phi(x)\} = R\{\cos x\} + R\left\{\int_0^x \sin(x-t)\phi(t)\,dt\right\}$$

$$\Rightarrow R\{\phi(x)\} = \frac{\sigma^2}{(\varepsilon^2+\sigma^2)} + R\{\sin x * \phi(x)\}$$

(7)

Now using the Rishi transform convolution theorem on (7),

$$R\{\phi(x)\} = \frac{\sigma^2}{(\varepsilon^2+\sigma^2)} + \left(\frac{\varepsilon}{\sigma}\right)R\{\sin x\} * R\{\phi(x)\}$$

$$R\{\phi(x)\} = \frac{\sigma^2}{(\varepsilon^2+\sigma^2)} + \left(\frac{\varepsilon}{\sigma}\right)\left[\frac{\sigma^3}{\varepsilon(\varepsilon^2+\sigma^2)}\right]R\{\phi(x)\}$$

$$\Rightarrow R\{\phi(x)\} = \frac{\sigma^2}{(\varepsilon^2+\sigma^2)} + \left[\frac{\sigma^2}{(\varepsilon^2+\sigma^2)}\right]R\{\phi(x)\}$$

$$\Rightarrow R\{\phi(x)\} - \left[\frac{\sigma^2}{(\varepsilon^2+\sigma^2)}\right]R\{\phi(x)\} = \frac{\sigma^2}{(\varepsilon^2+\sigma^2)}$$

$$\Rightarrow R\{\phi(x)\}\left[\frac{\varepsilon^2}{(\varepsilon^2+\sigma^2)}\right] = \frac{\sigma^2}{(\varepsilon^2+\sigma^2)}$$

$$\Rightarrow R\{\phi(x)\} = \left(\frac{\sigma}{\varepsilon}\right)^2 \quad (8)$$

Inverting the Rishi transform on both sides of the preceding equation (8), we obtain:

$$\Rightarrow \phi(x) = R^{-1}\left\{\frac{\sigma}{\varepsilon}\right\}^2 = 1$$

Therefore, the required solution of given integral equation is 1.

(B) Consider the following second type: Volterra integral equation

$$\phi(x) = x + \int_0^x \sin(x-t)\,\phi(t)\,dt \qquad (9)$$

The Rishi transform is applied to both sides of (9), yielding

$$R\{\phi(x)\} = R\{x\} + R\left\{\int_0^x \sin(x-t)\phi(t)\,dt\right\}$$

$$\Rightarrow R\{\phi(x)\} = \frac{\sigma^3}{\varepsilon^3} + R\{\sin x * \phi(x)\}$$

(10)

Now using the Rishi transform convolution theorem on (10),

$$R\{\phi(x)\} = \frac{\sigma^3}{\varepsilon^3} + \left(\frac{\varepsilon}{\sigma}\right) R\{\sin x\} * R\{\phi(x)\}$$

$$R\{\phi(x)\} = \frac{\sigma^3}{\varepsilon^3} + \left(\frac{\varepsilon}{\sigma}\right)\left[\frac{\sigma^3}{\varepsilon(\varepsilon^2+\sigma^2)}\right] R\{\phi(x)\}$$

$$\Rightarrow R\{\phi(x)\} = \frac{\sigma^3}{\varepsilon^3} + \left[\frac{\sigma^2}{(\varepsilon^2+\sigma^2)}\right] R\{\phi(x)\}$$

$$\Rightarrow R\{\phi(x)\} - \left[\frac{\sigma^2}{(\varepsilon^2+\sigma^2)}\right] R\{\phi(x)\} = \frac{\sigma^3}{\varepsilon^3}$$

$$\Rightarrow R\{\phi(x)\}\left[\frac{\varepsilon^2}{(\varepsilon^2+\sigma^2)}\right] = \frac{\sigma^3}{\varepsilon^3}$$

$$\Rightarrow R\{\phi(x)\} = \frac{\sigma^3}{\varepsilon^3} \times \frac{(\varepsilon^2+\sigma^2)}{\varepsilon^2}$$

$$\Rightarrow R\{\phi(x)\} = \frac{\sigma^3}{\varepsilon^3} + \frac{\sigma^5}{\varepsilon^5}$$

(11)

Inverting the Rishi transform on both sides of the preceding equation (11), we obtain:

$$\Rightarrow \phi(x) = R^{-1}\left[\frac{\sigma^3}{\varepsilon^3} + \frac{\sigma^5}{\varepsilon^5}\right] = x + \frac{x^3}{6}$$

Therefore, the required solution of given integral equation is $x + \frac{x^3}{6}$.

