Quest Journals
Journal of Research in Applied Mathematics
Volume 8 ~ Issue 8 (2022) pp: 07-12
ISSN(Online) : 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Research Paper

A Tale of Two Sequences: Fibonacci and Perrin

Eric Choi, Anthony Delgado, Marty Lewinter, Derek Tan

Abstract

The Fibonacci sequence, $\{u_n\}$, is defined by: $u_0 = 0$, $u_1 = 1$, and $u_{n+2} = u_n + u_{n+1}$. The Perrin sequence, $\{P_n\}$, is defined by: $P_0 = 3$, $P_1 = 0$, $P_2 = 2$, and $P_{n+3} = P_n + P_{n+1}$. We compare and contrast these important sequences in this largely expository article.

R**eceived 02 August, 2022; **R**evised 14 August, 2022; **A**ccepted 16 August, 2022* *© The author(s) 2022. **P**ublished with open access at **www.questjournals.org

The Fibonacci Sequence

The Fibonacci sequence, $\{u_n\}$, is defined by: $u_0 = 0$, $u_1 = 1$, and $u_{n+2} = u_n + u_{n+1}$. The first few Fibonacci numbers are 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, and 610. The following identities are quite interesting.

1. $gcd(u_n, u_{n+1}) = 1$
2. **(a)** $u_1 + u_2 + \ldots + u_n = u_{n+2} - 1$ **(b)** $u_1 + u_3 + \ldots + u_{2k-1} = u_{2k}$ **(c)** $u_2 + u_4 + \ldots + u_{2k} = u_{2k+1} - 1$
3. $u_1^2 + u_2^2 + u_3^2 + \ldots + u_n^2 = u_n u_{n+1}$
4. *Binet* formula: $u_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$, where $\left|\frac{1-\sqrt{5}}{2}\right| < 1$.
5. For large n, $u_n \approx \frac{1}{\sqrt{5}}\phi^n$, where $\phi = \frac{1+\sqrt{5}}{2} = 1 + \cfrac{1}{1+\cfrac{1}{1+\cdots}}$. ϕ is a root of $\lambda^2 - \lambda - 1 = 0$.
6. $u_n^2 - u_{n+1}u_{n-1} = (-1)^{n+1}$
7. **(a)** $u_{m+n} = u_{m-1}u_n + u_m u_{n+1}$ **(b)** $u_{n+1}^2 - u_{n-1}^2 = u_{2n}$ **(c)** $u_{n+1}^2 + u_n^2 = u_{2n+1}$
8. Let $n \geq m \geq 3$. Then $u_m \mid u_n$ if and only if $m \mid n$.
9. Let p be prime such that $p \neq 5$. Then either $p \mid u_{p-1}$ or $p \mid u_{p+1}$. When $p = 5$, we have $p \mid u_5$.

Proof of #6: Verify that $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} u_n & u_{n-1} \\ u_{n+1} & u_n \end{pmatrix} = \begin{pmatrix} u_{n+1} & u_n \\ u_{n+2} & u_{n+1} \end{pmatrix}$. The determinant of a product of square matrices is the product of their determinants. So

$$\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} \times \begin{vmatrix} u_n & u_{n-1} \\ u_{n+1} & u_n \end{vmatrix} = \begin{vmatrix} u_{n+1} & u_n \\ u_{n+2} & u_{n+1} \end{vmatrix} \quad \Longrightarrow \quad -(u_n^2 - u_{n-1}u_{n+1}) = u_{n+1}^2 - u_n u_{n+2}$$

Letting $F(n) = u_n^2 - u_{n-1}u_{n+1}$, our last equation (exchanging the left and right sides) becomes $F(n+1) = -F(n)$. Now $F(1) = u_1^2 - u_0u_2 = 1$, implying that $F(n) = (-1)^{n+1}$. Done! Note that the characteristic equation of $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ is $\begin{vmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0$, or $\lambda^2 - \lambda - 1 = 0$. The roots are $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$, the heroes of the Binet equation!

