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Research Paper

A Review of Historical Background and Current Developments inFractional Calculus

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Abstract: *The mathematical growth of Fractional Calculus is the main focus of this review study. In the field of fractional operators, the contributions of Riemann, Liouville, and Laurent are discussed. Fractional calculus has recently received a lot of attention due to its importance in various disciplines of engineering and science.*

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I. INTRODUCTION

Non-integer order derivatives and integrals are studied and applied in fractional calculus, a new discipline of mathematics. It has more than a 320-years history. Its progress has primarily been centered on the pure mathematical discipline. Liouville, Riemann, Leibniz, and others appear to have conducted the first systematic research in the 19thcentury. Fractional differential equations (FDEs) have been utilized to explain a variety of stable physical phenomena with anomalous degradation throughout the last two decades. Many mathematical models of real-world problems thatarise in engineering and research are either linear or non-linear systems. With the discovery of fractional calculus, it has been shown that differential systems may be used to describe the role of many systems. It's worth noting that thefractional calculus may be used to describe a variety of physical processes with memory and genetic features. In reality, fractional order systems make up the majority of real-world processes. As a result of unique materials and chemical features, many physical systems exhibit fractional dynamical behaviour. In the fields of bio-physics, physics, biology, chemistry, economics, control theory, signal and image processing, polymer archaeology, and aerodynamics, fractional differential equations are used to represent systems and processes. The model problem is extremely tough to manage and solve analytically. Furthermore, due of the nonlinearity and complex geometry of these modelled problems, solving them might be difficult at times. Researchers have recently devised fractional order finite difference schemes and iterative approaches for solving fractional differential equations [10,31]. To tackle such modelled problems, certain finite difference methods (FDMs) and decomposition approaches exist.

II. HISTORICAL DEVELOPMENT

L' Hospital wrote to Leibniz on September 30th, 1695, inquiring about a particular notation he had used in his writings for the nth-derivative of the linear function $f(x) = x, \frac{d^n x}{ax^n}$. L'Hopital's raised the question to Leibniz, what will happen if $n = \frac{1}{2}$? Leibniz's response: "An apparent paradox, from which one day useful consequences will be drawn"[19].

As a result, fractional calculus was born on that date, and it is now known as fractional calculus' birthday. Many famous mathematicians, such as Fourier, Euler, and Laplace, contributed to the development of fractional calculus. Many mathematicians also established the idea of non-integer order integrals and derivatives using their own notations. The majority of fractional calculus mathematical theory is produced in the twentieth century. However, themost fascinating jumps in engineering and scientific application have been discovered in the last 100 years. The Riemann Liouville and Grunwald-Letnikov definitions are the most well-known of these definitions in the realm of fractional calculus (but not yet in the rest of the world). Caputo also revised the more "traditional" concept of the Riemann-Liouville fractional derivative in order to solve his fractional order differential equations using integer orderinitial conditions [25]. In order to differentiate no-where differentiable fractal functions, Kolowankar rewrote the Riemann-Liouville fractional derivative in 1996 [15]. Many notable mathematicians, including Euler, Laplace (1812), Fourier J. B. J. (1822), Abel (1823-1826), Liouville (18321873), Riemann (1847), Holmgren (186567), Grunwald

(18671872), Letnikov (18681872), Laurent (1884), Nekrassov (1888) are contributing to its development. However, there has been a recent attempt to define fractional derivative as a local operator in the context of fractal scientific theory. Fractional Calculus is maybe the calculus of the 21st century. Heaviside in 1922, Buss in 1929, Goldman in 1949, Starkey in 1954, Holbrook in 1966, Oldham and Spanier in 1974, Debnath in 1992, Miller and Ross in 1993, Saxena in 2002, Podlubny in 2003, Ray in 2005, and several others devoted additionally very reliable contributions to the subject. Sonin in 1869, Krug in 1890, Hadmard in 1892, Pinchorle in 1902, Hardy. and Littlewood in 1917-28,Weyl in 1917, Levy in 1923, Marchaud in 1927, Davis in 1924-36, Post in 1930, Zygmund 1935- 45, Love in 1938-

