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Research Paper

A computational approach for nonlinear variable order fuzzy fractional partial differential equations

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Abstract
This researchwork employs the Bernstein spectral technique based on Bernstein polynomialstoanalyze and to obtain the approximate numerical solution of a class of variable order fuzzy partial differential equations (PDEs) and its some particular cases using the basic properties offuzzy theory. We analyze a variable order mathematical fuzzy model where the coefficients,unknown functions, initial and boundary conditions are some fuzzy numbers and fuzzy valuedfunctions. The variable order fuzzy operational matrix of Bernstein polynomials is derivedfor fuzzy fractional derivatives with respect to space and time where the fuzzy derivative istaken in Caputo sense. The Bernstein fuzzy operational matrix is applied to concerned nonlinearfuzzy space-time fractional variable order reaction–diffusion equations which reduceinto a system of non-linear fuzzy algebraic equations and can be deal with using the method given in the literature. To validate the high efficiency and capability of the proposed numericalscheme fewtest examples are reported with computation of the absolute error for the obtainednumerical solution.

Key words: *Fuzzy calculus, Fractional calculus, Variable order derivatives, Fuzzy partial differential equations,Diffusion equation*

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I. Introduction

The calculus theory inwhich the concept of any arbitrary order differentiation and integration is discussed that can be a generalization of classical calculus theory. This generalized calculustheory (fractional calculus) has diverse and widely spreads in applied mathematical sciences,engineering, fluid mechanics, electromagnetic, etc. This approach is increasingly applied tomathematical modeling of several complex physical phenomena viz., fluid flow, viscoelasticity,dynamical systems, control, groundwater contamination, transports of molecules via pores etc. Due to its wide application and feasibility fractional calculus seeks the attention of many researchers, scientists, engineers and applied mathematicians. The concepts ofgeneralizedcalculus theory were first introduced by Abel and Liouville. The diverse application of fractional calculus theory leads us to deal with the fractional differential and partial differential equations [1,2]To find the exact solution of fractional partial differential equations is a tough task as the analytical solution of many fractional partial differential equations does not exist. To overcome the lack of exact solution many researchers developed various techniques for the approximate analytical solution of fractional partial differential equations [3,4]. Few of these numerical techniques can also be used to find the numerical solution of integro-differential equations and integral equations. Many numerical techniques have been constructed to compute the approximate solution of fractional partial differential equations[5,6]. These numerical schemes are mainly based on transform techniques[7], homotopy perturbation method[8], Adomain decomposition scheme[9], etc. Various numerical techniques have been constructed using polynomials and wavelets with operational matrix for computation of approximate solution of fractional partial differential equations. These techniques basically based on the Chebyshev wavelet[10], Sin wavelet [11], Haar wavelet[12], Legendre wavelet[13]etc. Few operational matrices are developed on polynomials such as Genocchi polynomial[14], Chebyshev polynomial, Laguerre polynomial[15],Luca polynomial[16], Fibonacci polynomials, etc. The definition and property of fractional integration and differentiation is given by many researchers [17,18]. The variable order differentiation is a hot topic and has vast applications in many branches of physical sciences and engineering than fractional order differentiation of constant order, etc. [19,20,21]. The computation of approximate solution for a mathematical model containing variable order differentiation is very tough because thevariable order operator contains complex kernel terms for its power. In recent years, variableorder fractional derivative operators have been introduced.Variable order fractional calculushas been used in study and analysis of random order models. Fractional PDEs can be utilized in modeling many linear and non-linear physical processes. The fractional order systems have vast applications and utilization in the mathematical modeling of various phenomena but it fails to exact and efficient modeling of some complex realistic phenomena. The fractional systems in fuzzy environment is an useful tools to deal such types of complex realistic phenomena, this approach is constructed by Zadeh. The Fuzzy analytic theory has a very useful and significant part as fuzzy differential equations (DEs)[22]. The applications of fuzzy fractional differential equations fuzzy fractional PDEs are rapidly spread in last few years because of its wide presence in modeling the several physical industrial processes like mass and heat transfer, bio-mechanics, electromagnetic fields etc. Many researchers have been developed some numerical scheme for the approximate solutions of fractional fuzzy PDEs [23,24,25]. Although there are many numerical schemes are available but the field of fractional fuzzy PDEs yet to be tackled more accurately and needs some more efficient and valid numerical schemes.