The Fibonacci sequence, $\{u_n\}$, may be extended for negative values of n by rewriting the recursive relation as $u_n = u_{n+2} - u_{n+1}$. We have

$u_{-1} = u_1 - u_0 = 1 - 0 = 1$

$u_{-2} = u_0 - u_{-1} = 0 - 1 = -1$

$u_{-3} = u_{-1} - u_{-2} = 1 - (-1) = 2$

$u_{-4} = u_{-2} - u_{-3} = -1 - 2 = -3$

$u_{-5} = u_{-3} - u_{-4} = 2 - (-3) = 5$

$u_{-6} = u_{-2} - u_{-3} = -3 - 5 = -8$

Fact: For $n < 0$, $u_n = (-1)^{1-n}u_{-n}$. That is, the signs strictly alternate, and $|u_n| = u_{-n}$.

Proof: The Fact is true for the first few (negative) values of n. Assume that $u_n = (-1)^{1-n}u_{-n}$ and $u_{n-1} = (-1)^{2-n}u_{-n+1}$. Then by the recursion,

$$u_{n-2} = u_n - u_{n-1} = (-1)^{1-n}u_{-n} - (-1)^{2-n}u_{-n+1} = (-1)^{1-n}[u_{-n} - (-1)u_{-n+1}] = (-1)^{1-n}[u_{-n} + u_{-n+1}] =$$

$$(-1)^{1-n}[u_{|n|} + u_{|n|+1}] = (-1)^{1-n}u_{|n|+2} = (-1)^{1-n}u_{-n+2} = (-1)^{1-n}u_{2-n}.$$

The Perrin Sequence

The Perrin sequence, $\{P_n\}$, is defined by:

$P_0 = 3$ $\quad$ $P_1 = 0$ $\quad$ $P_2 = 2$ $\quad$ $\boxed{P_{n+3} = P_n + P_{n+1}}$

The first few Perrin numbers are 3 0 **2** **3** 2 **5** 5 **7** 10 12 17 **22** 29 **39** 51 68 90 **119**. The bold-font values have prime indices.

Fact: Given any prime number, p, we have $p \mid P_p$.

Examples: $2 \mid 2$ $\quad 3 \mid 3$ $\quad 5 \mid 5$ $\quad 7 \mid 7$ $\quad 11 \mid 22$ $\quad 13 \mid 39$ $\quad 17 \mid 119$

A composite number, n, for which $n \mid P_n$, is called a *Perrin pseudoprime*. The smallest Perrin pseudoprime is 271441.

For $n \geq 3$, P_n counts the number of maximal independent sets of the cycle, C_n.

Consider the matrix-vector equation

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} P_n \\ P_{n+1} \\ P_{n+2} \end{pmatrix} \qquad (*)$$

When $n = 0$, the matrix is the identity, so (*) yields our three initial values. That is,

$$\begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} P_0 \\ P_1 \\ P_2 \end{pmatrix}$$

When $n = 1$, (*) becomes

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix}$$

*Corresponding Author: Eric Choi

As a consequence of (*), we have

$$\begin{pmatrix}0&1&0\\0&0&1\\1&1&0\end{pmatrix}^n \begin{pmatrix}3&0&2\\0&2&3\\2&3&2\end{pmatrix} = \begin{pmatrix}P_n&P_{n+1}&P_{n+2}\\P_{n+1}&P_{n+2}&P_{n+3}\\P_{n+2}&P_{n+3}&P_{n+4}\end{pmatrix} \Rightarrow \begin{vmatrix}0&1&0\\0&0&1\\1&1&0\end{vmatrix}^n \times \begin{vmatrix}3&0&2\\0&2&3\\2&3&2\end{vmatrix} = \begin{vmatrix}P_n&P_{n+1}&P_{n+2}\\P_{n+1}&P_{n+2}&P_{n+3}\\P_{n+2}&P_{n+3}&P_{n+4}\end{vmatrix}$$

$$\Rightarrow \begin{vmatrix}P_n&P_{n+1}&P_{n+2}\\P_{n+1}&P_{n+2}&P_{n+3}\\P_{n+2}&P_{n+3}&P_{n+4}\end{vmatrix} = -23 \qquad (**)$$

since $\begin{vmatrix}0&1&0\\0&0&1\\1&1&0\end{vmatrix}=1$ and $\begin{vmatrix}3&0&2\\0&2&3\\2&3&2\end{vmatrix}=3\begin{vmatrix}2&3\\3&2\end{vmatrix}+2\begin{vmatrix}0&2\\2&3\end{vmatrix}=3(-5)+2(-4)=-23$. Done!

