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Research Paper

Application of Residue Theorem in Evaluating Integrals of Rational Trigonometric Functions of Order "3 & 4" in the Denominator

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ABSTRACT
In this research work, integral of rational function of sine and cosine of order three (3) and four (4) in the denominator are evaluated over the unit circle using the residue theorem. Transforming sine and cosine of these orders (3 & 4) to the complex field usually give rise to polynomials of degree six and above. In this work, a polynomial root-calculator was used to determine the poles inside the unit circle. The approximated roots were used to calculate the residues inside the unit circle. This process is more complex than that of the lower orders (1 and 2). It shows that the higher the order of sine and cosine, the more the approximation in the roots and the weaker the accuracy of the final value of the integral.
Keywords*: Application; Residue; Integrals of Rational; Trigonometric functions; Denominator*

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I. INTRODUCTION

Complex integration has been advanced by a reason of being evaluated by methods of it in many complicated real and complex integrals in applications. The main methods are: Cauchy theorem, Cauchy integral theorem and the residues theorem. The Cauchy theorem is applicable to the integral of a complex function over a closed contour and states that $\oint_c f(z)dz = 0$ this theorem does not presuppose the likelihood of existence of a singular point inside the region. The Cauchy integral formula $f(x) = \frac{1}{2\pi i}\int_c \frac{f(\xi)}{\xi - z}d\xi$ was later developed to handle isolated singularity inside the region where the function fails to be analytic. However, the big question is 'what happens if there are more than one singular points within and on the contour'? This is where the residues theorem comes to play. The residue theorem sometimes called Cauchy's residue theorem, named after Augustine Louis Cauchy (1789-1857) is a powerful tool for evaluating certain real and complex integrals.

Residue Theorem has been used to evaluate integrals of analytic functions over closed curves. (Meerut, 2006); (Pk Mittal, 2008) and many other authors have used the theorem to evaluate integral of trigonometric functions of order one and two over the unit circle. The theorem states that provided $f(z)$ is analytic at all points inside and on the simple closed contour C, apart from the isolated singularities at Zk(k = 0, 1, ……. n) which is interior to C, then $\oint_C f(z)dz = 2\pi i \sum \mathrm{Re}\, s$ where $\sum \mathrm{Re}\, s$ is the summation of the residues of the poles inside the contour. This shows that the residue theorem extends the Cauchy integral formula to cases where the contour contains a finite number of singularities. Some of the areas of application of residue in evaluating integrals are Rational Trigonometric functions of the form $F(\cos\theta, \sin\theta)$ over a unit circle.

Rational functions over a real line Rational and trigonometric functions over a real line.
Bending Round a Singularity.
Integrand with Branch points.

Residue method is employed particularly when it is not possible to find indefinite integral explicitly. Even in cases where ordinary methods of calculus can be applied, the use of residues often proves to be a labour saving device.We usually want to integrate some real functions which can be extended to the complex domain. It must be recognized that the techniques of complex integration apply to closed curves while a real integral is over an interval. It then follows that we need a device to reduce our problem to one which concerns integration over closed curves. The residue theorem is one such device which combines results from Cauchy theorem and Cauchy integral formula to make our work easy. It is intended to illustrate in this work how residue theorem can be used to evaluate integrals of rational trigonometric functions of order three and four.

II. Materials and Methods

In this chapter we shall restate and prove the residue theorem and examine some basic concepts involved in evaluating integral by means of residue.

1. Residue Theorem

If $f(z)$ be analytic in a simply connected domain D except at a finite number of singular points $z_{1,} z_{2,} \ldots\ldots\ldots\ldots\ldots\ldots z_n$ and be continuous on the boundary of C which is a rectifiable Jordan curve, then $\oint f(z)dz = 2\pi i \sum_{i=1}^{n} \operatorname{Re} s(z_i)$

