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Review Paper

An Optimal Fourth-Order Iterative Methods Based on a Variant of Newton's Method

H.Y. Alharib[1], I.A. Al-Subaihi[2*]
[1]*(Department of Architectural Engineering, College of Engineering, University of Prince Mugrin, Saudi Arabia)*
[2]*(Department of General Studies, University of Prince Mugrin, Saudi Arabia)*
Corresponding Author: I.A. Al-Subaihi

ABSTRACT : *In this research, we provide a simple and easy-to-apply method for constructing a fourth-order modification based on the Weerakoon- Fernando's method, which is a third-order form of Newton's method. The new technique is based on the weight function approach, and it achieves the optimal order four through two function evaluations and one first derivative evaluation. To validate the theory stated in this research, the method's accuracy is tested on a variety of numerical examples.*
KEYWORDS : *Non-linear equations, Newton's method, Fourth order, Efficiency index.*

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I. INTRODUCTION

Numerical analysis has valuable applications in various branches in all fields of science and engineering. Finding the simple root of a given nonlinear equation $f(r) = 0,$ is crucial. Where f: D ⊆ R → R is a scalar function for an open interval D [1–4]. Newton's method is a basic, common, and important iterative method to find the solution r. It is given by:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{1}$$

which is quadratically convergent. Two crucial components of the iterative method are the order of convergence and the efficiency index. Considering Kung–Traub conjecture [4, 5], the efficiency index I is determined by, $I = p^{\frac{1}{d}}$ where p is the order of the iterative method and d is the number of function evaluations per iteration. The optimal convergence order p depends on the function evaluations and is given by $p = 2^{d-1}$. This illustrates that Newton's method is an optimal method with an efficiency index $I = 2^{\frac{1}{2}} = 1.41421$.

Many researchers have modified Newton's method to improve its convergence as can be seen in [6 − 8]. A Variant of Newton's Method that is established by Weerakoon and Fernando's[7] has a convergence order equals to three. The method is not optimal, and it uses three function evaluations, one function and two first derivatives:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - \frac{2f(x_n)}{[f'(x_n) + f'(y_n)]}. \end{aligned} \tag{2}$$

Another suggested modification to Newton's method obtained by Frontini and Sormani [9] resulting in a third-order convergence:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'\left(\frac{x_n + y_n}{2}\right)}, \tag{3}$$

where, $y_n = x_n - \frac{f(x_n)}{f'(x_n)}$.

Several strategies have been used to suggest and analyse improvements to Newton's method to obtain a fourth order with greater efficiency, see for example $[1-14]$. The strategy used in Chun and Ham [10] was to build the fourth-order method using existing higher-order methods.

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{2f(x_n) + (2\beta - 1)f(y_n)}{2f(x_n) + (2\beta - 5)f(y_n)} \frac{f(y_n)}{f'(x_n)}, \tag{4}$$

where $\beta \in R$.

In this research, we use an approach that involves applying weight function to Weerakoon and Fernando's iterative method (2) to achieve a higher order of convergence and efficiency index than origins. The new approach achieves fourth-order convergence while only requiring three evaluations per computation iterative.

II. DEVELOPMENT OF THE NEW METHOD AND CONVERGENCE ANALYSIS

This section demonstrates the development of a new improved form of Weerakoon and Fernando's approach (2). In light of Kung and Traub's conjecture [5], it is clear that Weerakoon and Fernando's [7] approach is not optimal. It has a third order convergence and uses three functional evaluations for each iteration.

To obtain an optimal method of (2), we start by applying appropriate approximation of $f'(y_n)$ that has been developed by [3] to utilize two functions and one of its first derivative instead of one function and two of its first derivatives without loss of order.

$$f'(y_n) \approx \frac{f'(x_n)\, f(x_n)^2}{\left(f(x_n) + f(y_n)\right)^2}. \tag{5}$$

By substituting (5) in the second step of (2) we will get,

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = x_n - \left(\frac{2f(x_n)[f(x_n) + f(y_n)]^2}{[f(x_n) + f(y_n)]^2 f'(x_n) + f'(x_n) f(x_n)^2}\right). \tag{6}$$

Increase the convergence order of an iterative approach can be done using one or more of several available techniques. In this method, a weight function $H(t_n, s_n)$ is used. Similar approach to this weight function can be seen in $[4, 11, 12]$. It is expressed in the form of partial derivative at the origin as:

$$H_{t_n, s_n} = \frac{\partial^{i+j} H(t_n, s_n)}{\partial t^i\, \partial s^j}|(t_n, s_n) = (0,0), \tag{7}$$

where $s_n = \frac{f(y_n)}{f(x_n)}$ and $t_n = \frac{f(x_n)}{f'(x_n)}$.