In the present article, we have proposed an accurate and efficient numerical technique to solve variable order fractional reaction–diffusion equation arising in porous media as

$$\frac{\partial^{\eta(x.t)}\tilde{\zeta}(x.t)}{\partial t^{\eta(x.t)}} = \tilde{d}\frac{\partial^2\tilde{\zeta}(x.t)}{\partial x^2} + \tilde{\gamma}\zeta^a(\tilde{x}.t)\left(\frac{\partial\tilde{\zeta}(x.t)}{\partial x}\right)^b + \tilde{f}\left(\tilde{\zeta}(x.t)\right) + \tilde{h}(x.t), \quad (1)$$

subject to the following constraints:

where $0 \leq x, t \leq 1$, $\eta(x, t)$ denotes the fractional variable order of fuzzy derivative. The field variable $\zeta(x,t)$denotes a fuzzy values function w.r.to the crisp variables x, t.The fractiona order derivatives$\frac{\partial^{\eta(x,t)}\tilde{\zeta}(x,t)}{\partial t^{\eta(x,t)}}$ is considered w.r.to the Hukuhara derivatives .The constant coefficients viz., $\tilde{d}$and $\tilde{\gamma}$denotes some fixed fuzzy numbers. The unknown functions viz., $\tilde{f}\left(\tilde{\zeta}(x,t)\right)\tilde{h}_i'$sand known functions $\tilde{h}(x, t)$ represents some known fuzzy valued functions. In the present article, authors are going to construct a numerical technique to find the approximate solution of the concerned fuzzy mathematical model. The Bernstein spectral numerical method is an effective and efficient numerical technique over other knownschemes[26]. developed a method for the solution of a system of fuzzy algebraic equations. This idea of solving the system of equations has been used in the present numerical technique. The accuracy and effectiveness of the numerical technique is justified with the help of few test examples. Here we consider reaction–diffusion model in fuzzyenvironment.Those models in which parameters does not have a particular value but it havea range of values, for those models we use fuzzy approach as its parameter values are overa range. Using the constructed numerical scheme in this article, we will able to deal thesetype of fuzzy models. There are also some difficulties due to fuzzy logic as it increase thecomplexity by increasing the number of predictors. There is also some loss of informationbecause of the date discretisation.This paper is organized as: basic concepts and properties of fuzzy set theory, fuzzy fractional derivatives in Caputo sense is discussed in Sect. 2. Section 3 contains definitions and fundamental results of Bernstein polynomials. The functions approximation is given in the next Sect. 4. Derivation of novel fuzzy operational matrix for fractional order and proposed algorithm of the present scheme are discussed in Sects. 5. The last one is the conclusion section.

II. Preliminaries and notations

This part of the work contains some basic introduction and properties of fuzzy calculus. Definitions of fractional fuzzy derivatives and integrals are given in Caputo sense an Riemann–Liouville sense which can be further used in the manuscript

2.1. Fuzzy calculus

In [27] his work has provided the concept to overcome uncertainty due to vagueness and imprecision. Let Z a non-empty set and element of the set Z is related to some membership grade, Z is a base set. Any non-empty collection of elements of Z ×[0, 1] is named as fuzzy subset of Z [27]. Let ξ be a mapping such that $\xi: Z \rightarrow [0, 1]$ then a fuzzy set F is a subset of $\{(b, \xi(b)) : b \in Z\}$.