1. Proof of the Residue Theorem

We enclose the singular points by positively oriented circles $C_{1,} C_2 \ldots\ldots\ldots\ldots\ldots C_n$ Such that the circles along with their interior belong to D and do not overlap. Then $f(z)$ is analytic in the multiply-connected domain D obtained by removing from D these circles and their interiors. The boundary D[1] obtained by removing from D these circles and their interiors also consists of the curves $c, -c_{1,} -c_{2,} \ldots\ldots\ldots\ldots\ldots\ldots c_n$ so that

$$\frac{1}{2\pi i}\int_c f(z)dz + \frac{1}{2\pi i}\int_{-c_1} f(z)dz + \ldots\ldots\ldots\ldots + \frac{1}{2\pi i}\int_{-c_n} f(z)dz = 0$$

$$\frac{1}{2\pi i}\int_c f(z)dz = \frac{1}{2\pi i}\int_{C_1} f(z)dz + \ldots\ldots\ldots\ldots\ldots\ldots + \frac{1}{2\pi i}\int_{C_n} f(z)dz$$

$$= \operatorname{Re} s(z_i) + \ldots\ldots\ldots\ldots\ldots + \operatorname{Re} s(z_n) = \sum_{i=1}^{n} \operatorname{Re} s(z_i)$$

$$\int_c f(z)dz = 2\pi i \sum_{i=1}^{n} \operatorname{Re} s(z_i) \quad \sum \operatorname{Re} s(z_i)$$

Theorem 3: If f has an isolated singularity at a then f has Laurent series expansion near a (punctured neighborhood of a) or in some neighborhood of a (annuli).

Alternatively,

If a function f(z) is analytic throughout a doubly-connected domain D defined by $r < |z-a| < R$, which is a circular ring with centre $''a''$ then it can be represented by an expansion of $f(z)$. This means that we can write

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n.$$

4. Computation of Residues:

Residue computation constitutes an important step in the application of the subject of residue. We shall now see how the residue at a point can be computed.

Suppose that $\frac{\varphi(z)}{\psi(z)}$ is any given function and that $z = a$ is a zero of any order m of $\psi(z)$ but not a zero o $\varphi(z)$. Then a is a pole of order m of the function. We first consider the case of a simple pole in which case $m = 1$. We then have $\psi(z) = (z-a) f(z)$ where $f(a) \neq 0$.

Thus $\frac{\varphi(z)}{\psi(z)} = \frac{1}{z-a}\frac{\varphi(z)}{f(z)}$, [$f(a) \neq 0$]

$= \frac{1}{z-a}F(z)$ [$F(z) = \frac{\varphi(z)}{f(z)}$]

Hence F(z) is analytic at a and $F(a) \neq 0$

By Taylor's theorem, we have in a neighborhood of a, $F(z) = F(a) + (z-a)F'(a) + ...$ so that we have the Laurent series

$$\frac{\varphi(z)}{\psi(z)} = \frac{F(a)}{(z-a)} + F'(a) + \frac{(z-a)}{2!}F''(a) + ...$$

Thus

$$\operatorname{Re}s(a) = F(a) = \frac{\varphi(a)}{f(a)}.$$

We now state this result in a form involving $\varphi(z)$ and $\psi(z)$ only as follows:

The residue of

$$\frac{\varphi(z)}{\psi(z)}$$

At $z = a$ which is a zero of order one of $\psi(z)$ but not a zero of $\varphi(z)$ is

$$\lim_{z\to a}\frac{(z-a)\varphi(z)}{\psi(z)} \; or \; \frac{\varphi(a)}{\psi'(a)}.$$

Since

$$\frac{(z-a)\varphi(z)}{\psi(z)} = \frac{(z-a)\varphi(z)}{(z-a)f(z)} = F(z), \text{ when } z \neq a$$

Therefore,

$$\lim_{z\to a}\frac{(z-a)\varphi(z)}{\psi(z)} = \lim_{z\to a}F(z) = F(a) = \operatorname{Re}s(a).$$

Also

$$\psi'(z) = f(z) + (z-a)f'(z)$$

So that

$$\psi'(a) = f(a)$$

And

$$\frac{\varphi(z)}{\psi'(z)} = \frac{\varphi(a)}{f(a)} = F(a) = \operatorname{Re}s(a).$$

Suppose that a is a pole of any order m, we have then a neighborhood of a

$$\psi(z) = (z-a)^m f(z) \text{ where } f(a) \neq 0$$

Now

$$\frac{\varphi(z)}{\psi(z)} = \frac{1}{(z-a)^m}.\frac{\psi(z)}{f(z)} = \frac{1}{(z-a)^m}F(z), ...(1) \text{ where } F(a) \neq 0$$