(C) Consider the following second type: Volterra integral equation

$$\phi(x) = 1 + x^2 + \int_0^x \sin(x-t)\,\phi(t)\,dt$$

(12)

The Rishi transform is applied to both sides of the preceding equation (12), yielding

$$R\{\phi(x)\} = R\{1\} + R\{x^2\} + R\left\{\int_0^x \sin(x-t)\phi(t)\,dt\right\}$$

$$\Rightarrow R\{\phi(x)\} = \frac{\sigma^2}{\varepsilon^2} + 2\left(\frac{\sigma^4}{\varepsilon^4}\right) + R\{\sin x * \phi(x)\}$$

(13)

Now using the Rishi transform convolution theorem on (13),

$$R\{\phi(x)\} = \frac{\sigma^2}{\varepsilon^2} + 2\left(\frac{\sigma^4}{\varepsilon^4}\right) + \left(\frac{\varepsilon}{\sigma}\right) R\{\sin x\}\, R\{\phi(x)\}$$

$$R\{\phi(x)\}=\frac{\sigma^2}{\varepsilon^2}+2\left(\frac{\sigma^4}{\varepsilon^4}\right)+\left(\frac{\varepsilon}{\sigma}\right)\left[\frac{\sigma^3}{\varepsilon(\varepsilon^2+\sigma^2)}\right]R\{\phi(x)\}$$

$$\Rightarrow R\{\phi(x)\}=\frac{\sigma^2}{\varepsilon^2}+2\left(\frac{\sigma^4}{\varepsilon^4}\right)+\left[\frac{\sigma^2}{(\varepsilon^2+\sigma^2)}\right]R\{\phi(x)\}$$

$$\Rightarrow R\{\phi(x)\}-\left[\frac{\sigma^2}{(\varepsilon^2+\sigma^2)}\right]R\{\phi(x)\}=\frac{\sigma^2}{\varepsilon^2}+2\left(\frac{\sigma^4}{\varepsilon^4}\right)$$

$$\Rightarrow R\{\phi(x)\}\left[\frac{\varepsilon^2}{(\varepsilon^2+\sigma^2)}\right]=\frac{\sigma^2}{\varepsilon^2}+2\left(\frac{\sigma^4}{\varepsilon^4}\right)$$

$$\Rightarrow R\{\phi(x)\}=\frac{\sigma^2}{\varepsilon^2}\left(\frac{(\varepsilon^2+\sigma^2)}{\varepsilon^2}\right)+2\left(\frac{\sigma^4}{\varepsilon^4}\right)\times\left(\frac{(\varepsilon^2+\sigma^2)}{\varepsilon^2}\right)$$

$$\Rightarrow R\{\phi(x)\}=\frac{\sigma^2}{\varepsilon^2}+\frac{\sigma^4}{\varepsilon^4}+2\left(\frac{\sigma^4}{\varepsilon^4}+\frac{\sigma^6}{\varepsilon^6}\right)$$

(14)

Inverting the Rishi transform on both sides of the preceding equation (14), we obtain:

$$\phi(x)=R^{-1}\left(\frac{\sigma^2}{\varepsilon^2}\right)+R^{-1}\left(\frac{\sigma^4}{\varepsilon^4}\right)+2R^{-1}\left(\frac{\sigma^4}{\varepsilon^4}+\frac{\sigma^6}{\varepsilon^6}\right)$$

$$\Rightarrow \phi(x)=1+\frac{x^2}{2}+2\left(\frac{x^2}{2}+\frac{x^4}{24}\right)=1+\frac{x^2}{2}+x^2+\frac{x^4}{12}$$

Therefore, the required solution of given integral equation is $1+\frac{x^2}{2}+x^2+\frac{x^4}{12}$.

(D) Consider the following second type: Volterra integral equation

$$\phi(x)=x+\int_0^x e^{-(x-t)}\,\phi(t)\,dt$$

(15)

The Rishi transform is applied to both sides of the preceding equation (15), yielding

$$R\{\phi(x)\}=R\{x\}+R\left\{\int_0^x e^{-(x-t)}\phi(t)\,dt\right\}$$

$$\Rightarrow R\{\phi(x)\}=R\{x\}+R\{e^{-x}*\phi(x)\}$$

(16)