96, Erdelyi in 1939-65, Kober in 1940, Wider in 1941, Riesz in 1949, Feller in 1952, Nishimoto in 1987, Caputo in 1967. Many authors have looked at fractional differential equations in current years. To estimate the solution of fractional equations, Rawashdeh employed the collection spline approach. Momani found a solution to the integro- differential equation that was both local and global, as well as unique. Yong Zhou was dedicated to a rapidly developing field of study in qualitative theory of fractional differential equations. He was particularly interested in fractional differential equations' fundamental theory. Further study into the control, dynamics, numerical analysis, andapplications of fractional differential equations should begin with such fundamental theory. He discusses various methods for investigating fractional evolution equations that are regulated by the C0 semigroup [33]. He's also working on fractional Euler-Lagrange equations, temporal fractional diffusion equations, fractional Hamiltonian systems, and fractional Schrodinger equations, among other recent breakthroughs in theory for fractional partial differential equations. Sonin N.Ya: The first study that eventually led to what is now known as the Riemann-Liouville definition comes in the paper "On Differentiation with Arbitrary Index" by Sonin N.Ya. (1869), as highlighted in thebook (Miller and Ross 1993). Cauchy's integral formula was his starting point. From 1868 through 1872, Letnikov published four papers on the subject. Sonin's paper was extended in his article “An explanation on the essential notion of the theory of differentiation of arbitrary index (1872)”. Abel and Liouville, Leibniz, Euler, Laplace, and Lacroix Fourier all mentioned arbitrary order derivatives, but Niels Henrik Abel was the first to apply fractional operations in1823 [Abel 1881]. In order to solve an integral equation that occurs in the formulation of tautochrone (isochrone) problems, Abel used fractional calculus. In contrast to Sonin and Letnikov's closet circuit C0, Laurent (1884) developed integration via an open circuit C on the Riemann surface. Nishimoto (1984-1994) [32] represents the final version of this notion. The fundamental definition was also derived from Cauchy's integral formula by Nekrassov (1888) and Krug (1890), however their methods differed in the choice of an integration contour. However, it remainsa peculiar fact that these extended operators of integration and their link with the Cauchy integral formula have only received brief mentions in classic works in the theory of analytic function [34]. The present fractional differential equation stability results, as well as the analytical technique utilised, are the focus of S.Priyadharsini's study[35]. Liouville (1832a) was expanded functions in series of exponentials and defined the q^{th} derivative of such a series by operating term-by-term as though q were a positive integer. A different definition, presented by Riemann (1953), involved a definite integral and was applicable to power series with non-integer exponents. Evidently, it was Grunwald(1867) and Krug who first unified the results of Liouville and Riemann. Grunwald (1867) disturbed by the restrictionsof Liouville’s approach, adopted on his starting point the definition of a derivative as a limit of difference quotient and arrived as definite-integral formulas for the q th derivative. Krug (1890), working through Cauchy’s integral formula for ordinary derivatives, showed that Riemann’s definite integral had to be interpreted as having a finite lower limit while Liouville’s definition, in which no distinguishable lower limit appeared, corresponded to a lower limit $-\infty$ [23]. Leibniz, letter from Hanover, Germany, May 28, 1697 to J. Wallis. In this letter Leibniz discusses Wallis infinite product for Π. Leibniz mentions differential calculus and uses of the notation $d^{1}/2y$ to denote a derivative of order ½.In 1819, Lacroix, “Traite du Calcul Differentiel et du Calcul Integral, 2nd ed., Vol. 3, pp 409-410 Courcier, Paris”, inthis 700 page text two paper was devoted to fractional calculus. Lacroix develops a formula for fractional differentiation for the n^{th} derivative of v^m by induction. Then, he formerly replaces n with the fraction ½. In 1839, S.S.Greatheed, "On General Differentiation No.I, Cambridge Math, J.I., pp.11-12". In the same issue are two more papers: "On General Differentiation No II, Cambridge Math, J.I.,pp.109-117"; "On the Expansion of the Function of a Binomial, Cambridge Math, J.I.,pp.67-74". In the first two papers above, Greatheed uses Liouville's definition to develop formulas for fractional differentiation. In 3rd paper he supplements Taylor's theorem by use of fractional derivatives. In 1847, B. Riemann,"Ver such einer Auffassung der Integration and Differentiation." Grsammelte Werke, "1876,ed. Publ. Postumously, p.331-344; 1892 ed. Pp.353-366", Teubner,Leipzig. Also, in collected works "(H.Weber ed.)pp.354-360, Dover New York 1953". Riemann developed the following definition for fractional integration as an extension of a Taylor's series expansion:

$$\frac{d^{-s}}{dz^{-s}} v(z) = \frac{1}{\Gamma(s)} \int_{c}^{z} (z-k)^{s-1} u(k) dk$$