By Taylor's theorem we have in some neighborhood of a

$$F(z) = F(a) + (z-a)F'(a) + ... + (z-a)^{m-1}\frac{F^{m-1}(a)}{(m-1)!} + (z-a)^m\frac{F^m(a)}{m!} + ...(2)$$

We see form (1) and (2) that the residue at a of $\varphi(z)/\psi(z)$ is

$$= \frac{F^{m-1}(a)}{(m-1)!}.$$

5. EVALUATION OF INTEGRAL

To evaluate integral of real- valued function using the residues, a number of steps are required

Transform the real valued function into the complex plane using the relation $z = e^{i\theta}$, $\sin\theta = \frac{1}{2i}(z - \frac{1}{z})$ and $\cos\theta = \frac{1}{2}(z + \frac{1}{z})$.

Find the singularities of the function F(z)

Calculate the residues at the poles inside and on the contour.

Multiply the sum of the residues by 2πi and simplify the result.

Numerical Experiments

We have seen the theorems related to residues theorem and their proofs. We shall proceed in this to look at the application of residue theorem in evaluating integrals of rational trig functions within the unit circle.

Examples 1:

1. Find the residue for the function $(z^4+1)z^{-1}(z^2-1)^{-1}, z = 0$.

Solution:

$$\operatorname{Re}s(0) = \lim_{z\to 0}\left[z.\frac{z^4+1}{z(z^2-1)}\right] = -1$$

2. let $f(z) = \frac{1}{z^2(z^2+1)}$, find the residues of all isolated singularities of f .

Solution:

Note that $z^2+1 = (z-i)(z+i).$ thus f has three singularities, $0, i$ and $-i$. 0 is a pole of order 2 and $\pm i$ are poles of the first order.

The residue at 0 can be computed as shown

$$\operatorname{Re}s(0) = \lim_{z\to 0}\frac{d}{dz}\left(z^2 f(z)\right) = \lim_{z\to 0}\frac{d}{dz}\left(\frac{1}{z^2+1}\right) = \lim_{z\to 0}\frac{-2z}{(z^2+1)^2} = 0 .$$

$$\operatorname{Re}s(i) = \lim_{z\to i}(z-i)f(z) = \lim_{z\to i}\left(\frac{1}{z^2(z+i)}\right) = \frac{1}{-1(i+i)} = \frac{-1}{2i} = \frac{i}{2} .$$

$$\operatorname{Re}s(-i) = \lim_{z\to -i}(z+i)f(z) = \lim_{z\to -i}\left(\frac{1}{z^2(z-i)}\right) = \frac{1}{-1(-i-i)} = \frac{1}{2i} = \frac{-i}{2} .$$

Problem 1

Problem 1: Evaluate $\int_{-\pi}^{\pi}\frac{1}{1+3(\cos t)^2}dt$

Solution 1:

Let $z = e^{it}$

Recall that $\cos t = \frac{1}{2}\left(e^{it} + e^{-it}\right) = \frac{1}{2}\left(z + \frac{1}{z}\right)$

and $\frac{dz}{dt} = iz, dt = \frac{dz}{iz}$

Taking c to be a unit circle, we substitute to get

$$\oint_c \frac{1}{1+3\left(\frac{1}{2}\left(z+\frac{1}{z}\right)\right)^2}\frac{dz}{iz}$$

$$=\oint_c \left(\frac{-i}{1+\frac{3}{4}\left(z+\frac{1}{z}\right)^2}\frac{1}{iz}dz\right)$$

$$=\oint_c \frac{-i}{z+\frac{3}{4}z\left(z^2+2z+\frac{1}{z^2}\right)}dz=-i\oint\frac{1}{z+\frac{3}{4}z\left(z^3+2z+\frac{1}{z}\right)}dz$$

$$=-i\oint\frac{1}{\frac{3}{4}z^3+\frac{5}{2}z+\frac{3}{4}z}dz=-i\oint\frac{4}{3z^3+10z+\frac{3}{z}}dz$$

$$=-4i\oint\frac{1}{3z^3+10z+\frac{3}{z}}dz=-4i\oint\frac{z}{3z^4+10z^2+3}dz$$

$$=-4i\oint\frac{z}{3\left(z+\sqrt{3i}\right)\left(z-\sqrt{3i}\right)\left(z+\frac{i}{\sqrt{3}}\right)\left(z-\frac{i}{\sqrt{3}}\right)}dz$$

$$=-\frac{4}{3}i\oint\frac{z}{\left(z+\sqrt{3i}\right)\left(z-\sqrt{3i}\right)\left(z+\frac{i}{\sqrt{3}}\right)\left(z-\frac{i}{\sqrt{3}}\right)}dz$$