Fuzzy Number: A mapping$\tilde{f}$: R → [0, 1]satisfying the conditions as given below is called a fuzzy number. These conditions are
(a) $\tilde{f}$satisfies the upper semi continuity criteria.
(b) $\tilde{f}(ab_1+ (1 -a)b_2) \geq \min\{ \tilde{f}(b_1), \tilde{f}(b_2)\}$,where $0 \leq a \leq 1$ and a_1,a_2 are real numbers.

(c) $\tilde{f}(b_0) = 1$ for some $b_0 \in R$.

(c) Closure of support of the mapping $\tilde{f}$should be a compact set.
v-**Level set of fuzzy number:**Let fuzzy numbers on R is R_F , then v-level set of fuzzy number $[\tilde{f}_v]$ for all $0 \leq v \leq 1$ is given by

$$[\tilde{f}] = \begin{cases} \{b \in \mathbb{R}: \tilde{f}(b) \leq v\}, 0 < v \leq 1 \\ \text{closure } (supp(\tilde{f})), v = 0. \end{cases} \tag{3}$$

Here, it is clear that v−level set [$\tilde{f}$] is a bounded and closed set. The v-level fuzzy interval can be written as [$\tilde{f}$] = [$\tilde{f}-(v)$, $\tilde{f}+(v)$].
Parametric interval form: The representation of parametric interval form for fuzzy number $\tilde{f}$is $\tilde{f}[v]$ = [$\tilde{f}-(v)$, $\tilde{f}+(v)$], $0 \leq v \leq 1$. Following properties are satisfied by parametric interval form

(a) $\tilde{f}+(v)$ is greater than or equal to $\tilde{f}-(v)$, for every $0 \leq v \leq 1$.
(b) $\tilde{f}-(v)$ is left continuous and nondecreasing function.
(c) $\tilde{f}+(v)$ is left continuous and nonincreasing function.

The basic binary operations in fuzzy environment for two fuzzy numbers are given by the following equations:

$$\begin{aligned} (\tilde{f} \oplus \tilde{g})(v) &= [\tilde{f}_- + \tilde{g}_-(v), \tilde{f}_+ + \tilde{f}_+(v)], \\ (a \odot \tilde{f})(v) &= \begin{cases} [a\tilde{f}_-(v), a\tilde{f}_+(v)], a \geq 0, \\ [\tilde{f}_+(v), a\tilde{f}_-(v)], a < 0. \end{cases} \end{aligned} \tag{4}$$

gH-Difference: The gH-Difference (Generalized Hukuhara Difference) of two non-void compact sets S and T as another compact set U is given by the following expression:

$$S \ominus_{gH} T = U \leftrightarrow \begin{cases} (a) S = T + U \\ \text{or } (b) T = S - U. \end{cases} \tag{5}$$

gH-Derivatives: A function ξ : (c, d) → R_F is H-Differentiable at $c_0 \in$ (c, d) if

(a) Case 1 (1-Differentiation): Let the H-Difference of $\xi(c_0 + \epsilon \ominus \xi(c_0))$ and $\xi(c_0) \ominus \xi(c_0 - \epsilon)$ exist then we can write the derivative in following form:

$$\xi'(c_0) = \lim_{\epsilon \to 0^+} \frac{\xi(c_0 + \epsilon \ominus \zeta(c_0))}{\epsilon} = \lim_{\epsilon \to 0^+} \frac{\xi(c_0) \ominus \xi(c_0 - \epsilon)}{\epsilon}. \tag{6}$$

b) **Case 2 (2-Differentiation):** Let the H-Difference of $\xi(c_0 \ominus \xi(c_0 + \epsilon))$ and $\xi(c_0 - \epsilon) \ominus \xi(c_0)$ exist then we can write derivative in the following form:

$\xi^{'}(c_0)$

The another form of the gH-Derivative written in the form of following equation.

$$\xi^{'}(c_0) = \lim_{\epsilon\to 0}\frac{\xi\left(c_0+\epsilon\Theta_{gH}\,\xi(c_0)\right)}{\epsilon}. \quad (8)$$