To find the characteristic equation of $\begin{pmatrix}0&1&0\\0&0&1\\1&1&0\end{pmatrix}$, we evaluate $\begin{vmatrix}-\lambda&1&0\\0&-\lambda&1\\1&1&-\lambda\end{vmatrix}$ by cofactor expansion around the first row, obtaining

$$\begin{vmatrix}-\lambda&1&0\\0&-\lambda&1\\1&1&-\lambda\end{vmatrix} = -\lambda\begin{vmatrix}-\lambda&1\\1&-\lambda\end{vmatrix} -1\begin{vmatrix}0&1\\1&-\lambda\end{vmatrix} = \lambda(\lambda^2-1)-1=\lambda^3-\lambda-1$$

$\lambda^3-\lambda-1$ is an example of a *depressed cubic*, since it is missing a square term, and its leading coefficient is 1. The discriminant of the depressed cubic, x^3+bx+c, is $-4b^3-27c^2$. The depressed cubic has a repeated root if and only if the discriminant is 0. The discriminant of $\lambda^3-\lambda-1$ is $-4(-1)^3-27(-1)^2=4-27=-23$, so the roots are distinct. (The identity (**) features -23.)

Let the (distinct) roots of $\lambda^3-\lambda-1$ be r, s, and t, where

$$r=\sqrt[3]{\frac{9+\sqrt{69}}{18}}+\sqrt[3]{\frac{9-\sqrt{69}}{18}}\approx 1.3247 \qquad s\approx -.66236-.56228i \qquad t\approx -.66236+.56228i$$

(s and t are complex conjugates.) Then we have a formula for P_n, reminiscent of the Binet formula for the Fibonacci numbers, namely

$$\boxed{P_n=r^n+s^n+t^n}$$

Since $|s|<1$ and $|t|<1$, we see that as n gets large, $P_n\approx r^n$, implying that $\lim_{n\to\infty}\frac{P_{n+1}}{P_n}=r$.

Here is another way to obtain $\lim_{n\to\infty}\frac{P_{n+1}}{P_n}=r$, where r is the real solution of $x^3-x-1=0$.

$$P_{n+3}=P_n+P_{n+1} \quad\Rightarrow\quad \frac{P_{n+3}}{P_{n+2}}=\frac{P_n}{P_{n+2}}+\frac{P_{n+1}}{P_{n+2}} \quad\Rightarrow\quad \frac{P_{n+3}}{P_{n+2}}=\frac{P_n}{P_{n+1}}\cdot\frac{P_{n+1}}{P_{n+2}}+\frac{P_{n+1}}{P_{n+2}}$$

Letting $\lim_{n\to\infty}\frac{P_{n+1}}{P_n}=x$, the last equation above becomes

$$x=\frac{1}{x}\cdot\frac{1}{x}+\frac{1}{x}=\frac{1}{x^2}+\frac{1}{x} \quad\Rightarrow\quad x=\frac{1}{x^2}+\frac{1}{x} \quad\Rightarrow\quad x^3=1+x \quad\Rightarrow\quad x^3-x-1=0 \qquad \text{Done!}$$

Fact: $r = -2\text{re}(s)$

Proof: Since $P_n = r^n + s^n + t^n$, we have $r + s + t = P_1 = 0$. Since s and t are complex conjugates, $0 = r + s + t = r + 2\text{re}(s)$, so $r = -2\text{re}(s)$.

Interesting Formula for r: $r = \sqrt[3]{1+\sqrt[3]{1+\sqrt[3]{1+\sqrt[3]{1+\cdots}}}}$

Heuristic Proof: Assuming that the right side converges, we can rewrite it as $r = \sqrt[3]{1+r}$ which becomes $r^3 = 1 + r$, or $r^3 - r - 1 = 0$.

Fact: $P_0 + P_1 + P_2 + \ldots + P_n = P_{n+2} + P_{n+3} - 2$

Example: When $n = 4$, $P_0 + P_1 + P_2 + P_3 + P_4 = P_6 + P_7 - 2$, or $3 + 0 + 2 + 3 + 2 = 5 + 7 - 2$.