The singularities to be considered are $3^{\frac{1}{2}}i$ and $-3^{-\frac{1}{2}}i;$ let c_1 be θ about $3^{\frac{1}{2}}i$ and c_2 be about $-3^{-\frac{1}{2}}i;$ then,

Then, $$-\frac{4}{3}i\left[\oint_{c_1}\frac{z}{\left(z+\sqrt{3i}\right)\left(z-\sqrt{3i}\right)\left(z+\frac{i}{\sqrt{3}}\right)\left(z+\frac{i}{\sqrt{3}}\right)}dz+\oint_{c_2}\frac{\frac{z}{\left(z+\sqrt{3i}\right)\left(z-\sqrt{3i}\right)\left(z-\frac{i}{\sqrt{3}}\right)}}{z+\frac{i}{\sqrt{3}}}dz\right]$$

$$=-\frac{4}{3}i\left[2\pi i\left(\left.\frac{z}{\left(z+\sqrt{3i}\right)\left(z-\sqrt{3i}\right)\left(z-\frac{i}{\sqrt{3}}\right)}\right|_{z=\frac{i}{3}}\right)+2\pi i\left(\left.\frac{z}{\left(z+\sqrt{3i}\right)\left(z-\sqrt{3i}\right)\left(z-\frac{i}{\sqrt{3}}\right)}\right|_{z=-\frac{i}{3}}\right)\right]$$

$$=\frac{8\pi}{3}\left[\frac{\frac{i}{\sqrt{3}}}{\left(\frac{i}{\sqrt{3}}+\sqrt{3i}\right)\left(\frac{i}{\sqrt{3}}+\frac{i}{\sqrt{3}}\right)\left(\frac{i}{\sqrt{3}}-\sqrt{3i}\right)}+\frac{-\frac{i}{\sqrt{3}}}{\left(-\frac{i}{\sqrt{3}}-\sqrt{3i}\right)\left(-\frac{i}{\sqrt{3}}+\sqrt{3i}\right)\left(-\frac{i}{\sqrt{3}}-\frac{i}{\sqrt{3}}\right)}\right]$$

$$=\frac{8\pi}{3}\left[\frac{\frac{i}{\sqrt{3}}}{\left(\frac{4}{\sqrt{3}}i\right)\left(-\frac{2}{\sqrt{3}}\right)\left(\frac{2}{\sqrt{3}}i\right)}+\frac{-\frac{i}{\sqrt{3}}}{\left(\frac{2}{\sqrt{3}}i\right)\left(-\frac{4}{\sqrt{3}}i\right)\left(-\frac{2}{\sqrt{3i}}\right)}\right]$$

$$=\frac{8\pi}{3}\left[\frac{\frac{i}{\sqrt{3}}}{i\left(\frac{4}{\sqrt{3}}\right)\left(\frac{2}{\sqrt{3}}\right)\left(\frac{2}{\sqrt{3}}\right)}+\frac{-\frac{i}{\sqrt{3}}}{-i\left(\frac{2}{\sqrt{3}}\right)\left(\frac{4}{\sqrt{3}}\right)\left(\frac{2}{\sqrt{3}}\right)}\right]$$

$$=\frac{8\pi}{3}\left[\frac{\frac{1}{\sqrt{3}}}{\left(\frac{4}{\sqrt{3}}\right)\left(\frac{2}{\sqrt{3}}\right)\left(\frac{2}{\sqrt{3}}\right)}+\frac{\frac{1}{\sqrt{3}}}{\left(\frac{2}{\sqrt{3}}\right)\left(\frac{4}{\sqrt{3}}\right)\left(\frac{2}{\sqrt{3}}\right)}\right]$$

$$=\frac{8\pi}{3}\left[\frac{\frac{1}{\sqrt{3}}}{\left(\frac{16}{3\sqrt{3}}\right)}+\frac{\frac{1}{\sqrt{3}}}{\left(\frac{16}{3\sqrt{3}}\right)}\right]$$

$$=\frac{8\pi}{3}\left[\frac{3}{16}+\frac{3}{16}\right]$$

$$=\pi$$

Problem 2: Evaluate $\int_0^{2\pi}\frac{d\theta}{5\cos\theta-13}$

Solution 2:

