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Research Paper

The New Exponential- Exponential Distribution: Theory and Properties

[1]Ogunwale Olukunle Daniel, [1]Ayeni Taiwo Michael, [2]Osunronbi, Fatai Alani, [1]Adeleke Raphael Ayan,[1]Odukoya Elijah Ayooluwa
[1](Department of Statistics Ekiti State University, Ado-Ekiti, Nigeria)
[2](Department Statistics, Federal Polytechnic Ado-Ekiti, Nigeria)
Corresponding Author: Ayeni Taiwo Michael

ABSTRACT: *Statistical distributions are in general, used in describing real-world occurrences. In line with the usefulness of these statistical distributions, their theory is largely studied and applied to many scenarios in real life. In this paper, we propose a new probability distribution called Exponential-Exponential distribution, provide a comprehensive study of its theory and derive appropriate expressions for its statistical properties. The method of maximum likelihood was employed to estimate its parameter.*
KEYWORDS: *Exponential Distribution, Survival function, Hazard function, Maximum Likelihood Estimate, Exponential-Exponential Distribution*

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I. INTRODUCTION

Statistical distributions are in general, used in describing real-world occurrences. In line with the usefulness of these statistical distributions, their theory is largely studied and applied to many scenarios in real life. One of the main challenges of statisticians is to find the appropriate and efficient statistical distribution in modeling natural life events in the form of known probability distributions. Probability distributions posed important usefulness in modeling natural life phenomena that are characterized by uncertainty and riskiness. It was noted in [4] that the interest in developing and modifying existing distributions to form more flexible statistical distributions remains an arguable topic in the statistics profession; numerous generalized forms and classes of distributions have emerged and applied to describe various phenomena a common feature of these generalized distributions is that they have more parameters. In several fields of studies such as medicine, engineering, and finance, estimating and analyzing lifetime data is fundamental, some lifetime distributions have been employed to model such kinds of data, and the feature of the techniques used in a statistical analysis depends heavily on the assumed probability distributions given this, significant efforts have been used in the development of large classes of typical probability distributions along with pertinent statistical procedures [11]. Researchers in many fields of study have contributed to the development and application of new probability distribution models. The following is a brief kind of literature review on the development of new probability distribution and their applications by different scholars such as [1], [2],[3],[4],[5],[6], and some contributions in term of applications of probability into various areas of life such as [7],[8],[9],[10],[11],[13],[14]

II. METHODS

2.1 The Derivation of the new model Exponential-Exponential Distribution (EED)

The probability density function of the EED is derived in this section

Theorem 1: Let x be continuous independent random variables such that; $x \sim \mathrm{E}(x,\lambda)$ follows an Exponential distribution and, let $f(x)$ and $F(x)$ be the probability density function and CDF of exponential distribution given as;

$$f(x)=\lambda e^{-\lambda x},\quad x>0,\lambda>0 \tag{1}$$

and $F(x)=1-e^{-\lambda x},\quad x>0,\lambda>0$ (2)

similarly, from [4], let,

$$g(x) = \lambda f(x)(1 - F(x))^{\lambda-1}, \quad x > 0 \tag{3}$$

be the pdf of the ED-X, where $f(x)$ and $1 - F(x)$ is the pdf and the survival function of the baseline distribution then inserting (1) and (2) into (3) above, then pdf of the Exponential-Exponential distribution is given as

$$g(x) = \lambda^2 e^{-\lambda^2 x}, \; x > 0, \lambda > 0 \tag{4}$$

2.2 Statistical Properties of the Exponential-Exponential Distribution (EED)

The statistical properties of EED especially the first four moments, the variance, coefficient of variation, moment generating function, characteristic function, skewness, and kurtosis are obtained in this section as follows

(i) Moments

Theorem 2: If X is a random variable distributed as an EED $(x;\lambda)$, then the rth non-central moment is given by $\mu_r^{'} = \dfrac{\Gamma(r+1)}{\lambda^{2r}}$

Proof:

$$\mu_r^{'} = \int_0^\infty x^r f(x;\lambda)dx$$

$$= \int_0^\infty x^r \lambda^2 e^{-\lambda^2 x} dx$$

$$= \lambda^2 \int_0^\infty x^r e^{-\lambda^2 x} dx$$

(5)

Let $u = \lambda^2 x, \; x = \dfrac{u}{\lambda^2}, \; dx = \dfrac{du}{\lambda^2}$ then, so that (5) reduces to:

$$= \lambda^2 \int_0^\infty \left(\frac{u}{\lambda^2}\right)^r e^{-u} \frac{du}{\lambda^2}$$

$$= \frac{\lambda^2}{\lambda^2} \times \frac{1}{(\lambda)^{2r}} \int_0^\infty u^r e^{-u} du$$

But $\int_0^\infty u^r e^{-u} du = \Gamma(r+1)$, then

$$\mu_r^{'} = \frac{\Gamma(r+1)}{\lambda^{2r}} \tag{6}$$

the first(mean), second, third, and fourth moment is obtained by substituting r=1,2,3 and 4 respectively as follows;