Proof: Let's rewrite the recursive equation, $P_{n+3} = P_n + P_{n+1}$, as $P_n = P_{n+3} - P_{n+1}$. Now insert, successively, the values $0, 1, 2, \ldots, n$, yielding the $n + 1$ equations

$$
\begin{aligned}
P_0 &= P_3 - P_1 \\
P_1 &= P_4 - P_2 \\
P_2 &= P_5 - P_3 \\
&\vdots \\
P_{n-2} &= P_{n+1} - P_{n-1} \\
P_{n-1} &= P_{n+2} - P_n \\
P_n &= P_{n+3} - P_{n+1}
\end{aligned}
$$

Adding the left and right sides of these equations and cancelling all but a few terms on the right-side sum confirms the Fact. (This is called *telescoping*.) Using telescoping, we also have:

(1) $P_0 + P_2 + P_4 + \ldots + P_{2n} = P_{2n+3}$ **(2)** $P_1 + P_3 + P_5 + \ldots + P_{2n-1} = P_{2n+2} - 2$

Fact: $P_n + P_{n+4} = P_{n+2} + P_{n+3}$

Proof: Let a, b, c, $a + b$, $b + c$ be five consecutive Perrin numbers starting with P_n. That is, $a = P_n$, $b = P_{n+1}$, etc. Then $P_n + P_{n+4} = a + (b + c)$, while $P_{n+2} + P_{n+3} = c + (a + b)$. Done.

Fact: For $n \geq 5$, $\Delta_n = P_n - P_{n-1} = P_{n-5}$.

Proof: This is true for $n = 5$, 6, and 7. That is, **(1)** $P_5 - P_4 = P_0$ and **(2)** $P_6 - P_5 = P_1$. Then add these two equations to obtain $P_8 - P_7 = P_3$. Now $P_7 - P_6 = P_2$, that is, $7 - 5 = 2$. So $P_n - P_{n-1} = P_{n-5}$ is true for $n = 5$, 6, 7, and 8. Now generalize this *inductive* procedure for all $n \geq 5$. Done.

Fact 1: Given any k, we have: $P_k - P_{k+2} + P_{k+4} - P_{k+6} = -P_{k+4}$.

Proof: $P_k - P_{k+2} + P_{k+4} - P_{k+6} = (P_k + P_{k+4}) - (P_{k+2} + P_{k+6}) = P_{k+5} - P_{k+7} = -P_{k+4}$.

Fact 2: Given $a \geq 3$, we have: $\sum_{k=0}^{2^a-1} (-1)^k P_{2k} = -\sum_{k=0}^{2^{a-2}-1} P_{8k+4}$

Proof: The series on the left side of the equality has 2^a terms, so it can be partitioned into $\frac{2^a}{4} = 2^{a-2}$ sums which we evaluate using Fact 1, as follows.

1[st] sum: $P_0 - P_2 + P_4 - P_6 = -P_4 = -P_{(0)8+4}$

2[nd] sum: $P_8 - P_{10} + P_{12} - P_{14} = -P_{12} = -P_{(1)8+4}$

3[rd] sum: $P_{16} - P_{18} + P_{20} - P_{22} = -P_{20} = -P_{(2)8+4}$

$$\vdots$$

2^{a-2} sum: $P_{(2^{a-2}-1)8} - P_{(2^{a-2}-1)8+2} + P_{(2^{a-2}-1)8+4} - P_{(2^{a-2}-1)8+6} = -P_{(2^{a-2}-1)8+4}$

Adding both sides of these equations yields $\sum_{k=0}^{2^a-1}(-1)^k P_{2k} = -\sum_{k=0}^{2^{a-2}-1} P_{8k+4}$. Using a similar argument we have Fact 3.

Fact 3: Given $a \geq 3$, we have: $\boxed{\sum_{k=0}^{2^{a-1}-1}(-1)^k P_{2k+1} = -\sum_{k=0}^{2^{a-2}-1} P_{8k+5}}$

Proof: Using Facts 2 and 3, we have $\sum_{k=0}^{2^a-1} P_k i^k = P_0 + P_1 i + P_2 i^2 \ldots + P_{2^a-1} i^{2^a-1} =$

$$\left(P_0 - P_2 + P_4 - \ldots + P_{2^a-4} - P_{2^a-2}\right) + \left(P_1 - P_3 + P_5 - \ldots + P_{2^a-3} - P_{2^a-1}\right)i = \sum_{k=0}^{2^{a-1}-1}(-1)^k P_{2k} + i\sum_{k=0}^{2^{a-1}-1}(-1)^k P_{2k+1}$$

$$= -\sum_{k=0}^{2^{a-2}-1} P_{8k+4} - i\sum_{k=0}^{2^{a-2}-1} P_{8k+5} = -\sum_{k=0}^{2^{a-2}-1}\left(P_{8k+4} + P_{8k+5} i\right).$$

Fact: $P_n + P_{n+1} + P_{n+2} = P_{n+5}$.

