Quest Journals
Journal of Research in Applied Mathematics
Volume 9 ~ Issue 1 (2023) pp: 74-78
ISSN(Online) : 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Research Paper

Existence Result for First Order Nonlinear Quadratic Functional Differential Equation

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ABSTRACT: *In this paper, we discuss the existence Result for Fractional Order Nonlinear Quadratic Functional Differential Equation in $\mathcal{R}_+$ by using hybrid fixed point theorem due to B.C.Dhage. For this we consider the first order nonlinear quadratic functional differential equation.*

KEYWORDS: *Banach algebras, hybridfixed point theorem, Quadratic functional differential equation, and existence result.*

Received 06 Jan., 2023; Revised 17 Jan., 2023; Accepted 19 Jan., 2023

I. INTRODUCTION:

In Literature several authors for various aspects of the solutions are studying the nonlinear differential and integral equations. Fractional order differential and integral equations play a very important role in many applications of real word problem. The study of nonlinear fractional differential equations had been made extensively in the literature by several authors all over the world and now it has become the core part of the nonlinear analysis. The development of nonlinear fractional differential and integral equations though vast growing topic in the subject of nonlinear differential and integral functions [20-25].

In this paper we will study the existence the solution of first order nonlinear quadratic functional differential equation. The result has been obtained by using hybrid fixed point theorem for two operators in Banach space due to Dhage. The main result is well illustrated with the help of example.

We consider the following first order nonlinear quadratic functional differential equations:

$$\left.\begin{aligned} \mathcal{D}\left[\frac{x(t)}{f\big(t,x(\alpha(t))\big)}\right] &= g[t,x(\mu(t))],\quad t\in\mathcal{R}_+ \\ x(0) &= 0 \end{aligned}\right\} \quad (2.1.1)$$

Where, $f(t,x){:}\,\mathcal{R}_+\times\mathcal{R}\to\mathcal{R}-\{0\}$, $g(t,x){:}\,\mathcal{R}_+\times\mathcal{R}\to\mathcal{R}$ and $\alpha,\mu{:}\,\mathcal{R}_+\to\mathcal{R}$

Here the solution of nonlinear differential equations (2.1.1) we mean a function $x\in BC(\mathcal{R}_+,\mathcal{R})$ such that:

(i) The function $t\to\left[\frac{x(t)}{f\left(t,x(\alpha(t))\right)}\right]$ is bounded and continuous for each $x\in\mathcal{R}$.

(ii) x satisfies (2.1.1)

2.2 PRELIMINARIES:

Let $X=BC(\mathcal{R}_+,\mathcal{R})$ be the space of bounded real valued continuous function on $\mathcal{R}_+$ and S be a subset of X. Let a mapping $\mathcal{A}{:}\,X\to X$ be an operator and consider the following operator equation in X, namely,

$x(t)=(\mathcal{A}x)(t)$, for all $t\in\mathcal{R}_+$(2.2.1)

We require the following definitions.

Definition 2.2.1[22]: Let X be a Banach space. A mapping $\mathcal{A}{:}\,X\to X$ is called Lipschitz if there is a constant $\alpha>0$ such that, $\|\mathcal{A}x-\mathcal{A}y\|\le\alpha\|x-y\|$ for all $x,y\in X$. If $\alpha<1$, then $\mathcal{A}$ is called a contraction on X with the contraction constant α.

Definition 2.2.2[18]:.An operator $\mathcal{U}$ on a Banach space X into itself is called compact if for any bounded subset S of X, $\mathcal{U}(S)$ is relatively compact subset of X. If $\mathcal{U}$ is continuous and compact, then it is called completely continuous on X.

Definition 2.2.3[18]: Let X be a Banach space with the norm $\|\cdot\|$ and let $\mathcal{U}{:}\,X\to X$ be an operator (in general nonlinear). Then $\mathcal{U}$ is called

i. Compact if $\mathcal{U}(X)$ is relatively compact subset of X.
ii. Totally bounded if $\mathcal{U}(S)$ is totally bounded subset of X for any bounded subset S of X.
iii. Completely continuous if it is continuous and totally bounded operator on X

It is clear that every compact operator is totally bounded but the converse need not be true.

Theorem 2.2.1 [6] :(Arzela-Ascoli Theorem) If every uniformly bounded and equicontinuous sequence $\{f_n\}$ of functions in $C(\mathcal{R}_+, \mathcal{R})$, then it has a convergent subsequence.

Theorem 2.2.2[6]: A metric space X is compact iff every sequence in X has a convergent subsequence.