Using the transformation $z=e^{i\theta}, d\theta=\frac{dz}{iz}$

$\cos\theta=\frac{z+\frac{1}{z}}{2}$, we transform the definite integral to a contour of unit circle

$$\therefore\int_0^{2\pi}\frac{d\theta}{5\cos\theta-13}=\oint_c\frac{dz}{5\left(\frac{z+z^{-1}}{2}\right)-13}id\quad=\oint\frac{dz}{5\left(\frac{z^2+1}{2z}\right)-13}.iz$$

$$=-i\oint\frac{2dz}{5z^2-26z+5}\quad=-i\oint\frac{2dz}{5z^2-25z-z+5}\quad=-i\oint\frac{2dz}{5z(z-5)-1(z-5)}$$

$$=-i\oint\frac{2dz}{(5z-1)(z-5)}$$

The poles are $z=\frac{1}{5}$ and $z=5$, $z=\frac{1}{5}$ is inside the circle while $z=5$ is outside

:. Calculating residue at $z=\frac{1}{5}$ we have

$$\operatorname{Re}s\left(f,\frac{1}{5}\right)=\lim_{as z\to\frac{1}{5}}\left\{\left(z-\frac{1}{5}\right)\left(\frac{2}{(5z-1)(z-5)}\right)\right\}$$

$$=\frac{5z-1}{5}\cdot\frac{2}{(5z-1)(z-5)}=\frac{2}{5}\left(\frac{1}{\frac{1}{5}-5}\right)$$

$$\operatorname{Re}s=\frac{2}{5}\left(\frac{1}{\frac{1-25}{5}}\right)=\frac{2}{5}\left(\frac{5}{-24}\right)=-\frac{1}{12}$$

$$\therefore\int_0^{2\pi}\frac{d\theta}{5\cos\theta-13}=\int_c\frac{\frac{dz}{iz}}{5z^2-26z+5}$$

$$=i\int_c\frac{2dz}{(5z-1)(z-5)}=i\times 2\pi i\times\left(-\frac{1}{12}\right)$$

$$=-2\pi\left(-\frac{1}{12}\right)$$

$$=\frac{\pi}{6}.$$

Problem 3: Evaluate $\int_0^{2\pi}\frac{d\theta}{2+\sin\theta}$

Solution 3:

Using, $z=e^{i\theta}, d\theta=\frac{dz}{iz}\, and \sin\theta=\frac{z-z^{-1}}{2i}$

$$\int_0^{2\pi}\frac{d\theta}{2+\sin\theta}=\oint_c\frac{dz}{2+\frac{z-z^{-1}}{2i}\cdot iz}$$

$$=\oint\frac{dz}{2+\frac{z-z^{-1}}{2i}\cdot iz}=\oint\frac{dz}{\frac{4z+z^2-1}{2zi}\cdot iz}$$

$$=\int_c\frac{2dz}{z^2+4zi-1}$$ taking the denominator

$$z+4zi=1$$

$$z^2+4zi+(2i)^2=1+(2i)^2$$ by completing the square

$$(z+2i)^2=1-4=-3$$

$$z+2i=\pm\sqrt{-3}=\pm\sqrt{3i}$$

$$\therefore z=-2i\pm\sqrt{3i}$$

$= -2i + \sqrt{3i}$ or $-2i - \sqrt{3i}$

$-2i + \sqrt{3i}$ is inside the circle while $-2i - \sqrt{3i}$ is outside the circle.

