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Research Paper

On extended Salem's criterion for the zeta function

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ABSTRACT: *Based on the results of Salem [1], in this paper we derive an equivalent of the Riemann hypothesis (RH) using the theory of k-special functions.*
KEYWORDS: *k-special functions, k-gamma function, Salem's criterion.*

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I. INTRODUCTION

Salem in his paper [1] proved that for RH to be true it is necessary and sufficient that for $0<\sigma<1$, the following integral, which is a function of x

$$\int_{-\infty}^{+\infty}\frac{e^{-\sigma y}\varphi(y)}{e^{e^{x-y}}+1}dy=0$$

among all possible bounded and measurable solutions has only the trivial solution $\varphi(y)\equiv 0$. This is now known as the Salem's criterion. Based on the above result, in this paper we are going to construct similar but modified equivalence of RH from the theory of $k-$special functions. The theory of $k-$special functions was first introduced by Diaz and Pariguan [3] when they saw the repeated appearance of expressions of the form $x(x+k)(x+2k)...(x+(n-1)k)$ in a variety of contexts, such as, the combinatorics of creation and annihilation operators and perturbative computation of Feynman integrals [4]. Based on the above expression, they constructed the corresponding Pochhammer symbol which they called Pochhammer $k-$symbol. And from it, they constructed the corresponding gamma function which is now known as the $k-$gamma function. For $\Re s>0$ and $k>0$, the $k-$gamma function is defined as

$$\Gamma_k(s)=\int_0^{\infty}x^{s-1}e^{-\frac{x^k}{k}}dx. \tag{1}$$

The $k-$gamma function is related to the ordinary gamma function as $\Gamma_k(x)=k^{\frac{x}{k}-1}\Gamma\left(\frac{x}{k}\right)$.

II. EXTENDED SALEM'S CRITERION

Let $\sigma>0$. Using the integral representation of $k-$gamma function, for $n>0$ we have

$$\frac{\Gamma_k(s)}{n^{\frac{s}{k}}}=\int_0^{\infty}x^{s-1}e^{-\frac{x^k}{k}n}dx \tag{2}$$

Since $0\le e^{-\frac{x^k}{k}}-e^{-\frac{2x^k}{k}}+...+(-1)^{n+1}e^{-\frac{nx^k}{k}}\le e^{-\frac{x^k}{k}}$ and $x^{\sigma-1}e^{-\frac{x^k}{k}}$ is Lebesgue integrable on $(0,\infty)$, we have

$$\Gamma_k(s)\left(1-2^{1-\frac{s}{k}}\right)\zeta\left(\frac{s}{k}\right)=\Gamma_k(s)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^{\frac{s}{k}}}=\sum_{n=1}^{\infty}(-1)^{n+1}\int_0^{\infty}x^{s-1}e^{-\frac{x^k}{k}n}dx$$
$$=\int_0^{\infty}x^{s-1}\left(\sum_{n=1}^{\infty}(-1)^{n+1}e^{-\frac{x^k}{k}n}\right)dx. \tag{3}$$

Using Lebesgue's dominated convergence theorem, we get

$$\Gamma_k(s)\left(1-2^{1-\frac{s}{k}}\right)\zeta\left(\frac{s}{k}\right)=\int_0^{\infty}\frac{x^{s-1}}{e^{\frac{x^k}{k}}+1}dx=\int_0^{\infty}\frac{x^{\sigma-1}x^{it}}{e^{\frac{x^k}{k}}+1}dx=\int_{-\infty}^{\infty}\frac{e^{u\sigma}e^{iut}}{e^{\frac{e^{uk}}{k}}+1}du \tag{4}$$

where in the last step we have substituted $x=e^u$. Notice that the last integral is Fourier transform of the function

$$\psi_{\sigma,k}(u)=\frac{e^{u\sigma}}{e^{\frac{e^{uk}}{k}}+1} \tag{5}$$

$\psi_{\sigma,k}(u)$ is Lebesgue integrable on $\mathbb{R}$.

Let $c,\lambda\in\mathbb{R}$ and $f(x),h\in L_1(\mathbb{R})$ such that the holomorphic Fourier transform

$$\hat{H}f(u):=\int_{-\infty}^{+\infty}f(x)e^{ixu}dx=0 \tag{6}$$

and

$$\hat{H}h(u):=\int_{-\infty}^{+\infty}h(x)e^{ixu}dx\neq 0. \tag{7}$$

Define the translate $f_\lambda(x)=f(x+\lambda)$ which implies $\hat{H}f_\lambda(x)=e^{-i\lambda x}\hat{H}f(x)$. Thus, for any $\tilde{f}\in L_1\mathbb{R}$ for the form $\tilde{f}(x)=\sum_{n=1}^{N}a_n f(x+\lambda_n)$ for $a_n\in\mathbb{R}$ and $1\le n\le N$ we have $\hat{H}g=0$ with

$$\left|\int_{-\infty}^{+\infty}\left(\tilde{f}(x)-h(x)\right)e^{iux}dx\right|=\left|\hat{H}h(u)\right|\ge 0 \tag{8}$$

which implies that $\hat{H}\tilde{f}(u)$ is not in $L_1(\mathbb{R})$ closure of the sums of them from $\tilde{f}$. Thus, if $f\in L_1(\mathbb{R})$ then the linear span of the set translations of $f(x)$ is dense in $L_1(\mathbb{R})$ if and only if the holomorphic Fourier transform of f has no real zeros. This is also known as Wiener's theorem [5].