The Lebesgue integration of ξ (c) in form of fuzzy parametric interval form is

$$\left[\int_0^t \xi^{'}(c)\mathrm{d}c\right]_v = \int_0^t [\xi^{'}(c)]_v \mathrm{d}c = \begin{cases}\left[\int_0^t \xi_-^{'}(c;v)\mathrm{d}c, \int_0^t \xi_+^{'}(c;v)\mathrm{d}c\right], \text{for case-1,}\\ \left[\int_0^t \xi_-^{'}(c;v)\mathrm{d}c, \int_0^t \xi_+^{'}(c;v)\mathrm{d}c\right], \text{for case-2}\end{cases} \quad (9)$$

This equation is useful in the establishment of Caputo derivative and RL derivative.

2.2. Fractional fuzzy differentiation

The fractional fuzzy differentiation is the generalization of ordinary fractional derivatives taken in the crisp sense.

Caputo Fractional g-Derivative: The Caputo fractional g-Derivatives of a measurable continuous fuzzy function ξ (t) can be written as

$${}^{g}D_{a^+}^{\mu}\xi(t) = \lim_{\epsilon\to 0}\frac{K(t+\epsilon)\ominus_g K(t)}{\epsilon}. \quad (10)$$

The function K(t) in the above equation is expressed by

$$K(t) = \frac{1}{\Gamma(-\mu+1)}\int_a^t (-p+t)^{-\mu}\xi(p)dp. \quad (11)$$

Now we consider ξ(t) as an absolute continuous function so the fuzzy fractional differentiation in Caputo sense is given by

$$\begin{aligned}\left[\ {}_{a^+}^{C}D_t^{i,\mu}\xi(t)\right] &= \left[\ {}_{a^+}^{C}D_t^{i,\mu}\xi_-(t;v),\quad {}_{a^+}^{C}D_t^{i,\mu}\xi_+(t;v)\right] \text{ Caso-1,}\\ \left[\ {}_{a^+}^{C}D_t^{ii,\mu}\xi(t)\right] &= \left[\ {}_{a^+}^{C}D_t^{ii,\mu}\xi_+(t;v),\quad {}_{a^+}^{C}D_t^{ii,\mu}\xi_-(t;v)\right] \text{ Case-2.}\end{aligned} \quad (12)$$

The terms ${}_{a}^{C} + D_t^{i,u}\xi - (t;v)$, ${}_{a^+}^{C}D_t^{i,\mathrm{u}}\xi + (t;v)$, ${}_{a^+}^{C}D_t^{ii.u}\xi - (t;v)$and ${}_{a^+}^{C}D_t^{i,\mathrm{i},\mathrm{u}}\xi + (t;v)$ are expressed as

$$\begin{aligned}{}_{a}^{c}+D_t^{i,\mu}\xi_-(t;v) &= \frac{1}{\Gamma(-\mu+1)}\int_a^t (-p+t)^{-\mu}\xi_-^{'}(p)\mathrm{d}p.\\ {}_{a}^{c}+D_t^{i,\mu}\xi_+(t;v) &= \frac{1}{\Gamma(-\mu+1)}\int_a^t (-p+t)^{-\mu}\xi_+^{'}(p)\mathrm{d}p.\\ {}_{a}^{C}+D_t^{ii,\mu}\xi_+(t;v) &= \frac{1}{\Gamma(-\mu+1)}\int_a^t (-p+t)^{-\mu}\xi_+^{'}(p)\mathrm{d}p.\\ {}_{a}^{C}+D_t^{ii,\mu}\xi_-(t;v) &= \frac{1}{\Gamma(-\mu+1)}\int_a^t (-p+t)^{-\mu}\xi_-^{'}(p)\mathrm{d}p.\end{aligned} \quad (13)$$

