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Research Paper

Interesting Facts About Triangular Numbers

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Abstract
The n-th triangular number, t_n, equals 1 + 2 + 3 + ... + n.Gauss showed that every positive integer can be expressed as the sum of three or fewer triangular numbers. Positive integers that are sums of two triangular numbers are characterized. Numbers that can be written in two different ways as the sum of two triangular numbers are considered. Additional known properties are presented.

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I. Introduction

The n-th *triangular number*,t_n, equals 1 + 2 + 3 + ... + n. See [1,2]. The closed form is given by

$$\boxed{t_n = \frac{n(n+1)}{2}} \tag{1}$$

Triangular numbers were of interest to ancient Greek mathematicians such as Pythagoras and Euclid.There are many beautiful identities involving them, requiring nothing more than high school algebra. Triangular numbers can be depicted as triangular arrays of dots.

A given number m is triangular if and only if 1 + 8m is a perfect square. Furthermore, the greatest triangular number less than or equal to m is $\left\lfloor \frac{-1+\sqrt{1+8m}}{2} \right\rfloor$.

The identity $\boxed{t_n + t_{n-1} = n^2}$ (2)

will be useful. It follows directly from (1), that is, $t_n + t_{n-1} = \frac{n(n+1)}{2} + \frac{(n-1)n}{2} = \frac{2n^2}{2} = n^2$. The geometric proof cuts a square array of n^2 dots with a line just over the main diagonal of dots, thereby partitioning the dots into two right triangular configurations that represent the two consecutive triangular numbers, t_n and t_{n-1}.

Theorem: $\left\lfloor \frac{t_{n-1} + t_{n+1}}{2} \right\rfloor = t_n$

Proof: $t_{n-1} + t_{n+1} = \frac{(n-1)n}{2} + \frac{(n+1)(n+2)}{2} = \frac{n^2-n}{2} + \frac{n^2+3n+2}{2} = \frac{2n^2+2n+2}{2} = n^2+n+1$

$$\Rightarrow \left\lfloor \frac{t_{n-1}+t_{n+1}}{2} \right\rfloor = \left\lfloor \frac{n^2+n+1}{2} \right\rfloor = \left\lfloor \frac{n^2+n}{2} + \frac{1}{2} \right\rfloor = \left\lfloor \frac{n(n+1)}{2} + \frac{1}{2} \right\rfloor = \left\lfloor t_n + \frac{1}{2} \right\rfloor = t_n \blacksquare$$

Contrast this with squares: $\dfrac{(n-1)^2+(n+1)^2}{2}=\dfrac{2n^2+2}{2}=n^2+1$.

Theorem: Since $t_n - t_{n-1} = n$ and $t_n + t_{n-1} = n^2$, we equate the products of the left and right sides of these two equations to obtain $\boxed{(t_n)^2 - (t_{n-1})^2 = n^3}$ ■

There are infinitely many triangular numbers such as $t_8 = 36$ that are squares. The sum of the reciprocals of the triangular numbers is 2.

Theorem: Since $t_1=\binom{2}{2},\ t_2=\binom{3}{2},\ t_3=\binom{4}{2},\ \cdots,\ t_n=\binom{n+1}{2}$, we have, using the Hockey Stick Theorem for Pascal's Triangle,

$$t_1+t_2+\ldots+t_n=\binom{2}{2}+\binom{3}{2}+\cdots+\binom{n+1}{2}=\binom{n+2}{3}=\frac{n(n+1)(n+2)}{6} \quad ■$$

Theorem: Given a sphere with radius, r, let there be a partition of the diameter into $n+1$ segments, each with length $\Delta_n=\dfrac{2r}{n+1}$. Then the volume, V, is given by

$$V=\lim_{n\to\infty}\pi\Delta_n{}^3(t_1+t_2+\ldots+t_n)$$

Proof: $\Delta_n{}^3=\left(\dfrac{2r}{n+1}\right)^3=\dfrac{8r^3}{(n+1)^3}$, where r is the radius of the sphere. By the previous Theorem, $t_1+t_2+\ldots+t_n=\dfrac{n(n+1)(n+2)}{6}$. Then

$$\lim_{n\to\infty}\pi\Delta_n{}^3(t_1+t_2+\ldots+t_n)=\lim_{n\to\infty}\pi\left(\frac{8r^3}{(n+1)^3}\right)\left(\frac{n(n+1)(n+2)}{6}\right)=\lim_{n\to\infty}\left(\frac{8r^3\pi}{6}\right)\left(\frac{n(n+1)(n+2)}{(n+1)^3}\right)=V.$$

■

Theorem: The n-th square and the n-th oblong number add up to a triangular number.