Now using the Rishi transform convolution theorem on (16),

$$R\{\phi(x)\}=R\{x\}+\left(\frac{\varepsilon}{\sigma}\right)R\{e^{-x}\}\,R\{\phi(x)\}$$

$$R\{\phi(x)\}=\left(\frac{\sigma^3}{\varepsilon^3}\right)+\frac{\varepsilon}{\sigma}\left[\frac{\sigma^2}{\varepsilon(\varepsilon+\sigma)}\right]R\{\phi(x)\}$$

$$\Rightarrow R\{\phi(x)\}=\left(\frac{\sigma^3}{\varepsilon^3}\right)+\left[\frac{\sigma}{(\varepsilon+\sigma)}\right]R\{\phi(x)\}$$

$$\Rightarrow R\{\phi(x)\}-\left[\frac{\sigma}{(\varepsilon+\sigma)}\right]R\{\phi(x)\}=\frac{\sigma^3}{\varepsilon^3}$$

$$\Rightarrow R\{\phi(x)\}\left[\frac{\varepsilon}{(\varepsilon+\sigma)}\right]=\frac{\sigma^3}{\varepsilon^3}$$

$$\Rightarrow R\{\phi(x)\} = \frac{\sigma^3}{\varepsilon^3} \times \left(\frac{\varepsilon + \sigma}{\varepsilon} \right)$$

(17)

Inverting the Rishi transform on both sides of the preceding equation (17), we obtain:

$$\Rightarrow \phi(x) = R^{-1}\left[\frac{\sigma^3}{\varepsilon^3} + \left(\frac{\sigma^4}{\varepsilon^4} \right) \right]$$

$$\Rightarrow \phi(x) = x + \frac{x^2}{2}$$

Therefore, the required solution of given integral equation is $x + \frac{x^2}{2}$.

IX. CONCLUSION

In this study, the Rishi transform was effectively studied for the solution of second kind linear Volterra integral equations. The entire approach was provided by examining four numerical problems. The Rishi transform is a very useful integral transform for getting numerical solutions to the second kind of Volterra integral equations.

REFERENCES

[1]. Rahman, M. 2007. Integral equations and their application, WIT press.

[2]. Linz, P. 1984. Analytical and numerical methods for Volterra equations, SIAM, Philadelphia.

[3]. Jerri, A. 1999. Introduction to integral equations with applications, Wiley, New York, 1999.

[4]. Song, Y. and Kim, H. 2014. The Solution of Volterra Integral Equation of the Second kind by using the Elzaki Transform, Applied Mathematical Sciences, 8(11), 525-530.

[5]. Fadhil, R. (2017).Convolution for Kamal and Mahgoub transforms, *Bulletin of Mathematics and Statistics Research*, 5(4), 11-16.

[6]. Kumar, P.S., Saranya, C., Gnanavel, M.G. and Viswanathan, A.2018. Applications of Mohand transform for solving linear Volterra integral equations of first kind, *International Journal of Research in Advent Technology*, 6(10), 2786-2789.

[7]. Aggarwal, S., Sharma, N. and Chauhan, R. 2018. Application of Kamal transform for solving linear Volterra integral equations of first kind, *International Journal of Research in Advent Technology*, 6(8), 2081- 2088.

[8]. Aggarwal, S., Sharma, N. and Chauhan, R. 2018. Solution of linear Volterra integral equations of Second kind using Mohand transform, *International Journal of Research in Advent Technology*, 6(11), 3098- 3102.

[9]. Aggarwal, S., Singh, A., Kumar, A. and Kumar, N. 2019. Application of Laplace transform for solving improper integrals whose integrand consisting error function. *Journal of Advanced Research in Applied Mathematics and Statistics*, 4(2), 1-7.

[10]. Chauhan, R. and Aggarwal, S. 2019. Laplace transform for convolution type linear Volterra integral equation of Second kind. *Journal of Advanced Research in Applied Mathematics and Statistics*, 4(3-4), 1–7.

[11]. Aggarwal, S., Gupta, A.R. and Sharma, S.D. 2019. A New Application of Shehu Transform for Handling Volterra Integral Equations of First Kind, *International Journal of Research in Advent Technology*, 7(4), 439-445.

[12]. Higazy, M., Aggarwal, S. and Nofal, T.A. 2020. Sawi decomposition method for Volterra integral equation with application, Journal *of Mathematics*, 2020, 13.

[13]. Aggarwal, S., Vyas, A. and Sharma, S.D. 2020. Primitive of Second kind linear Volterra integral equation using Shehu transform, *International Journal of Latest Technology in Engineering, Management and Applied Science*, 9(8), 26-32.

[14]. Aggarwal, S. and Kumar, S. 2021. Laplace transform for system of Second kind linear Volterra integro-differential equations, *Journal of Emerging Technologies and Innovative Research*, 8(6), a769-a786.

[15]. Kumar, R., Chandel, J. and Aggarwal, S. 2022. A New integral transform Rishi Transform with application, *Journal of Scientific Research*, 14(2), 521-532.