Fuzzy differentiation of variable order: Let ξ (t) be an absolute continuous function so the fuzzy fractional differentiation of variable order in Caputo sense is given by

$$\left[\ {}_{a^+}^{C}D_t^{i,\eta(x,t)}\xi(t)\right] = \left[\ {}_{a^+}^{C}D_t^{i,\eta(x,t)}\xi_-(t;v),\quad {}_{a}^{C}D_t^{i,\eta(x,t)}\xi_+(t;v)\right] \text{ Case} - 1, \quad (14)$$

=

The terms ${}_{a^+}^{C}D_t^{i,\eta(x,t)}\xi_-(t;v)$, ${}_{a^+}^{C}D_t^{i,\eta(x,t)}\xi_+(t;v)$, ${}_{a^+}^{C}D_t^{ii,\eta(x,t)}\xi_-(t;v)$ *and* ${}_{a^+}^{C}D_t^{ii,\eta(x,t)}\xi_+(t;v)$are expressed as

$$\begin{aligned} C_{a^+}D_t^{i,\eta(x,t)}\xi_-(t;v) &= \frac{1}{\Gamma(-\eta(x,t)+g)}\int_a^t (-p+t)^{-\eta(x,t)+g-1}\xi_-^{(g)}(p)\mathrm{d}p, \\ C_{a^+}D_t^{i,\eta(x,t)}\xi_+(t;v) &= \frac{1}{\Gamma(-\eta(x,t)+g)}\int_a^t (-p+t)^{-\eta(x,t)+g-1}\xi_+^{(g)}(p)\mathrm{d}p, \\ C_{a^+}D_t^{ii,\eta(x,t)}\xi_+(t;v) &= \frac{1}{\Gamma(-\eta(x,t)+g)}\int_a^t (-p+t)^{-\eta(x,t)+g-1}\xi_+^{(g)}(p)\mathrm{d}p, \\ C_{a^+}D_t^{ii,\eta(x,t)}\xi_-(t;v) &= \frac{1}{\Gamma(-\eta(x,t)+g)}\int_a^t (-p+t)^{-\eta(x,t)+g-1}\xi_-^{(g)}(p)\mathrm{d}p. \end{aligned} \quad (15)$$

Here g is an integer such that $g-1<\eta(x,t)<g$.

3. Bernstein Polynomial

The Bernstein polynomial has been useful in many aspects of science [28,29]. The Bernstein polynomial of degree l in [0,1] is defined by

$$B_{p,l}(x)=\binom{l}{p}x^p(1-x)^{l-p}, 0\le p\le l. \quad (16)$$

As $0 \le x \le 1$, we can use Binomial expansion in the above equation, we have

$$B_{p,l}(x)=\binom{l}{p}x^p\left(\sum_{s=0}^{l-p}(-1)^s\binom{l-p}{s}x^s\right), 0\le p\le l, \quad (17)$$

Or,

$$B_{p,l}(x)=\sum_{s=0}^{l-p}(-1)^s\binom{l}{p}\binom{l-p}{s}x^{p+s}, 0\le p\le l, \quad (18)$$

The Bernstein polynomial can be written in following form:

$$\vartheta(x)=MP_l(x). \quad (19)$$

The term$\vartheta(x)=\left[B_{0,l}(x),B_{1,l}(x),\dots,B_{l,l}(x)\right]^{\mathrm{T}}$, and $P_l(x)=[1,x,x^2,\dots,x^l]^{\mathrm{T}}$. M is a matrix of following type.

$$M=\begin{bmatrix} (-1)^0\binom{l}{0} & (-1)^1\binom{l}{0}\binom{l-0}{1-0} & \dots & \dots & \dots & (-1)^{m-0}\binom{l}{0}\binom{l-0}{1-0} \\ 0 & (-1)^0\binom{l}{1} & \dots & \dots & \dots & (-1)^{m-1}\binom{l}{1}\binom{l-1}{l-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & (-1)^0\binom{l}{i} & \dots & (-1)^{m-i}\binom{l}{i}\binom{l-i}{l-i} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & \dots & (-1)^0\binom{l}{l} \end{bmatrix} \quad (20)$$

The above expression of M shows that M is non-singular matrix i.e., $|M|\neq 0$. Few properties of Bernstein polynomials are given as:

(i) $B_{p,l}(x)\ge 0$, $\forall x\in[0,1]$, i.e., Bernstein polynomials are always positive in their domain.