Proof: $P_{n+5} = P_{n+2} + P_{n+3} = P_{n+2} + (P_n + P_{n+1}) = P_n + P_{n+1} + P_{n+2}$

Perrin numbers with negative indices are obtained by working backwards. We rewrite the recursion, $P_{n+3} = P_n + P_{n+1}$, as $\boxed{P_n = P_{n+3} - P_{n+1}}$. For $n = -1$, we have $P_{-1} = P_2 - P_0 = 2 - 3 = -1$, and for $n = -2$, we have $P_{-2} = P_1 - P_{-1} = 0 - (-1) = 1$. Here are the first few Perrin numbers with negative indices: $P_{-1} = -1$, $P_{-2} = 1$, $P_{-3} = 2$, $P_{-4} = -3$, $P_{-5} = 4$, $P_{-6} = -2$, $P_{-7} = -1$, $P_{-8} = 5$, $P_{-9} = -7$, $P_{-10} = 6$, $P_{-11} = -1$.

For $n > 1$, $P_n = \sum_{m=-3}^{n-5} P_m = P_{-3} + P_{-2} + \cdots + P_{n-5}$. For $n < 1$, $P_n = \sum_{m=4}^{4-n} P_{-m} = P_{-4} + P_{-3} + P_{-2} + \cdots + P_{4-n}$.

Non-linear Identities: **(1)** $P_{2n} = P_n^2 - 2P_{-n}$ **(2)** $P_{2n+1} = P_n P_{n+1} + P_{1-n}$ **(3)** $P_{3n} = P_n^3 - 3P_n P_{-n} + 3$

Examples: For $n = 3$, **(1)** $P_6 = P_3^2 - 2P_{-3}$, or $5 = 3^2 - 2\cdot 2$; **(2)** $P_7 = P_3 P_4 + P_{-2}$, or $7 = 3\cdot 2 + 1$; and **(3)** $P_9 = P_3^3 - 3P_3 P_{-3} + 3$, or $12 = 3^3 - 3\cdot 3\cdot 2 + 3$.

Surprising Fact: When p is prime, $P_{-p} = -1 \pmod{p}$.

*Corresponding Author: Eric Choi

Examples: $P_{-2} = 1 \equiv -1 \pmod 2$, $P_{-3} = 2 \equiv -1 \pmod 3$, $P_{-5} = 4 \equiv -1 \pmod 5$, $P_{-7} \equiv -1 \pmod 7$, and $P_{-11} \equiv -1 \pmod{11}$.

The Perrin sequence, *mod* 3, starts **0 0 2** 0 2 2 2 1 1 0 2 1 2 **0 0 2** … The last three numbers (in bold font) are identical to the first three numbers, so the period of the Perrin numbers, mod 3, consists of the thirteen values, 0 0 2 0 2 2 2 1 1 0 2 1 2. The period length is 13. Out of every thirteen Perrin numbers, six are 2 (mod 3). Note that we can obtain the Perrin sequence mod 3, by applying the recursive relation directly from the revised seed, 0 0 3 provided that we reduce our sums mod 3.

References

1. M.Lewinter, J.Meyer, *Elementary Number Theory with Programming*, Wiley & Sons. 2015.
2. https://www.cut-the-knot.org/arithmetic/combinatorics/PerrinSequence.shtml
3. https://en.wikipedia.org/wiki/Perrin_number
4. https://mathworld.wolfram.com/PerrinSequence.html
5. https://www.geeksforgeeks.org/program-perrin-number/
6. https://cp4space.hatsya.com/2013/09/15/perrin-sequence/
7. https://www.had2know.org/academics/perrin-padovan-sequence-plastic-constant-calculator.html