Substituting $\mathrm{r} = 1,2,3,4$, in (6) we obtain the mean, the second moment, third, and the fourth moment for EED. We also obtained the Variance from the association

$$V(x) = \mu_2^{'} - \left(\mu_1^{'}\right)^2$$

The mean is given as

$$mean = \mu_1^{'} = \frac{1}{\lambda^2} \quad (7)$$

$$\mu_2^{'} = \frac{2}{\lambda^4}$$

$$Variance = V(x) = \frac{1}{\lambda^4} \quad (8)$$

The third and the fourth moment is given as

$$\mu_3^{'} = \frac{6}{\lambda^6} \quad (9)$$

$$\mu_4^{'} = \frac{24}{\lambda^8} \quad (10)$$

(ii) Coefficient of variation (C.V) is a standardized measure of the dispersion of a probability distribution and is given as;

$$C.V = \frac{\sigma}{\mu}$$

$$= \frac{\sqrt{\frac{1}{\lambda^4}}}{\frac{1}{\lambda^2}}$$

$$C.V = 1 \quad (11)$$

(iii) Moment generating function (M.G.F)

Theorem 3: If X is a continuous random variable distributed as an EED $(x;\lambda)$, then the moment generating function is defined as $M_x(t) = \frac{\lambda^2}{\lambda^2 - t}$

Proof:

$$\mathrm{M}_x(t) = E\left(e^{tx}\right) = \int_0^\infty e^{tx} f(x;\lambda)dx$$

$$M_x(t) = \int_0^\infty \lambda^2 e^{-\lambda^2 x} e^{tx} dx$$

$$= \lambda^2 \int_0^\infty e^{-x(\lambda^2 - t)} dx \quad (12)$$

Let $u = x(\lambda^2 - t),\ x = \frac{u}{(\lambda^2 - t)}$, then $dx = \frac{du}{(\lambda^2 - t)}$ so that (12) is reduced to

$$= \lambda^2 \int_0^\infty e^{-u} \frac{du}{(\lambda^2 - t)}$$

$$= \frac{\lambda^2}{(\lambda^2 - \mathrm{t})}\left[-e^{-\infty} - (-e^{-0})\right]$$

$$M_x(t) = \frac{\lambda^2}{(\lambda^2 - \mathrm{t})} \tag{13}$$

(iv) Characteristic function (C.F)

Theorem 5: If X is a random variable distributed as an EED $(x;\lambda)$, then the characteristics function $\phi_x(it)$ is defined as $\phi_x(it) = \frac{\lambda^2}{(\lambda^2 - it)}$

Proof:

$$\phi_x(it) = E\left(e^{itx}\right) = \int_0^\infty e^{itx} f(x;\lambda)dx$$

$$\phi_x(it) = \int_0^\infty \lambda^2 e^{-\lambda^2 x} e^{itx} dx$$

$$= \lambda^2 \int_0^\infty e^{-x(\lambda^2 - it)} dx \tag{14}$$

Let $u = x(\lambda^2 - it),\ x = \frac{u}{(\lambda^2 - it)}$, then $dx = \frac{du}{(\lambda^2 - it)}$ so that (14) is reduced to

$$= \lambda^2 \int_0^\infty e^{-u} \frac{du}{(\lambda^2 - it)}$$

$$\phi_x(it) = \frac{\lambda^2}{(\lambda^2 - i\,\mathrm{t})} \tag{15}$$

(v) Skewness and Kurtosis

Skewness is a measure of the asymmetry of the probability distribution of a real-valued random variable about its mean and is given as;

$$SK = \frac{E(x-\mu)^3}{\sigma^3} = \frac{\mu_3}{\sigma^3} = \frac{\mu_3' - 3\mu_1'\mu_2' + 2\mu_1'^3}{\sigma^3}$$

$$= \frac{\dfrac{\dfrac{6}{\lambda^6} - 3\left(\dfrac{1}{\lambda^2}\right)\left(\dfrac{2}{\lambda^4}\right) + 2\left(\dfrac{1}{\lambda^2}\right)^3}{[1]^{3/2}}}{\left[\lambda^4\right]^{3/2}}$$

$$= \frac{\dfrac{6}{\lambda^6} - 3\left(\dfrac{2}{\lambda^6}\right) + 2\left(\dfrac{1}{\lambda^6}\right)}{\left(\dfrac{1}{\lambda^4}\right)^{3/2}}$$

$$Sk = 2 \tag{16}$$

Kurtosis is a descriptor of the shape of a probability distribution and is given as;

$$K = \frac{E(x-\mu)^4}{\sigma^4} = \frac{\mu_4}{\sigma^4} = \frac{\mu_4' - 4\mu_1'\mu_3' + 6\mu_1'^2\mu_2' - 3\mu_1'^4}{\sigma^4}$$

$$= \frac{\dfrac{\dfrac{24}{\lambda^8} - 4\left(\dfrac{1}{\lambda^2}\right)\left(\dfrac{6}{\lambda^6}\right) + 6\left(\dfrac{\alpha}{\lambda^2}\right)^2\left(\dfrac{2}{\lambda^4}\right) - 3\left(\dfrac{1}{\lambda^2}\right)^4}{\left[\alpha\left(\alpha 2^\alpha - \alpha\lambda + 2^\alpha\right)\right]^2}}{\left[\dfrac{1}{\lambda}\right]^2}$$