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = x_n - H(t_n, s_n)\left(\frac{2f(x_n)[f(x_n) + f(y_n)]^2}{[f(x_n) + f(y_n)]^2 f'(x_n) + f'(x_n) f(x_n)^2}\right). \tag{8}$$

The proposed method (8) has an optimal convergence of order four with three evaluations and efficiency index $4^{\frac{1}{3}} = 1.5874$ under the $H(t_n, s_n)$ requirements stated in the subsequent theorem.

Theorem 1. Let r be a simple root of $f(x_n) = 0$ in an open interval D. If x_0 is close enough to r then the method (8) has a convergence order of four with the following error equation:

$$x_{n+1} = \left(\frac{5}{2}c_2^3 - c_2c_3 - \frac{1}{6}H_{(ttt)_n} - \frac{1}{2}H_{(tt)_n s_n}\, c_2 - \frac{1}{2}H_{t_n (ss)_n}\, c_2^2 - \frac{1}{6}H_{(sss)_n}\, c_2^3\right) e_n^4 + O(e_n^5),$$

if the following conditions hold:

$H(0,0) = 1,\ H_{s_n}(0,0) = 1,\ H_{t_n}(0,0) = 1, H_{(ss)_n}(0,0) = 5, H_{(tt)_n}(0,0) = 0,\ H_{t_n s_n}(0,) = 0$. Where $e_n = x_n - r$ is the error at the nth iteration and $c_k = \frac{f^{(k)}(r)}{k!\,f'(r)}, k = 2,3,4, \dots$.

Proof. Using Taylor's expansion of $f(x_n)$ about r and accounting for the fact that $f(r) = 0$, we have:

$$f(x_n) = f'(r)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4\ O(e_n^5)]. \tag{9}$$

Further, repeat the Taylor expansion in the same manner this time with $f'(x_n)$ we get,

$$f'(x_n) = f'(r)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + O\ (e_n^5)]. \tag{10}$$

Dividing (9) by (10) yields,

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2e_n^2 + (2c_2^2 - 2c_3)e_n^3 + (-4c_2^3 + 7c_2c_3 - 3c_4)e_n^4 + O\ (e_n^5). \tag{11}$$

Using (11) in the first step of (8):

$$y_n = r + c_2e_n^2 + (-2\,c_2^2 + 2\,c_3)e_n^3 + (4\,c_2^3 - 7\,c_2c_3 + 3\,c_4)e_n^4 + O\ (e_n^5). \tag{12}$$

Now, $f(y_n)$ is expanded around r to give,

$$f(y_n) = f'(r)[c_2e_n^2 + (-2c_2^2 + 2c_3)e_n^3 + (5c_2^3 - 7c_2c_3 + 3c_4)e_n^4 + O\ (e_n^5). \tag{13}$$

Using the 2-variable Taylor expansion of $H(t_n, s_n)$:

$$H(t_n, s_n) = H(0,0) + \sum_{m=1}^{\infty} \sum_{i+j=m} \frac{H_{i,j}}{i!\,j!} {t_n}^i {s_n}^j,$$

where $i, j = 0, 1, 2, \dots$, we get,

$$\begin{aligned} H(t_n, s_n) &= H(0,0) + H_{t_n}t_n + H_{s_n}s_n + \frac{1}{2}\left(H_{(tt)_n}t_n^2 + H_{(ss)_n}s_n^2 + 2H_{t_n s_n}t_n s_n\right) \\ &\quad + \frac{1}{6}\left(H_{(ttt)_n}t_n^3 + H_{(sss)_n}s_n^3 + 3H_{(tt)_n s_n}{t_n}^2 s_n + 3H_{t_n (ss)_n}t_n {s_n}^2\right). \end{aligned} \tag{14}$$