However, he saw fit to add a complimentary function to the above definition. Today this definition is in common useas a definition for fractional integration but with the complimentary function taken to be identically zero and the lower limit of integration c is usually zero. In 1880, A. Caley, "Note on Riemann's paper, Math.Ann.16.81-82" referring toRiemann's paper (1847) he says, "The greatest difficulty in Riemann's theory, it appears to me, is the interpretation of a complimentary function containing infinity of arbitrary constants." The presence of a complementary function was a source of significant debate. Liouville and Peacock were led in to errors and Riemann became inextricably entangled in his concept of a complimentary function. Oliver Heaviside independently created operational calculus in 1880, a procedure for transforming problems with differential equations into algebraic equations with a differential operator p. Heaviside defined also fractional powers of p, thus establishing a connection between operation calculus and fractional calculus.[6]. In 1884, H. Laurent, "Surle Calcul des derivees a idices quelconques, Nouc. Ann. Math.[3],3, 240-252", Laurent generalizes Cauchy's integral formula. He does work on the generalized product rule of Leibniz but leaves the result in integral form. In 1917, Hardy published the paper "On Some Properties of Integrals of Fractional Order, Messenger Math. 47, 145-150". In 1919, Post, "Discussion of problems #360 and #433, Amer. Math.Monthly 26, 37-39". When two distinct answers to issue #433 are offered, Post uses the chance to answer problem #360 as well. He explains that both solutions are correct, but that they are based on distinct definitions. The proposer,in his solution, used Liouville's definition of integration of fractional order which is equivalent to the definite integral

$${}_bD_z^{-s} = \frac{1}{\Gamma(s)} \int_{b}^{z} (z-k)^{s-1} g(t) dt$$

With the lower limit of integration c equal to negative infinity, Post employed Riemann's definition, which is the above integral with b equal to zero, in his answer. Although, Post makes no reference to Center (1848[a]), it is clear why Center, with $f(x)$ equal to constant, would have to different results for the arbitrary derivative. In 1924, H.T.Davis,"Fractional operations as applied to a class of Voltera Integral Equations." Amer.J.Math, 46, 95-109. A review of fractional calculus theory before the theory is applied to the solution of particular integral equations compensates for the absence of extensive explanation, which is understood in a journal article. This work and Davis's1927 essay, in the opinion of the current author, are notable not just for their contributions to fractional calculus theory and applications, but also as models of how a mathematics paper should be written. In 1928, G.H.Hardy and J.E.Littlewood, "Some Properties of Fractional Integrals-I." Math.Z.27, 565- 606 (1928): "Some Properties of Fractional Integrals-II." Math.Z. 34, 403-439 (1932). In part-I, their purpose is to develop properties of the Riemann-Liouville integral and derivative of arbitrary order of functions of certain standard classes, in particular the "Lebesgue class Lp". Part-II is an extension of the first paper to the complex field. In 1935, A.Zygmund, Trignometric series, Vol.II, Ist ed.Z. Subwencji Funduszu Kultury rarodewej, Warsaw; IInd ed. Cambridge University Press, Cambridge,1959,pp.132-142. Zygmund believes a notion of fractional integration proposed by Weyl to be more suitable for trigonometric series in the section labelled "Fractional Integration". It is, nevertheless, a fresh issue, as it has only been the subject of specialist conferences and treatises for a little more than two decades. B. Ross, who organised the first conference on fractional calculus and its applications at the University of New Haven in June 1974and edited the papers, is credited with the inaugural meeting, see [26]. The first monograph is credited to K.B. Oldhamand J. Spanier, see [24], who released a book on fractional calculus in 1974 following a cooperative work that beganin 1968. Currently, there are around a dozen texts and proceedings devoted wholly or partially to fractional calculus and its applications [11,12,13,16,17,18,21,22,24,26,28,30,31], the most renowned of which is Samko, Kilbas, and Marichev's encyclopaedic treatise [30]. We also call attention to the treatises by Davis [5, Erd'elyi [7], Gel'fand & Shilov [9], Djrbashian [3,4], Caputo [2], Babenko [1], Gorenflo & Vessella [8], which, despite their titles not explicitlymentioning it, contain a detailed analysis of some mathematical aspects and/or physical applications of fractional calculus. Only a few books on fractional calculus were available from 1975 to 1985, and they are listed below: 1. Keith, B. Oldham, and J. Spanier, Dover Books on Mathematics, 1974, "The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order." 2. Fractional Calculus and its Applications: Proceedings of the International Conference held at the University of New Haven in June 1974 (Lecture Notes in Mathematics), 1975 (B. Ross, Editor). The Use of Fractional Integration Operators in Applied Mathematics (AppliedMathematics Series), Polish Scientific Publishers, 1979. 3. Ian N. Sneddon, The Use of Fractional Integration Operators in Applied Mathematics (Applied Mathematics

Series), Polish Scientific Publishers, 1979.

