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Review Paper

On the Kuznetsov's Polynomials

R. Sivaraman [1],J. López-Bonilla[2], E. N. Macin-Moran [2],
[1]*Department of Mathematics, Dwaraka Doss Goverdhan Doss Vaishnav College, Chennai 600 106, Tamil Nadu, India*
[2]*ESIME-Zacatenco, Instituto Politécnico Nacional, Edif. 4, 1er. Piso, Col. Lindavista CP 0778, CDMX, México*

Abstract: *We show that the Kuznetsov's polynomials $p_j(x)$ can be written in terms of the complete Bell polynomials, and we deduce a direct relationship between the $p_j(k)$ and the Bernoulli numbers.*
Keywords: *Complete Bell polynomials, Recurrence relations, Kuznetsov's polynomials, Bernoulli numbers, Euler-Mascheroni's constant.*

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I. Introduction

The Kuznetsov polynomials $p_k(x)$ can be defined via a generating function [1]:

$$\sum_{m=0}^{\infty} \frac{1}{(2m)!} p_m(x) u^{2m} = \left(\frac{u}{\sin u}\right)^x , \qquad (1)$$

or through a recurrence relation:

$$2n\, p_n(x) = x \sum_{j=1}^{n} \binom{2n}{2j} 4^j |B_{2j}| p_{n-j}(x), \qquad p_0(x) = 1, \qquad (2)$$

where B_{2k} are Bernoulli numbers [2]:

$$B_0 = 1, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30}, \quad B_{10} = \frac{5}{66}, \quad B_{12} = -\frac{691}{2730}, \ldots \qquad (3)$$

In Sec. 2 we show the solution of (2), that is, $p_n(x)$ in terms of the B_{2m} via the complete Bell polynomials [3-9] and the corresponding inversion gives the Bernoulli numbers in terms of Kuznetsov polynomials. The Sec. 3 has an expression to determine the Euler-Mascheroni's constant [10-13].

II. Explicit expression for the Kuznetsov polynomials

The relation (2) can be written in the form:

$$n \frac{p_n(x)}{(2n)!} = \sum_{j=1}^{n} \frac{2^{2j-1}}{(2j)!} |B_{2j}|\, x \frac{p_{n-j}(x)}{(2(n-j))!}, \qquad (4)$$

which has the structure of the recurrence relation studied in [14], therefore:

$$p_n(x) = \frac{(2n)!}{n!} B_n(x_1, x_2, \dots, x_n), \quad x_j = \frac{(j-1)!\, 2^{2j-1}}{(2j)!} |B_{2j}|\, x, \quad j = 1, 2, \dots, n, \tag{5}$$

involving the complete Bell polynomials [3-9]:

$$B_1(x_1) = x_1, \quad B_2(x_1, x_2) = x_1^2 + x_2, \quad B_3(x_1, x_2, x_3) = x_1^3 + 3\, x_1 x_2 + x_3,$$

$$B_4(x_1, \dots, x_4) = x_1^4 + 6\, x_1^2 x_2 + 4\, x_1 x_3 + 3\, x_2^2 + x_4, \tag{6}$$

$$B_5(x_1, \dots, x_5) = x_1^5 + 10\, x_1^3 x_2 + 15\, x_1 x_2^2 + 10\, x_1^2 x_3 + 10\, x_2 x_3 + 5\, x_1 x_4 + x_5, \dots,$$

hence from (3), (5) and (6) we reproduce the results of Kuznetsov [1]:

$$p_1(x) = \frac{x}{3}, \quad p_2(x) = \frac{x}{15}(5x + 2), \quad p_3(x) = \frac{x}{63}(35\, x^2 + 42\, x + 16),$$

$$p_4(x) = \frac{x}{135}(175\, x^3 + 420\, x^2 + 404\, x + 144), \tag{7}$$

$$p_5(x) = \frac{x}{99}(385\, x^4 + 1540\, x^3 + 2684\, x^2 + 2288\, x + 768), \dots$$

On the other hand, the inversion of expressions of the type (5) is given by [15]:

$$|B_{2n}| = -\frac{n}{2^{2n-1}} \sum_{j=1}^{n} \frac{(-1)^j}{j} \binom{n}{j} p_n(j), \tag{8}$$

that is, the polynomials (7) allow determine the absolute value of Bernoulli numbers.

III. Euler-Mascheroni's constant

The work of Kuznetsov [1] has connection with the gamma approximation obtained by Lanczos [16, 17], then it is natural to search for formulae to determine quantities related to the gamma function, for example, the Euler-Mascheroni constant γ_0 [10-13]. We know the following expression of Ulgenes [18]:

$$\Gamma(x) = x^{x-1} \sum_{k=1}^{\infty} (-1)^k \binom{x}{k} \sum_{j=1}^{k} \frac{(-1)^j\, j!}{j^j} \binom{k}{j}, \quad x \geq 1, \tag{9}$$

therefore:

$$\gamma_0 = -1 - \sum_{k=2}^{\infty} (k-2)! \sum_{j=1}^{k} \frac{(-1)^j}{(k-j)!\, j^j}, \tag{10}$$

where were applied the relations [2, 13]:

$$\gamma_0 = -\Gamma'(1), \quad \left[\frac{d}{dx}\binom{x}{r}\right](x = 1) = \begin{cases} 0, & r = 0, \\ 1, & r = 1, \\ \frac{(-1)^r}{r\,(r-1)}, & r \geq 2. \end{cases} \tag{11}$$

Similarly, we have the Hermite's formula [18, 19]:

$$Ln\, \Gamma(1+z) = \sum_{k=2}^{\infty} \binom{z}{k} Ln\left[\frac{k}{(k-1)^{k-1}}\right], \tag{12}$$

therefore:

$$\gamma_0 = \lim_{n\to\infty} \frac{1}{n} \sum_{k=2}^{2n+1} (-1)^k\, Ln\left(\frac{k^{\frac{1}{k}}}{(k-1)^{\frac{k-1}{k}}}\right), \tag{13}$$

where we used the property:

$$\left[\frac{d}{dz}\binom{z}{k}\right](z=0)=\frac{(-1)^{k-1}}{k}\,, \qquad k\geq 1;(14)$$

the formula (13) is alternative to the approximation deduced in [20].

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