Theorem 2.2.3[5, 6, and 17]: Let S be a non-empty, bounded and closed-convex subset of the Banach space X and let $\mathcal{A}: X \to X$ and $\mathcal{B}: S \to X$ are two operators satisfying

a) $\mathcal{A}$ is Lipschitz with a lipschitz constant α,
b) $\mathcal{B}$ is completely continuous, and
c) $\mathcal{A}x\mathcal{B}x \in S$ for all $x \in S$, and
d) $\alpha M < 1$,where $M = \|\mathcal{B}(S)\|: \sup\{\|\mathcal{B}x\|: x \in S\}$.

Then the operator equation $\mathcal{A}x\mathcal{B}x = x$ has a solution in S.

2.3 EXISTENCE THEORY:

Now we want the solution of (2.2.1) in the space $BC(\mathcal{R}_+, \mathcal{R})$of bounded and continuous realvalued functions defined on $\mathcal{R}_+$. Define a standard norm $\|\cdot\|$ and a multiplication "$\cdot$" in $BC(\mathcal{R}_+, \mathcal{R})$by,$\|x\| = sup\{|x(t)|: t \in \mathcal{R}_+\}$, $(xy)(t) = x(t)y(t)$, $t \in \mathcal{R}_+$

Clearly, $BC(\mathcal{R}_+, \mathcal{R})$ becomes a Banach space with respect to the above norm and the multiplication in it. By $\mathcal{L}^1(\mathcal{R}_+, \mathcal{R})$ we denote the space of Lebesgue-integrable function in$\mathcal{R}_+$with the norm $\|\cdot\|_{\mathcal{L}^1}$ defined by

$$\|x\|_{\mathcal{L}} = \int_0^\infty |x(t)|dt$$

2.4 MAIN RESULT:

We require the following hypothesis for existence of solution of FNFDE (2.1.1).

($\mathcal{H}_1$) The function $\alpha, \mu: \mathcal{R}_+ \to \mathcal{R}$ are continuous.

($\mathcal{H}_2$) The function $f(t,x): \mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}$ is continuous and bounded with bound F= $sup_{(t,x)\epsilon\mathcal{R}_+\times\mathcal{R}}|f(t,x)|$there exist a bounded function $l: \mathcal{R}_+ \to \mathcal{R}$ with bound L satisfying
$|f(t,x) - f(t,y)| \le l(t)\{|x(t) - y(t)|\}a.e.\ t \in \mathcal{R}_+$, for all $x, y \in \mathcal{R}$.

($\mathcal{H}_3$) The function $g(t,x) = g: \mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}$ is satisfying caratheodory condition with continuous function $h(t): \mathcal{R}_+ \to \mathcal{R}$ such that $g(t,x) \le h(t)\ \ \forall t \in \mathcal{R}_+$and $x, y \in \mathcal{R}$.

($\mathcal{H}_4$) The function $v: \mathcal{R}_+ \to \mathcal{R}$ defined by the formulas $v(t) = \int_0^t \frac{h(s)}{(t-s)^{1-\beta}} ds$ is bounded on $\mathcal{R}_+$ and vanish at infinity, that is $\lim_{t\to\infty} v(t) = 0$.

Remark 2.4.1: Note that the $(\mathcal{H}_3)$and$(\mathcal{H}_4)$hold, then there exists a constant $K_1 > 0$ such that $K_1 = \underbrace{\sup}_{t\ge0}\left\{\frac{v(t)}{\Gamma(\beta)}: t \in \mathcal{R}_+\right\}$

Lemma 2.4.1: Suppose that $\zeta \in (0,1)$ and the function f, g satisfying FNFDE (2.1.1) then x is the solution of the FNFDE (2.1.1) if and only if it is the solution of integral equation

$x(t) = [f(t, x(\alpha(t)))]\left[\int_0^t g\left(s, x(\mu(s))\right) ds\right],\ t \in \mathcal{R}_+$(2.4.1)

Proof: Integrating equation (2.1.1) of first order, we get

$$I\mathcal{D}\left[\frac{x(t)}{f(t, x(\alpha(t)))}\right]_0^t = I\left[g\left(s, x(\mu(s))\right)\right]$$

$x(t) = [f(t, x(\alpha(t)))]\left[\int_0^t g\left(s, x(\mu(s))\right) ds\right],\ , t \in \mathcal{R}_+$

Conversely differentiate (2.4.1) w.r.to t, we get,

$$\mathcal{D}\left[\frac{x(t)}{f(t, x(\alpha(t)))}\right] = \mathfrak{D}Ig\left(t, x(\mu(t))\right)$$

$$\mathcal{D}\left[\frac{x(t)}{f(t, x(\alpha(t)))}\right] = g\left(t, x(\mu(t))\right)$$

Theorem 2.4.1: Assume that condition ($\mathcal{H}_1$-$\mathcal{H}_4$) hold. Further if $LK_1 < 1$, where K_1 is defined in remark (2.4.1). Then FNFDE (2.1.1) has a solution in the space $BC(\mathcal{R}_+, \mathcal{R})$.