And (iii)$B'_{p,l}(x)=l\left(B_{p-1,l-1}(x)-B_{p,l-1}(x)\right)$.(ii) $B_{p,l}(1-x)=B_{l-p,l}(x)$

The considered Bernstein polynomials is used in the definition of Bezier surface and curves. These polynomials are fundamental to the theory of approximation because they are useful in the proof ofWeierstrass approximation theorem.We are going to use this polynomial in the approximation of the unknown functions because of its good accuracy and efficiency for the considered mathematical model.

4. Function approximation

The collection of Bernstein polynomial is a complete basis for the Hilbert space $L^2[0,1]$so every $\xi(x)\in L^2[0,1]$is written as

$$\xi(x) \simeq \xi_q(x) = \sum_{e=0}^{q} b_e B_{e,q}(x) = B^{\mathrm{T}} \cdot \chi(x). \quad (21)$$

The term$B^{\mathrm{T}} = [b_e]$is a matrix called as Bernstein coefficients matrix. The function $\xi(x,t) \in L^2[0,1]$is also written as

$$\xi(x,t) \simeq \xi_q(x,t) = \sum_{e=0}^{q} \sum_{f=0}^{q} b_{e,f} B_{e,q}(x) B_{f,q}(t) = \chi(x)^{\mathrm{T}} \cdot B \cdot \chi(t). \qquad (22)$$

The term $B = [b_e, f]$is a matrix called as Bernstein coefficients matrix.

4.1. Fuzzy function approximation

In this section we write a continuous, measurable fuzzy function in form of Bernstein polynomial as

$$\tilde{\xi}(x) \simeq \tilde{\xi}_q(x) = \sum_{e=0}^{q} \tilde{b}_e \odot B_{e,q}(x) = \tilde{B}^{\mathrm{T}} \odot \chi(x). \qquad (23)$$

The term $\tilde{B}^{\mathrm{T}} = \left[\tilde{b}_g\right]$is a matrix called as Bernstein coefficients matrix. The function $\tilde{\xi}(x,t) \in L^2[0,1]$is also written as

$$\tilde{\xi}(x,t) \simeq \tilde{\xi}_q(x,t) = \sum_{e=0}^{q} \sum_{f=0}^{q} \tilde{b}_{e,f} \odot B_{e,q}(x) \odot B_{f,q}(t) = \chi(x)^{\mathrm{T}} \odot \tilde{B} \odot \chi(t). \quad (24)$$

The term $\tilde{B} = \left[\tilde{b}_{e,f}\right]$is a matrix called as Bernstein coefficients matrix. All the binary operations, viz., multiplication, addition are taken in fuzzy environment denoted by $\odot$and$\oplus$ respectively.

5. Operational matrix

Here we will construct the variable order operational matrix. In view of the Eq. (19), the differentiation of χ(t) of order η(x, t) can be written as

$$\frac{\partial^{\eta(x,t)}\chi(t)}{\partial t^{\eta(x,t)}} = \frac{\partial^{\eta(x,t)} M \cdot P_q(t)}{\partial t^{\eta(x,t)}} = M \cdot \frac{\partial^{\eta(x,t)}}{\partial t^{\eta(x,t)}} \begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^{g-1} \\ t^g \\ \vdots \\ t^q \end{bmatrix}, \qquad (25)$$

where g − 1 ≤ η(x, t) ≤ g, we take $g = \lceil \eta(x,t) \rceil$and g <q.

