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Research Paper

A Note on the Exponential Polynomial and Its Generating Function

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Abstract
The basic exponential polynomial is simple but has no closed-form expression other than the polynomial itself. This polynomial is mostly studied in group theories and with prime numbers. However, some of its basic properties seem to have been ignored. In this paper are shown an integral representation for it with which one can generate the polynomial.
It is based on both a recursion relation and a differential equation. By using them various generating functions are solved. [1]
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I. Introduction

The exponential polynomial appears in some initial value problems and is also recognized as a special case of generalized Laguerre polynomials. As the number of terms increases, it approaches the corresponding exponential function. The polynomial resists all known methods [4] for creating a closed-form solution to it.

The exponential polynomial and its generalizations are commonly studied in connection with Galois groups and prime number theories [1], [2]. Recent examples in this subject are [3] and [5].

In the following are first introduced the basic recursion relations. Then a useful differential equation is developed. They are used to advantage in developing the generating function for the exponential polynomial. A few generalizations are developed in the same manner.

II. The Exponential Polynomial

The basic exponential polynomial having an argument $x \in C$ is defined as

$$y_j(x) = \sum_{n=0}^{j} \frac{x^n}{n!} \tag{1}$$

The polynomial is always finite as long as the argument is finite, even in the case of $j = \infty$. The following recursion is a trivial consequence of the definition

$$y_{j+1}(x) - y_j(x) = \frac{x^{j+1}}{(j+1)!} \tag{2}$$

The first task is to find a differential equation for the y_j. The following simple one is derived by differentiation of the definition. It is correct but it is not useful as such.

$$D_x y_{j+1}(x) = y_j(x) \tag{3}$$

However, if the recursion is implemented to this, one will get the following differential equation

$$D_x y_j - y_j = -\frac{x^j}{j!} \tag{4}$$

This is actually a linear first order differential equation with a general solution

$$y_j(x) = -e^x \int^x ds e^{-s} \frac{s^j}{j!} + C_0 e^x \tag{5}$$

The C_0 is solved to be zero leaving the solution

$$y_j(x) = -e^x \int^x dse^{-s}\frac{s^j}{j!} \tag{6}$$

With this integral the $y_j(x)$ can be constructed for all j.

3 The Generating Function for The Exponential Polynomial

One can multiply the integral above by t^j and sum it getting

$$\sum_{j=0}^{\infty} y_j(x)t^j = -e^x \sum_{j=0}^{\infty} t^j \int^x dse^{-s}\frac{s^j}{j!} \tag{7}$$

The order of summation and integration can be swapped as it seems proper for these functions.

$$\sum_{j=0}^{\infty} y_j(x)t^j = -e^x \int^x dse^{-s} \sum_{j=0}^{\infty} \frac{s^j t^j}{j!} \tag{8}$$

The integral can be solved after recognizing the series inside it to be a proper exponential function, obtaining

$$\sum_{j=0}^{\infty} y_j(x)t^j = \frac{e^{xt}}{1-t} \tag{9}$$

This is a simple generating function for the exponential polynomial. It will be finite as long as $|t| < 1$.

3.1 Generalizing the Generating Function

By multiplying the y_j by

$$(-atf(x))^j \tag{10}$$

instead of t^j in the development above, the following generating function will be a result

$$\sum_{j=0}^{\infty} y_j(x)(-atf(x))^j = \frac{e^{-atxf(x)}}{1+atf(x)} \tag{11}$$

Here $a \in C$ and standard requirements apply for the convergence of the series. $f(x)$ is an arbitrary function with limitations regarding the series convergence. One obvious remark is that

$$1 + atf(x) \neq 0 \tag{12}$$

to keep the series finite at all values of a, t and $f(x)$. If $|atf(x)| < 1$, no convergence issues should be waiting. While $atf(x) > 0$ the series will be alternating with weaker requirements on convergence. The result shows that the generating function can be modulated in an interesting way.

3.2 Simple Applications of the Generating Function

By multiplying the y_j by (with i being the imaginary constant) in (6)

$$(it/x)^j \tag{13}$$

and

$$(-it/x)^j \tag{14}$$

the following generating function pair is obtained

$$\sum_{j=0}^{\infty} y_j(x)(it/x)^j = \frac{e^{it}}{1-it/x} \tag{15}$$

and

$$\sum_{j=0}^{\infty} y_j(x)(-it/x)^j = \frac{e^{-it}}{1+it/x} \tag{16}$$

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They can be summed to get

$$\sum_{j=0}^{\infty} y_j(x)(t/x)^j(i^j + (-i)^j) = 2\frac{cos(t) - \frac{t}{x}sin(t)}{1 + (t/x)^2} \tag{17}$$

Subtraction would give

$$\sum_{j=0}^{\infty} y_j(x)(t/x)^j(i^j - (-i)^j) = 2i\frac{sin(t) + \frac{t}{x}cos(t)}{1 + (t/x)^2} \tag{18}$$

By using itx as the power multiplier, one will get the following analogous results

$$\sum_{j=0}^{\infty} y_j(x)(itx)^j = \frac{e^{itx^2}}{1 - itx} \tag{19}$$

$$\sum_{j=0}^{\infty} y_j(x)(-itx)^j = \frac{e^{-itx^2}}{1 + itx} \tag{20}$$

Summation of these will lead

$$\sum_{j=0}^{\infty} y_j(x)(tx)^j(i^j + (-i)^j) = 2\frac{cos(tx^2) - (tx)sin(tx^2)}{1 + (tx)^2} \tag{21}$$

More analogous results are given by continuing as above.

IV. Discussion

The exponential polynomial is mostly studied as an element in the group theory.

It is also linked to prime number theory. Elementary properties of the exponential polynomial are commonly ignored as it is deceivingly simple.

The original recursion relation for the exponential polynomial (2) is used for developing further the obvious differential (3). The resulting differential equation can be solved leading to an integral expression for the polynomial (6).

It is useful for generating more results. The rare exponential structure of the polynomial actually gives rise to various forms of the generating function, like (9), (17), (18) and (21).

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