Proof: $n^2+n(n+1)=2n^2+n=n(2n+1)=\dfrac{(2n)(2n+1)}{2}=t_{2n}$. ■

Theorem: $\boxed{\dfrac{t_1+t_2+\cdots+t_n}{n}=\dfrac{1}{3}\cdot t_{n+1}}$

Proof: $\dfrac{t_1+t_2+\cdots+t_n}{n}=\dfrac{1}{n}\binom{n+2}{3}=\dfrac{1}{n}\cdot\dfrac{(n+2)(n+1)n}{6}=\dfrac{(n+2)(n+1)}{6}=\dfrac{1}{3}\cdot\dfrac{(n+1)(n+2)}{2}=\dfrac{1}{3}\cdot t_{n+1}$

■

Compare this with the average of the first n squares.

Theorem: $\boxed{\dfrac{1^2+2^2+\cdots+n^2}{n}=\dfrac{1}{6}\cdot t_{2n+1}}$

Proof: $\dfrac{1^2+2^2+\cdots+n^2}{n}=\dfrac{1}{n}\cdot\dfrac{n(n+1)(2n+1)}{6}=\dfrac{(2n+1)(n+1)}{6}=\dfrac{1}{6}\cdot\dfrac{(2n+1)(2n+2)}{2}=\dfrac{1}{6}\cdot t_{2n+1}$ ■

Theorem: $\boxed{\dfrac{1^3+2^3+\cdots+n^3}{n}=\dfrac{(t_n)^2}{n}}$

Proof: This follows at once from the formula, $1^3+2^3+\cdots+n^3=\left[\dfrac{n(n+1)}{2}\right]^2=(t_n)^2$ ■

Theorem: $t_n = (2n-1)+(2n-5)+(2n-9)+(2n-13)+\ldots+[2+(-1)^n]$. The sum terminates with a 1 if n is odd, or with a 3, if n is even.■

Lemma: $t_{2m} < 4t_m < t_{2m+1}$

Proof: $4m^2+2m < 4m^2+4m < 4m^2+6m+2 \Rightarrow 2m(2m+1) < 4m(m+1) < (2m+1)(2m+2)$

$\Rightarrow \dfrac{2m(2m+1)}{2} < \dfrac{4m(m+1)}{2} < \dfrac{(2m+1)(2m+2)}{2} \Rightarrow t_{2m} < 4t_m < t_{2m+1}$ ■

Theorem: The equation, $t_n = 4t_m$, has no positive solution.

Proof: By the Lemma, $4t_m$ is trapped between two consecutive triangular numbers. ■

Alternate Proof: $t_n = 4t_m \Rightarrow n(n+1) = 4m(m+1) \Rightarrow n^2+n = 4(m^2+m) \Rightarrow$

$4n^2+4n = 4(4m^2+4m) \Rightarrow 4n^2+4n+1 = 4(4m^2+4m+1)-3 \Rightarrow$

$(2n+1)^2 = 4(2m+1)^2-3 \Rightarrow [2(2m+1)]^2-(2n+1)^2 = 3 \Rightarrow$

$[2(2m+1)]^2 = 4$ and $(2n+1)^2 = 1 \Rightarrow m = n = 0 \Rightarrow t_n = 4t_m$, has no positive solution. ■

Theorem: The equation, $t_n = 9t_m$, has no positive solution.

Proof: Observe that $9t_1 = t_4-1$, $9t_2 = t_7-1$, $9t_3 = t_{10}-1$, $9t_4 = t_{13}-1$, …., that is, $9t_m$is 1 less than t_{3m+1}. The Theorem follows at once. ■

Lemma: Given the positive integer k, the Diophantine equation, $a^2-b^2 = k$, has at most finitely many solutions.

Proof: Clearly, $a>b$. Then $a^2-b^2 = k \Rightarrow (a-b)(a+b) = k \Rightarrow a+b\le k \Rightarrow a\le kandb\le k$. ■

Theorem: For each$k$ = 10, 14, 18, … , the equation, $t_n = k^2t_m$, such that $n>m>1$, has **(a)** at least one solution, and **(b)** at most finitely many solutions.