Proof: By a solution of FNFDE (2.1.1) we mean a continuous function $x: \mathcal{R}_+ \to \mathcal{R}$ that satisfies FNFDE (2.1.1) on $\mathcal{R}_+$.Let$X = BC(\mathcal{R}_+, \mathcal{R})$

$B_r[0]$isthe closed ballin X centred at origin0 and radius r as

$B_r[0] = \{x \in X: \|x\| \le r\}$where r satisfies the inequality $FK_1 \le r$.

Let $X = BC(\mathcal{R}_+, \mathcal{R})$ be Banach algebras of all absolutely continuous real-valued function on $\mathcal{R}_+$ with the norm, $\|x\| = sup|x(t)|, t \in \mathcal{R}_+$(2.4.2)

Now the FNFDE (2.1.1) is equivalent to the FNFIE

$x(t) = \left[f\left(t, x(\alpha(t))\right)\right]\left[\int_0^t g\left(s, x(\mu(s))\right) ds\right]$(2.4.3)

Let us define the two mapping $\mathcal{A}: X \to X$ and $\mathcal{B}: B_r[0] \to X$ by

$\mathcal{A}x(t) = f\left(t, x(\alpha(t))\right), t \in \mathcal{R}_+$(2.4.4)

$\mathcal{B}x(t) = \int_0^t g\left(s, x(\mu(s))\right) ds \ , t \in \mathcal{R}_+$(2.4.5)

Thus from the FNDE (2.1.1), we obtain the operator equation as follows:

$x(t) = \mathcal{A}x(t)\mathcal{B}x(t),\ t \in \mathcal{R}_+$(2.4.6)

If the operator $\mathcal{A}$ and $\mathcal{B}$ satisfy all the hypothesis of theorem (2.2.3), then the operator equation (2.4.6) has a solution on $B_r[0]$.

Step I: Firstly we show that $\mathcal{A}$ is Lipschitz on $X = BC(\mathcal{R}_+, \mathcal{R})$. Let $x, y \in X$, then

$|\mathcal{A}x(t) - \mathcal{A}y(t)| = \left|f\left(t, x(\alpha(t))\right) - f\left(t, y(\alpha(t))\right)\right|$

$\leq l(t)\{|x(\alpha(t)) - y(\alpha(t))|\}$

$\leq L|x(t) - y(t)|$for all $t \in \mathcal{R}_+$

Taking supremumover t we get,

$\|\mathcal{A}x - \mathcal{A}y\| \leq L\|x - y\|$for all $x, y \in X$.

Thus, $\mathcal{A}$ is Lipchitz on X with Lipschitz constant L.

Step II: Now we show that $\mathcal{B}$ is completely continuous operator on $B_r[0]$using standard argument such as those in Granas at [18], it can be shown that $\mathcal{B}$ is continuous operator on $B_r[0]$.To do this, let us fix arbitrary $\epsilon > 0$ and take $x, y \in B_r[0]$such that$\|x - y\| \leq \epsilon$.

$$|\mathcal{B}x(t) - \mathcal{B}y(t)| = \left|\int_0^t g\left(s, x(\mu(s))\right) ds - \int_0^t g\left(s, y(\mu(s))\right) ds\right|$$

$$\leq \left|\int_0^t g\left(s, x(\mu(s))\right) ds\right| + \left|\int_0^t g\left(s, y(\mu(s))\right) ds\right| \leq \int_0^t h(s)\, ds + \int_0^t h(s)\, ds$$

$\leq 2v(t)$as $v(t) \leq \frac{\epsilon}{2}$

Therefore$|\mathcal{B}x(t) - \mathcal{B}y(t)| \leq \epsilon$

Thus $\mathcal{B}$ is continuous.