Fixing σ in Eqn. (5) and letting $\zeta(\sigma+it)\neq 0\forall t$, the Wiener's theorem implies that translates of $\psi_{\sigma,k}(u)$ are dense in $L_1(\mathbb{R})$. Now let RH to be true and $\vartheta(x)$ be a bounded and measurable function, $\vartheta(x)\le M\forall x\in\mathbb{R}$, and $\epsilon>0$ be given. Assuming that for all bounded measurable $\phi(y)$ we have

$$\int_{-\infty}^{+\infty}\psi_{\sigma,k}(x-y)\phi(y)dy=0. \tag{9}$$

Let $\vartheta_n^+(x):=\max\left(0,f(x)\right)\cdot\chi_{[-n,n]}$ and $\vartheta_n^-(x):=\min\left(0,f(x)\right)\cdot\chi_{[-n,n]}$ which implies that on $[-n,n]$ we have $\vartheta(x)=\vartheta_n^+(x)+\vartheta_n^-(x)$ and $\left|\vartheta_n^+(x)\right|,\left|\vartheta_n^-(x)\right|\le M$. We can write finite linear combination of translates of $\psi_{\sigma,k}(u)$ which would imply that $\vartheta(x)$ vanishes almost everywhere.

We know from Hahn–Banach theorem \cite{6} that for a normed space $\left(N,\|\cdot\|\right)$, subspace $N^{'} \subset M$ and a linear functional $F: N^{'} \to \mathbb{R}$ bounded on $N^{'}$ with norm $\|F\|_{N^{'}}$, the Fourier transform of $\psi_{\sigma,k}(x)$ vanishes at $t=\gamma$, there exists an another bounded linear functional $G: N^{'} \to \mathbb{R}$ which agrees with F on $N^{'}$ which satisfies $\|G\|_{N^{'}} = \|F\|_{N^{'}}$. Using this result, we can state that if $N^{'} \subset N$ and $\varphi \in M/N$ Then there is a bounded linear functional F on N such that $F(f)=0 \forall f \in N^{'}$ and $F(\varphi)=1$. Note that the f defined here is different from the f defined in previous paragraph. Now, let RH be false. Let $\rho \in \mathbb{C}$ such that $\zeta(\rho)=\zeta(\sigma+i\gamma)=0$ with $\sigma > \frac{1}{2}$. Since the Fourier transform of $\psi_{\sigma,k}(x)$ vanishes at $t=\gamma$, the linear span of the translates of $\psi_{\sigma,k}(x)$ is not dense in $L_1\mathbb{R}$. Let C be the closure of this set and let $\varphi \in L_1\mathbb{R}/C$, then there exists a linear functional $T: L_1\mathbb{R} \to \mathbb{R}$ such that $T(l)=0 \forall l \in C$ and $T(\varphi)=1$. Using Radon–Nikodym theorem [6], there is a $m \in L_\infty(\mathbb{R})$ such that m is measurable and bounded almost everywhere such that . Since $T(\varphi)=1$, we must have $m(x)$ non-zero on a set of positive measure. Hence
$\forall x$,

$$T(l)=\int_{-\infty}^{+\infty} \psi_{\sigma,k}(x-y)m(y)dy=0 \tag{10}$$

where $m(x)\neq 0$ almost everywhere. Thus, we can state that for RH to be true it is necessary and sufficient that for $0<\sigma<1$, the integral

$$\int_{-\infty}^{+\infty} \frac{e^{-\sigma y}m(y)}{e^{\frac{e^{(x-y)k}}{k}}+1}dy=0 \tag{11}$$

among all possible bounded and measurable solutions has only the trivial solution $m(y)\equiv 0$. This is the extended Salem's criterion.

REFERENCES

[1]. R. Salem, Sur une proposition equivalente alhypoth ese de Riemann. C. R. Acad. Sci. Paris 236 (1953), 127–128.

[2]. N. Levinson, On closure problems and zeros of the Riemann zeta function, Proc. Amer. Math. Soc. 7 (1956), 838–845.

[3]. R. Diaz, E. Pariguan, On hypergeometric functions and Pochhammer k-symbol. Divulgaciones Matematicas. Vol.15. No.2 (2007), pp. 179-192.

[4]. P. Deligne, P. Etingof, D. Freed, L. Jeffrey, D. Kazhdan, J. Morgan, D. Morrison, E. Witten, Quantum fields and strings: A course for mathematicians, American Mathematical Society, 1999.
N. Wiener, Tauberian theorems, Ann Math. 33 (1932), 1–100.

[5]. Equivalents of the Riemann Hypothesis: Volume 2, Analytic Equivalents, Kevin Broughan.