Proof of (a): The equation, $t_n = k^2t_m$, $n>m>1$ and $k>1$, has solutions parametrized by

$$n = 4(t+1)(t+2)k = 2(2t+3) \quad m = t+1 t>0$$

Proof of (b): $t_n = k^2t_m \Rightarrow n(n+1) = k^2m(m+1) \Rightarrow n^2+n = k^2(m^2+m) \Rightarrow$

$4n^2 + 4n = k^2(4m^2 + 4m) \implies 4n^2 + 4n + 1 = k^2(4m^2 + 4m + 1) - (k^2 - 1) \implies$

$(2n + 1)^2 = k^2(2m + 1)^2 - (k^2 - 1) \implies [k(2m + 1)]^2 - (2n + 1)^2 = (k^2 - 1)$. By the preceding Lemma, we have at most finitely many solutions to $t_n = k^2 t_m$. ■

■The following chart yields the first four solutions:

t	*k*	*n*	*m*
1	10	24	2
2	14	48	3
3	18	80	4
4	22	120	5

By contrast, we have

Theorem: The equation, $t_n = 2t_m$, has infinitely many positive solutions.

Proof: $n(n + 1) = 2m(m + 1) \implies n^2 + n = 2(m^2 + m) \implies$

$4n^2 + 4n = 2(4m^2 + 4m) \implies 4n^2 + 4n + 1 = 2(4m^2 + 4m + 1) - 1 \implies$

$(2n + 1)^2 = 2(2m + 1)^2 - 1$. Letting $x = 2n + 1$ and $y = 2m + 1$, this last equation becomes

$$x^2 - 2y^2 = -1$$

This is a *Pell equation* that has infinitely many solutions, (x, y), such as (7, 5) and (41, 19). Infinitely many solutions may be found from the following table in which $x_1 = y_1 = 1$, and

$$x_{n+1} = x_n + 2y_n y_{n+1} = x_n + y_n$$

The solutions to $x_n^2 - 2y_n^2 = -1$ are given for all odd values of n. Here are the first five rows of the chart. Note, however, that $x_1 = y_1 = 1$ implies that $n = m = 0$, which doesn't yield a positive solution.

n	x_n	y_n
1	1	1
2	3	2
3	**7**	**5**
4	17	12
5	**41**	**29**

■

Let's obtain the *generating function* of the triangular numbers, t_n. That is, we wish to find a function, $f(x)$ with Maclaurin series: ($t_0 = 0$, so we omit the constant term.)

$$f(x) = t_1 x + t_2 x^2 + t_3 x^3 + t_4 x^4 + t_5 x^5 + \dots = x + 3x^2 + 6x^3 + 10x^4 + 15x^5 + \dots$$

Recalling that the triangular number $t_n = \dfrac{n(n+1)}{2}$, we have, using sigma notation,

$$f(x) = \sum_{n=1}^{\infty} t_n x^n = \sum_{n=1}^{\infty} \frac{n(n+1)}{2} x^n = \frac{x}{2}\left[\sum_{n=1}^{\infty} n(n+1)x^{n-1}\right] = \frac{x}{2}\phi(x)$$

where $\phi(x) = \sum_{n=1}^{\infty} n(n+1)x^{n-1}$

Nowdefine $\psi(x)$ by $\psi(x) = \sum_{n=1}^{\infty} x^{n+1} = x^2 + x^3 + x^4 + x^5 + \cdots = \dfrac{x^2}{1-x}$

Then $\psi'(x) = \sum_{n=1}^{\infty}(n+1)x^n = 2x + 3x^2 + 4x^3 + 5x^4 + \cdots = \left(\frac{x^2}{1-x}\right)' =$

$$\frac{(1-x)2x - x^2(-1)}{(1-x)^2} = \frac{2x - 2x^2 + x^2}{(1-x)^2} = \frac{2x - x^2}{(1-x)^2}$$

Differentiating again yields $\psi''(x) = \sum_{n=1}^{\infty} n(n+1)x^{n-1} = \phi(x)$