$$\frac{\partial^{\eta(x,t)}\chi(t)}{\partial t^{\eta(x,t)}} = M \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \frac{\Gamma(g+1) \cdot t^{g-\eta(x,t)}}{\Gamma(g+1-\eta(x,t))} \\ \vdots \\ \frac{\Gamma(q+1) \cdot t^{q-\eta(x,t)}}{\Gamma(q+1-\eta(x,t))} \end{bmatrix}. \qquad (26)$$

In matrix form, it can also be represented as

$$\frac{\partial^{\eta(x,t)}\chi(t)}{\partial t^{\eta(x,t)}}=M.\begin{bmatrix} 0 & 0 & 0 & . & . & . & 0 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & 0 & . & . & . & 0 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & 0 & . & . & . & 0 & 0 & . & . & . & 0 & 0 \\ . & . & . & . & . & & . & . & & & . & . & \\ . & . & . & . & . & & . & . & & & . & . & \\ . & . & . & . & . & & . & . & & & . & . & \\ 0 & 0 & 0 & . & . & . & 0 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & 0 & . & . & . & 0 & \frac{\Gamma(g+1)\cdot t^{-\eta(x,t)}}{\Gamma(g+1-\eta(x,t))} & . & . & . & 0 & 0 \\ . & . & . & . & & & . & . & & & . & . & . \\ . & . & . & . & & & . & . & & & . & . & . \\ . & . & . & . & & & . & . & & & . & . & . \\ 0 & 0 & 0 & . & . & . & 0 & 0 & & & & 0 & \frac{\Gamma(q+1)\cdot t^{-\eta(x,t)}}{\Gamma(q+1-\eta(x,t))} \end{bmatrix}\cdot\begin{vmatrix} 1 \\ t \\ t^2 \\ . \\ . \\ . \\ t^{g-1} \\ t^g \\ . \\ . \\ . \\ t^q \end{vmatrix} \tag{27}$$

This expression can be rewritten as

$$\frac{\partial^{\eta(x,t)}\chi(t)}{\partial t^{\eta(x,t)}} = M \cdot \Omega \cdot P_q(t). \tag{28}$$

The expression for Ω is given by

$$\Omega = [a_{ab}]_{(q+1)\times(q+1)} = \begin{cases} 0, \text{ elsewhere}, \\ \frac{\Gamma(g+1).t^{-\eta(x,t)}}{\Gamma(g+1-\eta(x,t))}, \text{ when } a = b \geq g. \end{cases} \tag{29}$$

With the help of Eqs. (19) and(28), we have

$$\frac{\partial^{\eta(x,t)}\chi(t)}{\partial t^{\eta(x,t)}} = M \cdot \Omega \cdot M^{-1} \cdot \chi(t). \tag{30}$$

The term $M \cdot \Omega \cdot M^{-1}$ is called as operational matrix of fractional order differentiation with respect to temporal parameter. Using the same numerical approach we can construct the operational matrix of fractional order differentiation with respect to spatial parameter. To find the numerical solution of our considered mathematical model (1) we will collocate this model with given constraints (2). After this process we will get nonlinear systems of equations and on solving this system we get the unknown constant coefficients matrix B. Using the constant coefficients matrix in the Eq. (24) we will get numerical solution of our considered mathematical model.

6. Proposed algorithms

In this section of the article, we investigate the concerned fuzzy model under the environment of fuzzy calculus theory. Our variable order fractional fuzzy advection–diffusion equation is

$$\frac{\partial^{\eta(x,t)}\tilde{\zeta}}{\partial t^{\eta(x,t)}} = \tilde{d}\frac{\partial^2\tilde{\zeta}}{\partial x^2} + \tilde{\gamma}\tilde{\zeta^a}\left(\frac{\partial\tilde{\zeta}}{\partial x}\right)^b + \tilde{f}(\tilde{\zeta}(x,t)) + \tilde{h}(x,t), \tag{31}$$

under the given constraints

$$\tilde{\zeta}(0,t) = \tilde{h_1}(t), \tilde{\zeta}(x,0) = \tilde{h_2}(x), \tilde{\zeta}(1,t) = \tilde{h}_3(t). \tag{32}$$