Using (9),(10),(13), and (14) in (8) gives,

$$x_{n+1} = r + \left(1 - H(0,0)\right)e_n + \left(-H_{s_n}c_2 - H_{t_n}\right)e_n^2 + \tag{15}$$

$$\dots\left(\frac{1}{2}H_{t_n}\, {c_2}^2 + \cdots - \left(\frac{25}{2}\ H(0,0)\ {c_2}^3\right)\right)\ e_n^4 + O\ (e_n^5)\ .$$

Applying the weight function conditions $H(0,0) = 1$, $H_{s_n} = 1$, $H_{t_n} = 1$, $H_{(ss)_n} = 5$, $H_{(tt)_n} = 0$, $H_{t_n s_n} = 0$, and $|H_{(ttt)_n}|, |H_{(sss)_n}|, |H_{(tt)_n s_n}|, |H_{t_n (ss)_n}| < \infty$ gives the error expression,

$$x_{n+1} = \left(\frac{5}{2}c_2^3 - c_2c_3 - \frac{1}{6}H_{(ttt)_n} - \frac{1}{2}H_{(tt)_n s_n}\, c_2 - \frac{1}{2}H_{t_n(ss)_n}\, c_2^2 - \frac{1}{6}H_{(sss)_n}\, c_2^3\right)e_n^4 + O\,(e_n^5). \tag{16}$$

Thus, this indicates that the method defined by (8) has the fourth order of convergence which competes the proof.

III. THE CONCRETE ITERATIVE METHODS AND NUMARICAL RESULTS

In order to verify the accuracy of the theoretical conclusion reached in this paper and to illustrate it practically, we defined the (HM1) method using the function:

$$H(t_n, s_n) = \cos(t_n s_n) + \frac{5}{2}s_n^2, \tag{17}$$

and $(HM2)$ using the function:

$$H(t_n, s_n) = e^{s_n t_n} - t_n s_n + \frac{5}{2}s_n^2. \tag{18}$$

Furthermore, the effectiveness is compared with various closed competitor methods, such as Alnaser method (AM) [13]:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = y_n - \frac{4(y_n - x_n)f(y_n)}{(y_n - x_n)[f'(x_n) + f_1'(y_n)] + 2[f(y_n) - f(x_n)]}H(v), \tag{19}$$

where, $H(v) = \sin(v) + \cos(v)$ and $f_1'(y_n) \approx \frac{f'(x_n)f(x_n)^2}{(f(x_n)+f(y_n))^2}$.

Additionally, Chun and Ham method (CM), (4), where $\beta = 1$. Moreover, method proposed by Kuo et al. (KM) [14]:

$$y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = x_n - \left[1 - \left(1 - \frac{\frac{3}{4}\theta(f'(y_n) - f'(x_n))}{\beta f'(y_n) + (1-\beta)f'(x_n)}\right)\frac{\frac{3}{4}(f'(y_n) - f'(x_n))}{\alpha f'(y_n) + (1-\alpha)f'(x_n)}\right]\frac{f(x_n)}{f'(x_n)}, \tag{20}$$

where $\alpha, \beta, \theta \in R$.

In the following examples, they have been taken as: $\alpha = \beta = \frac{3}{2}$, and $\theta = 2 - \frac{4}{3}\alpha$.

Table 1 gives a list of the test functions examined, their roots r with just 16 decimal digits, and the initial values x_0.

Table 1. Test functions, their roots, and initial values

Function	Initial Value, (x_0)	Root, (r)
$f_1(x) = \sin(x^2) - x^2 + 1$	0.35	0.1437392592997536
$f_2(x) = e^{x^2+7x-30} - 1$	3.1	3.0
$f_3(x) = xe^{x^2} - \sin(x^2) + 3\cos(x) + 5$	−1.1	−1.2076478271309189
$f_4(x) = x^3 + \log(1+x)$	0.25	0.0
$f_5(x) = 2x\cos x + x - 3$	−4.8	−3.5322516915364759
$f_6(x) = e^{-x^2+x+2} - \cos(x+1) + x^3 + 1$	0	−1.0
$f_7(x) = x^2 - \sin x - 20$	4	4.3657177051597667

MATLAB (R2022a) was used to evaluate the implementations with 1000 digits accuracy. The stopping criteria are

i. $|x_n - r| \le 10^{-300}$

ii. $|f(x_n)| \le 10^{-300}$.