Since $\psi'(x) = \frac{2x - x^2}{(1-x)^2}$, we have

$$\phi(x) = \psi''(x) = \left(\frac{2x - x^2}{(1-x)^2}\right)' = \frac{(1-x)^2(2-2x) - (2x - x^2)2(1-x)(-1)}{(1-x)^4} =$$

$$\frac{(1-x)^2 2(1-x) + 2(2x - x^2)(1-x)}{(1-x)^4} = \frac{2(1-x)^3 + 2(2x - x^2)(1-x)}{(1-x)^4} =$$

$$\frac{2(1-x)^2 + 2(2x - x^2)}{(1-x)^3} = \frac{2(1 - 2x + x^2) + 2(2x - x^2)}{(1-x)^3} = \frac{2}{(1-x)^3}$$

or $\qquad \phi(x) = \frac{2}{(1-x)^3}$

Recalling that $f(x) = \frac{x}{2}\phi(x)$, we finally have

$$f(x) = \frac{x}{(1-x)^3}$$

We obtain an elegant consequence of the above equation by first letting $x = \frac{1}{2}$.

$$f\left(\frac{1}{2}\right) = \frac{\frac{1}{2}}{\left(1 - \frac{1}{2}\right)^3} = \frac{1}{2\left(\frac{1}{2}\right)^3} = \frac{1}{\left(\frac{1}{2}\right)^2} = 4$$

Now recalling that $f(x) = \sum_{n=1}^{\infty} t_n x^n$, the above equation becomes

$$\sum_{n=1}^{\infty} \frac{t_n}{2^n} = 4 \Leftrightarrow \boxed{\sum_{n=1}^{\infty} \frac{t_n}{2^n} = \frac{1}{2} + \frac{3}{4} + \frac{6}{8} + \frac{10}{16} + \frac{15}{32} + \cdots = 4}$$

Theorem: The sequence of triangular numbers, mod 4,is periodic, because the increments are periodic.

We will use a chart. The first column will list the triangular numbers, t_n, starting with the unconventional $t_0 = 0$. The second column will list them mod 4, and the third column will list the increments from the previous entries mod 4. (The increments, Δ, are 0, 1, 2, and 3. They repeat periodically.)

t_n	$t_n(\bmod 4)$	$\Delta(\bmod 4)$
0	0	
1	**1**	**1**
3	3	2
6	2	3
10	2	0
15	3	1
21	1	2
28	0	3
36	0	0
45	**1**	**1**

The entries in the rows that begin with entries 1 and 45 have identical **1**'s, thereby establishing periodicity. The period length is 8. ■

Theorem: The product of two *consecutive* triangular numbers is never a square.

Proof: By contradiction. Suppose $t_{n-1}t_n = m^2$. This becomes

$$\left(\frac{(n-1)n}{2}\right)\left(\frac{n(n+1)}{2}\right)=m^2 \Rightarrow n^2\left(n^2-1\right)=4m^2$$

which is impossible since $n^2 - 1$ isn't a square, while n^2 and $4m^2$ are squares. ■

On the other hand, the product of the three consecutive triangular numbers, $t_3 \times t_4 \times t_5 = 6\times 10\times 15 = 900 = 30^2$, a square.

II.A theorem of Gauss

The great mathematician, Gauss, proved that every positive integer can be expressed as the sum of three or fewer triangular numbers. See [1,2].

It is obvious that infinitely many numbers are sums of two triangular numbers. Simply add any two triangular numbers. In fact, the *n*-th oblong number, $O_n = n(n + 1) = t_n + t_n$.

Theorem: $n^2 - (n-1)^2 + (n-2)^2 - (n-3)^2 + \ldots + (-1)^{n+1} = t_n$ (3)

Proof: It is a direct result of (2). We have

$$n^2 - (n-1)^2 + (n-2)^2 - \ldots (-1)^{n+1} = (t_n + t_{n-1}) - (t_{n-1} + t_{n-2}) + (t_{n-2} + t_{n-3}) - \ldots + (-1)^{n+1} = t_n \blacksquare$$

III. The correspondence of two problems

Are there any numbers which may be written in more than one way as the sum of two triangular numbers? We answer this question by employing *consecutive partitions*.

From the consecutive partition 4 + 5 + 6 = 15, we add 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 to both sides, yielding

$$4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 = 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 + 15$$

or, $t_{14} - t_3 = t_{15} - t_6$. This becomes

$$t_{14} + t_6 = t_{15} + t_3$$

This says that 126 can be written in two ways as the sum of two triangular numbers, that is, 105 + 21 = 120 + 6.