$$= \frac{\dfrac{24}{\lambda^8} - \dfrac{24}{\lambda^8} + \dfrac{12}{\lambda^8} - \dfrac{3}{\lambda^8}}{\left[\dfrac{1}{\lambda^4}\right]^2}$$

$$K = 9 \tag{17}$$

(vi) Cumulative distribution function (CDF)
The cumulative distribution function of a random variable *X* evaluated at *x* is the probability that *X* will take a value less than or equal to *x* and is defined as;

$$F(x) = P\left(X \le x\right) = \int_0^x f(x)dx$$

Theorem 6: If X is a continuous random variable from the Exponential-Exponential distribution, the cumulative density function (CDF) is defined by $F(x) = 1 - e^{-\lambda^2 x},\ x, \lambda > 0$
Proof:

$$f(x) = \lambda^2 e^{-\lambda^2 x}, \qquad x, \alpha, \lambda > 0$$

$$F(x) = \int_0^x \lambda^2 e^{-\lambda^2 x} dx$$

$$= \lambda^2 \int_0^x e^{-\lambda^2 x} dx \tag{18}$$

let $u = -\lambda^2 x$, $x = \dfrac{u}{-\lambda^2}$, then $dx = \dfrac{du}{-\lambda^2}$ so that (18) is reduced to:

$$= -\frac{\lambda^2}{\lambda^2} \int_0^x e^u du$$

$$= -\int_0^x e^{-u} du = \left[-e^u\right]_0^x$$

recall that $u = -\lambda^2 x$ then,

$$= \left[-e^{-\lambda^2 x}\right]_0^x$$

applying the limits, then we have

$$= \left[-e^{-\lambda^2 x} - \left(-e^{-\lambda^2 (0)}\right)\right]$$

$$F(x) = 1 - e^{-\lambda^2 x}, \qquad \lambda > 0, \quad x > 0 \tag{19}$$

(vii) Survival function

The survival function is also known as the reliability function is a function that gives the probability that a patient, device, or other objects of interest will survive beyond any given specified time and is defined as; $S(x) = 1 - F(x)$ where; $F(x)$ is the cumulative distribution function of *x* then,

$$= 1 - \left(1 - e^{-\lambda^2 x}\right)$$

$$S(x) = e^{-\lambda^2 x} \qquad (20)$$

(viii) Hazard function

The **hazard function** also called the *force of mortality, instantaneous failure rate, instantaneous death rate*, or *age-specific failure rate* is the instantaneous risk that the event of interest happens, within a very narrow time frame and is defined as;

$$h(x) = \frac{f(x)}{S(x)}$$

Where $f(x)$ and $S(x)$ are pdf and survival function of EED then,

$$= \frac{\lambda^2 e^{-\lambda^2 x}}{e^{-\lambda^2 x}}$$

$$h(x) = \lambda^2 \qquad (21)$$

(ix) Cumulative hazard function

The cumulative hazard function is the integral of the hazard function. It can be interpreted as the probability of failure at time x given survival until time x and it can be defined as;

$$H(x) = W(F(x)) = -\log(1 - F(x)) \equiv \int_0^x h(x)dx$$

$$H(x) = \int_0^x \lambda^2 dx$$

$$= \lambda^2 \int_0^x dx \qquad (22)$$

$$H(x) = \lambda^2 x \qquad (23)$$

2.3 The Maximum Likelihood Estimator for Exponential-Exponential Distribution (EED)

Theorem 7: Let $\mathrm{X}_1, \mathrm{X}_2, ..., \mathrm{X}_n$ be a random sample of size n from Exponential-Exponential distribution(EED). Then the likelihood function is given by

$$\mathrm{L}(\alpha, \lambda; x) = \lambda^{2n} e^{-\lambda^2 \sum xi} \qquad (24)$$

By taking natural logarithm of (24), the log likelihood function is obtained as;

$$\frac{d}{d\lambda}\log_e = \frac{2n}{\lambda} - 2\lambda \sum xi = 0 \qquad (25)$$

differentiating (25) with respect to λ and equate it to zero(0) then we have

$$\frac{d}{d\lambda}\log_e = \frac{2n}{\lambda} - 2\lambda \sum xi = 0 \qquad (26)$$

make λ the subject of the formula from (26) above then,

$$\lambda^2 = \frac{n}{\sum xi}$$

Therefore, the MLE of which maximizes (26) is obtained as

$$\lambda = \left(\frac{n}{\sum xi} \right)^{1/2} \tag{27}$$

II. CONCLUSION

Numerous distributions have been developed, defined, and proved to be widely applied in the field of probability and statistics. In this study, we defined, studied, and established a new Exponential-Exponential distribution and derived various properties of the new distribution, including the first four moments, moment generating function, characteristics function cumulative distribution function, skewness, kurtosis, survival and hazard functions, and obtain its parameter estimate using the method of maximum likelihood estimation.

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