After the fuzzyfication above equation can be written for $v \in [0, 1]$ as

$$\begin{aligned}
&[\tilde{\zeta}(x,t)]^v = [\zeta_-(x,t,v), \zeta_+(x,t,v)],\\
&\left[\frac{\partial^{\eta(x,t)}\tilde{\zeta}(x,t)}{\partial t^{\eta(x,t)}}\right]^v = \left[\frac{\partial^{\eta(x,t)}\zeta_-(x,t,v)}{\partial t^{\eta(x,t)}}, \frac{\partial^{\eta(x,t)}\zeta_+(x,t,v)}{\partial t^{\eta(x,t)}}\right],\\
&\left[\frac{\partial^2\tilde{\zeta}(x,t)}{\partial x^2}\right]^v = \left[\frac{\partial^2\zeta_-(x,t,v)}{\partial x^2}, \frac{\partial^2\zeta_+(x,t,v)}{\partial x^2}\right],\\
&\left[\frac{\partial\tilde{\zeta}(x,t)}{\partial x}\right]^v = \left[\frac{\partial\zeta_-(x,t,v)}{\partial x}, \frac{\partial\zeta_+(x,t,v)}{\partial x}\right],\\
&[\tilde{h}(x,t)]^v = [h_-(x,t,v), h_+(x,t,v)].
\end{aligned} \tag{33}$$

Now we can rewrite concerned model in upper and lower approximations as

$$\begin{aligned}
\frac{\partial^{\eta(x,t)}\zeta_+(x,t,v)}{\partial t^{\eta(x,t)}} = d_+\frac{\partial^2\zeta_+(x,t,v)}{\partial x^2} + \gamma_+\zeta_+^a(x,t,v)\left(\frac{\partial\zeta_+(x,t,v)}{\partial x}\right)^b\\
+f_+\big(\zeta_+(x,t,v)\big) + h_+(x,t,v),
\end{aligned} \tag{34}$$

under given constraints

$$\zeta_+(0,t) = h_{1_+}(x), \zeta_+(x,0) = h_{2_+}(t), \zeta_+(1,t) = h_{3_+}(t). \tag{35}$$

and

$$\begin{aligned}
\frac{\partial^{\eta(x,t)}\zeta_-(x,t,v)}{\partial t^{\eta(x,t)}} = d_-\frac{\partial^2\zeta_-(x,t,v)}{\partial x^2} + \gamma_-\zeta_-^a(x,t,v)\left(\frac{\partial\zeta_-(x,t,v)}{\partial x}\right)^b\\
+f_-\big(\zeta_-(x,t,v)\big) + h_-(x,t,v)
\end{aligned} \tag{36}$$

under given constraints

$$\zeta_-(0,t) = h_{1_-}(x), \zeta_-(x,0) = h_{2_-}(t), \zeta_-(1,t) = h_{3_-}(t) \tag{37}$$

On solving above equation we can find the numerical solution of this non-linear fuzzy PDE for both upper and lower approximations.

VII. Conclusion

In this paper, the cluster of Bernstein polynomials is utilized for the development of a numerical technique to find the approximate analytical solution of non-linear variable orderfractional fuzzy partial differential equations. The theory of fuzzy calculus has been discussed and we approximated the fuzzy valued function in terms of Bernstein polynomials. The fuzzy operational matrix is developed and with the help of this matrix, we analyze the space-time fractional nonlinear fractional fuzzy reaction–advection–diffusion model with respect to variable order derivative for the first time. The numerical solution which is very close to exact solution is obtained after the fuzzification concerned model with proper crisp points. The proposed method can be utilized to investigate the behavior of system of fractional fuzzy PDEs.

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