In the spirit of the above example, consider the two consecutive partitions of equation (4) below, the one on the left having length 9 and the other of length 6.

$$3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 = 8 + 9 + 10 + 11 + 12 + 13 \quad (4)$$

Now, for any $k = 1, 2, 3, \ldots$, we add $2k$to each term on the left and add $3k$ to each term on the right. (This adds a total of $18k$ to both side.) We obtain the equality of two new consecutive partitions, given by

$$(2k + 3) + (2k + 4) + \ldots + (2k + 11) = (3k + 8) + (3k + 9) + \ldots + (3k + 13) \quad (5)$$

This is equivalent to $t_{2k+11}–t_{2k+2} = t_{3k+13}–t_{3k+7}$or

$$t_{2k+11} + t_{3k+7} = t_{3k+13} + t_{2k+2} \quad (6)$$

thereby yielding an infinite class of numbers that can be written as the sum of two triangular numbers in two different ways. Moreover, equations (5) and (6) imply each other, thereby establishing a correspondence between two seemingly unrelated problems. We conclude that if N has two consecutive partitions, then some other number, M, can be written in two different ways as the sum of two triangular numbers.

IV. Main Result

We have already seen (equation 2) that if n is a square, it is the sum of two (consecutive) triangular numbers. We have also seen that if n is an oblong number, it is the sum of two (equal) triangular numbers. We will, therefore, consider numbers of the form $t_n + t_m$, where $|n – m| \geq 2$. There are two cases:

Case 1: m and n have opposite parity. WLOG, assume $n>m$. Then

$$t_n + t_m= (t_n + t_{n–1}) – (t_{n–1} + t_{n–2}) + (t_{n–2} + t_{n–3}) – ... + (t_{m+1} + t_m) =$$

$$n^2– (n– 1)^2 + (n– 2)^2– (n– 3)^2 + ... + (m + 1)^2 \quad (7)$$

Note that the length (number of terms) of each side is $n – m$, which is odd.

Case 2:m and n have the same parity. Once again, we assume $n>m$. This case is more complicated since an attempt to use equation (7) will result in the final term of the left side being negative. Then the sum telescopes to $t_n–t_m$, and not $t_n + t_m$. To rectify this situation, we make the last term $(t_{m+1}–t_m)$ instead of $(t_{m+1} + t_m)$. So

$$t_n + t_m = (t_n + t_{n-1}) – (t_{n-1} + t_{n-2}) + (t_{n-2} + t_{n-3}) – ... – (t_{m+1}–t_m) =$$

$$n^2– (n– 1)^2 + (n– 2)^2– (n– 3)^2 + ... + (m + 2)^2 – (m +1). \quad (8)$$

The length (number of terms) of each side is even. We have proven the following theorem.

Theorem: A given positive integer, n, is the sum of two triangular numbers if and only if either

(1) n is squareor oblong,
(2) n is a descending, alternating sum of consecutive squares of odd length, or
(3) n is a descending, alternating sum of consecutive squares of even length, such that the first term is positive and the absolute value of the last (negative) term isone less than the square root of the previous term. ■

This chart displays all triangular partitions of length two for the 59 numbers up to 100 with this property.

2 = 1 + 1	27 = 6 + 21	55 = 10 + 45	79 = 1 + 78
4 = 1 + 3	29 = 1 + 28	56 = 1 + 55, 28 + 28	81 = 3 + 78, 15 + 66, 36 + 45
6 = 3 + 3	30 = 15 + 15	57 = 21 + 36	83 = 28 + 55
7 = 1 + 6	31 = 3 + 28, 10 + 21	58 = 3 + 55	84 = 6 + 78
9 = 3 + 6	34 = 6 + 28	60 = 15 + 45	87 = 21 + 66
11 = 1 + 10	36 = 15 + 21	61 = 6 + 55	88 = 10 + 78
12 = 6 + 6	37 = 1 + 36	64 = 28 + 36	90 = 45 + 45
13 = 3 + 10	38 = 10 + 28	65 = 10 + 55	91 = 91, 36 + 55
16 = 1 + 15, 6 + 10	39 = 3 + 36	66 = 21 + 45	92 = 1 + 91
18 = 3 + 15	42 = 6 + 36, 21 + 21	67 = 1 + 66	93 = 15 + 78
20 = 10 + 10	43 = 15 + 28	69 = 3 + 66	94 = 3 + 91 = 28 + 66
21 = 6 + 15	46 = 1 + 45, 10 + 36	70 = 15 + 55	97 = 6 + 91
22 = 1 + 21	48 = 3 + 45	72 = 6 + 66, 36 + 36	99 = 21 + 78
24 = 3 + 21	49 = 21 + 28	73 = 28 + 45	100 = 45 + 55
25 = 10 + 15	51 = 6 + 45, 15 + 36	76 = 10 + 66, 21 + 55	

References

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