Step III: Now we will show that$\mathcal{B}$ is compact on$B_r[0]$

a) First we prove that every sequence $\{\mathcal{B}x_n\}$ in $\mathcal{B}(B_r[0])$ has uniformly bounded sequence and $\{\mathcal{B}x_n\}$ is equicontinuous set in$\mathcal{B}(B_r[0])$. Since $g\left(t, x(\mu(t))\right)$ is $\mathcal{L}_X^1 -$ caratheodary, we have

$$|\mathcal{B}x_n(t)| = \left|\int_0^t g\left(s, x(\mu(s))\right) ds\right|$$

$$\leq \int_0^t \left|g\left(s, x(\mu(s))\right)\right| ds$$

$$\leq \int_0^t h(s) ds$$

$\leq v(t)$

Taking supremum over t, we obtain

$\|\mathcal{B}x_n\| \leq K_1$for all $x \in B_r[0]$where, $K_1 = sup_{t\in\mathcal{R}_+}\{v(t)\}$

This shows that $\{\mathcal{B}x_n\}$ is uniformly bounded sequence in $\mathcal{B}(B_r[0])$

To show that $\{\mathcal{B}x_n\}$ is an equicontinuous sequence, let $t_1, t_2 \in [0, T]$ be arbitrary. Then for any $x \in B_r[0]$ (2.4.5-2.4.6) implies

$$|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| = \left|\int_0^{t_2} g\left(s, x_n(\mu(s))\right) ds - \int_0^{t_1} g\left(s, x_n(\mu(s))\right) ds\right|$$

$$= \left|\int_0^{t_2} h(s)ds - \int_0^{t_1} h(s)ds\right| \leq |v(t_2) - v(t_1)|$$

The right hand side of the above inequality doesn't depend on x and tends to zero as $t_1 \to t_2$.There fore$|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| \to 0$ as $t_1 \to t_2$.

If $t_1, t_2 \geq T$ then we have similar procedure.
If $t_1, t_2 \in \mathcal{R}_+$ then we have
$|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| \leq |\mathcal{B}x_n(t_2) - \mathcal{B}x_n(T)| + |\mathcal{B}x_n(T) - \mathcal{B}x_n(t_1)|$
If $t_1 \to t_2$, then $t_1 \to T \; and \; T \to t_2$
Therefore $|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(T)| \to 0$ $\quad |\mathcal{B}x_n(T) - \mathcal{B}x_n(t_1)| \to 0$
So $|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| \to 0$ as $t_1 \to t_2$
Hence, $\{\mathcal{B}x_n\}$ is an equicontinuous sequence of functions in$\mathcal{B}(B_r[0])$ so $\mathcal{B}(B_r[0])$ is relatively compact. Hence$\mathcal{B}$ is compact.
so that$\mathcal{B}$ is compact and continuous operator on $B_r[0]$
Thus $\mathcal{B}$ is completely continuous on $B_r[0]$
Step IV: To show$\mathcal{A}x\mathcal{B}y \in B_r[0]$
Let $x, y \in B_r[0]$ such that $x = \mathcal{A}x\mathcal{B}x$
By assumptions $(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3)$

$$|x(t)| = |\mathcal{A}x(t)\mathcal{B}x(t)|$$
$$\leq |\mathcal{A}x(t)||\mathcal{B}x(t)|$$
$$\leq \left|f\left(t, x(\alpha(t))\right)\right| \left|\int_0^t g\left(s, x(\mu(s))\right) ds\right|$$
$$\leq \left|f\left(t, x(\alpha(t))\right)\right| \int_0^t \left|g\left(s, x(\mu(s))\right)\right| ds,$$
$$\leq F \int_o^t h(s) ds \leq Fv(t)$$

Taking supremum over ton $\mathcal{R}_+$, we obtain $\|\mathcal{A}x\mathcal{B}x\| \leq FK_1$, $\forall x \in B_r[0]$That is we have,$\|x\| = \|\mathcal{A}x\mathcal{B}x\| \leq r$,$\forall\, x \in B_r[0]$.
Which gives $\mathcal{A}x\mathcal{B}y \in B_r[0]$
Hence assumption(c) of theorem (2.2.3) is proved.
Step V: Also we have

$$M = \|\mathcal{B}(B_r[0])\| = sup\{\|\mathcal{B}x\| : x \in B_r[0]\}$$
$$= sup \left\{ sup_{t\in\mathcal{R}_+} \left[\int_0^t g\left(s, x(\mu(s))\right) ds\right] : x \in B_r[0] \right\}$$
$$\leq sup \left\{ sup_{t\in\mathcal{R}_+} \left[\int_0^t h(s) ds\right] : x \in B_r[0] \right\}$$
$$\leq sup\left\{sup_{t\in\mathcal{R}_+}[v(t)] : x \in B_r[0]\right\}$$
$$\leq K_1$$

and there fore $ML = LK_1 < 1$. Thus the condition (d) of theorem (2.2.3) is satisfied.
Hence all the conditions of theorem (2.2.3) are satisfied and therefore the operator equation $\mathcal{A}x\mathcal{B}x = x$ has a solution in $B_r[0]$.

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