III. CURRENT DEVELOPMENT

In recent years, a large number of international and national mathematicians have contributed to the advancement of fractional calculus. Many of them have developed iterative and finite difference methods to solve linear and non-linear fractional partial differential equations. In 2011, A.P. Bhadane and K. C. Takale were published the paper "Basic developments of fractional calculus and its applications", Bulletin of the Marathwada Mathematical Society, Vol.12, No. 2, Dec 2011, pp.1-7. They created the fundamental theory and applications of fractional calculus in this study. They were able to derive fractional integrals and fractional derivatives for a number of functions. The mathematical programme Mathematica simulates the fractional integral and fractional derivative of these functions. The paper "Recent Advancement in Fractional Calculus" by Manoj Kumar and Anuj Shankar Saxena was published in 2016. They focused on how differential equations arise from mathematical modelling of diverse real-life situations in engineering and science. Large mistakes may be introduced by these classic models based on integer order derivatives. The use of fractional derivatives in fractional calculus helps to reduce this inaccuracy, and it provides compatibilities that allow for effective depiction of memory and hereditary features of processes. They demonstrate the expressive capacity of fractional calculus in this review paper by examining two examples: the mortgage problem and the fractional oscillator. These examples demonstrate the superiority of fractional calculus over integer calculus. They also examine the state of the art of fractional calculus by looking at how quickly its applications have grown in various fields [14]. In 2016, Vasily E. Tarasov was published the paper "Local Fractional Derivatives of Differentiable Functions are Integer Order Derivative or Zero", International Journal of applied and computational Mathematics, 2016, Vol.2, No.2, pp. 195-201. In this research, he showed that total fractional derivatives of differentiable functionsare integer order derivatives or zero operators. He demonstrates that the Caputo fractional derivatives on the left-hand side have local fractional derivatives as limits. The Caputo derivative of fractional order α of function $f(x)$ is definedas a fractional integration of order n – α of the derivative $f\ (n)\ (x)$ of integer order n. Because it necessitates the presence of integer-order derivatives, we may deduce that the local fractional derivative is not the best technique to describe nowhere differentiable functions and fractal objects. He also proved that unviolated Leibniz rule cannot hold for derivatives of order $a \neq 1$ and etc. Following is list of books which are available on fractional calculus from 1985:

1) Denis Matignon, Gérard Montseny (Editors), Fractional Differential Systems: Models, Methods and Applications, European Society for Applied and Industrial Mathematics (ESAIM), Vol. 5, 1998.
2) Journal of Vibration and Control, Special Issue: Fractional Differentiation and its Applications, vol. 14, Sept.2008.
3) Physica Scripta Fractional Differentiation and its Applications, T136, 2009.

4) Computers and Mathematics with Applications, Special issues: - Advances in Fractional Differential Equations, vol. 59, Issue 3, pp. 1047-1376, February 2010.
5) Fractional Differentiation and its Applications, vol. 59, Issue 5, March 2010. 4

IV. APPLICATIONS OF FRACTIONAL CALCULUS

Abel 183 [18,23] makes the first use of semi-derivative (order1/2 derivative). The solution of the integral equation for the tautochrone issue is related to this application of fractional calculus. The goal of this challenge is to figure out how to determine the slope of a curve so that the time of gravity is independent of the beginning position. The previous several decades have demonstrated that arbitrary order derivatives and integrals are particularly useful for expressingcharacteristics of actual materials, such as polymers [25]. The new fractional order models are better than the previousinteger-order models. While such effects are ignored in integer-order models, fractional derivatives are a great tool for understanding the memory and heredity qualities of diverse materials and processes. Applications of fractional calculus have recently been discovered in a variety of fields, including electromagnetism, economics and finance, chaos and fractals, viscoelasticity and damping, diffusion and wave propagation, heat transfer, biology, electronics, robotics, system identification, traffic systems, percolation, signal processing, modelling and identification, telecommunications, genetic algorithms, chemistry, irreversibility, physics, control systems.

V. CONCLUSION AND SCOPE FOR FUTURE

Fractional calculus was first proposed almost 325 years ago, and significant research into it has just lately begun. Classical calculus is still far more familiar and favoured, maybe because its applications are more obvious. Classical calculus has a number of flaws that fractional calculus may fill. Fractional calculus, as a result, has the potential to present, integrate, and serve as a useful tool.

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