Table 2 lists the iterations IT, the absolute result of the function $|f(x_n)|$, and the absolute error $|x_n - r|$. Additionally, the approximated computational order of convergence (COC) that is approximated by [7]:

$$COC = \frac{ln|(x_n - r)/(x_{n-1} - r)|}{ln|(x_{n-1} - r)/(x_{n-2} - r)|}$$

Table 2. Comparison of various iterative methods.

Method	IT	$\|f(x_n)\|$	$\|x_n - r\|$	(COC)
$f_1(x) = \sin(x^2) - x^2 + 1$				
AM	6.0	$5.08347e-812$	$5.7072e-812$	4.0
CM	6.0	$8.09465e-875$	$9.08785e-875$	4.0
KM	6.0	$1.55126e-918$	$1.7416e-918$	4.0
HM1	6.0	$3.45841e-848$	$3.88275e-848$	4.0
HM2	6.0	$8.21999e-842$	$9.22857e-842$	4.0
$f_2(x) = e^{x^2+7x-30} - 1$				
AM	6.0	$3.63992e-803$	$2.79994e-804$	4.0
CM	Div.			
KM	5.0	$1.03178e-394$	$7.93675e-396$	4.0
HM1	6.0	$1.36001e-922$	$1.04616e-923$	4.0
HM2	6.0	$3.59251e-923$	$2.76347e-924$	4.0
$f_3(x) = xe^{x^2} - \sin(x^2) + 3\cos(x) + 5$				
AM	5.0	$5.35159e-593$	$2.63529e-594$	4.0
CM	5.0	$1.18345e-777$	$5.82768e-779$	4.0
KM	5.0	$2.26827e-1007$	0	4.0
HM1	5.0	$1.73786e-892$	$8.55775e-894$	4.0
HM2	5.0	$1.23752e-860$	$6.09392e-862$	4.0
$f_4(x) = x^3 + \log(1+x)$				
AM	5.0	$8.71539e-892$	$8.71539e-892$	4.0
CM	5.0	$7.2551e-846$	$7.2551e-846$	4.0
KM	5.0	$8.96563e-826$	$8.96563e-826$	4.0
Method	IT	$\|f(x_n)\|$	$\|x_n - r\|$	(COC)

HM1	5.0	$1.6584e-854$	$1.6584e-854$	4.0
HM2	5.0	$1.30224e-851$	$1.30224e-851$	4.0
$f_5(x)=2x\cos x+x-3$				
AM	6.0	$2.47219e-350$	$1.34297e-350$	4.0
CM	6.0	$7.74284e-450$	$4.20617e-450$	4.0
KM	6.0	$3.64813e-554$	$1.98178e-554$	4.0
HM1	6.0	$4.19911e-381$	$2.2811e-381$	4.0
HM2	6.0	$1.56569e-413$	$8.50535e-414$	4.0
$f_6(x)=e^{-x^2+x+2}-\cos(x+1)+x^3+1$				
AM	5.0	$7.00095e-625$	$1.16682e-625$	4.0
CM	5.0	$2.04714e-618$	$3.4119e-619$	4.0
KM	5.0	$1.90758e-500$	$3.17929e-501$	4.0
HM1	5.0	$2.50686e-613$	$4.1781e-614$	4.0
HM2	5.0	$1.87573e-626$	$3.12621e-627$	4.0
$f_7(x)=x^2-\sin x-20$				
AM	4.0	$4.10354e-346$	$4.5237e-347$	4.0
CM	4.0	$2.73821e-363$	$3.01857e-364$	4.0
KM	4.0	$2.4707e-357$	$2.72367e-358$	4.0
HM1	4.0	$3.8921e-391$	$4.29061e-392$	4.0
HM2	4.0	$4.85715e-345$	$5.35448e-346$	4.0

IV. CONCLUSION

To summarize, we created a novel optimum fourth order approach for solving nonlinear equations based on the non-optimal Weerakoon- Fernando's method. The key advantage of this technique is that it requires two evaluations of the function f and one evaluation of its first derivative per whole cycle. As a result, it supports the Kung and Traub hypothesis. Furthermore, when we examine the definition of efficiency index I, our suggested technique has $I = 1.5874$, which is better than Weerakoon- Fernando's method $I = 1.44225$. The created approach was compared against other known fourth-order methods and found to have at least equivalent performance.

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