$$\therefore \operatorname{Re}s\left(F, -2i - \sqrt{3i}\right) = \lim_{asz \to -2i+\sqrt{3i}} \left\{ \left(\frac{2}{\left(z + 2i - \sqrt{3i}\right)\left(z + 2i + \sqrt{3i}\right)} \right) \right\}$$

$$\operatorname{Re}s = \lim_{asz \to -2i+\sqrt{3i}} \left\{ \left(\frac{2}{z + 2i + \sqrt{3i}} \right) \right\}$$

$$= \frac{2}{-2i + \sqrt{3i} + 2i + \sqrt{3i}} = \frac{2}{2\sqrt{3i}} = \frac{1}{\sqrt{3i}}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{2 + \sin\theta} = \oint_c \frac{2dz}{\left(z + 2i - \sqrt{3i}\right)\left(z + 2i + \sqrt{3i}\right)}$$

$$= 2\pi i \cdot \frac{1}{\sqrt{3i}}$$

$$= \frac{2\pi}{\sqrt{3}}$$

Problem 4: Evaluate $\int_0^{2\pi} \frac{\sin^2 \theta d\theta}{5 - 4\cos\theta}$

Solution 4:

Let,

$$z = e^{i\theta}, d\theta = \frac{dz}{iz} \; and \; \sin\theta = \frac{z^2 - 1}{2iz}$$

So that,

$$\sin^2\theta = \left(\frac{z^2 - 1}{2iz}\right)^2 = \frac{\left(z^2 - 1\right)^2}{-4z^2}$$

$$\cos\theta = \frac{z^2 + 1}{2z}$$

Transforming into contour integral we have

$$\int_0^{2\pi} \frac{\sin^2 d\theta}{5 - 4\cos\theta} = \oint_c \frac{\frac{\left(z^2 - 1\right)^2}{-4z}}{5 - 4\frac{(z+1)}{2z} \cdot iz} dz$$

$$\oint_c \frac{\frac{\left(z^2 - 1\right)^2}{-4z}}{\frac{10z - 4\left(z^2 + 1\right)}{2z} \cdot iz} dz = -i \oint_c \frac{2\left(z^2 - 1\right)^2}{-4z\left(10z - 4z^2 - 4\right)}$$

$$= \frac{i}{2} \oint \frac{\left(z^2 - 1\right)}{2z^2(4z - 2)(z - 2)} dz$$

the poles are

$$z = 0, z = \frac{1}{2} \; and, z = 2$$

Here, z = 2 is outside the circle

$$\therefore \operatorname{Re} s(f,0) = \lim_{as z\to 0}\left\{ z\frac{(z^2-1)^2}{z^2(4z-2)(z-2)}\right\}$$

$$\therefore \operatorname{Re} s\left(f,\frac{1}{2}\right) = \lim_{as z\to 0}\left\{ z\frac{z(z^2-1)^2}{2z^2(2z-1)(z-2)}\right\}$$

$$= \frac{\left(\frac{1}{4}-1\right)^2}{4\left(\frac{1}{4}-\frac{1}{2}\right)\left(\frac{1}{2}-2\right)} = \frac{-\left(\frac{3}{4}\right)^2}{4\left(-\frac{1}{4}\right)\left(-\frac{3}{2}\right)}$$

$$= \frac{\frac{9}{16}}{\frac{3}{2}} = \frac{9}{16}\times\frac{2}{3} = \frac{3}{8}$$

$$\therefore i\oint_c \frac{(z^2-1)^2}{2z^2(2z-1)(z-2)}dz = -i\times 2\pi i\times\frac{3}{8}$$

$$= \frac{2\pi\times 3}{8} = \frac{3\pi}{4}$$

Problem 5: integrate $\int_0^{2\pi}\frac{\cos 3\theta d\theta}{(5-3\cos\theta)^4}$

Solution 5:

Using the transformation

$$z = e^{i\theta}, d\theta = \frac{dz}{iz}\ and \cos\theta = \frac{z^2+1}{2z}, \cos 3\theta = \frac{z^3+z^{-3}}{2} = \frac{z^6+1}{2z^3}$$

We transform from the real to contour integral as follows:

$$\int_0^{2\pi}\frac{\cos 3\theta}{(5-3\cos\theta)^4} = \oint_c \frac{\frac{z^6+1}{2z^3}}{\left(5-3\frac{(z^2+1)}{2z}\right)^4}\cdot\frac{dz}{iz}$$

$$= \oint_c \frac{z^6+1}{2z^3\left(\frac{10z-3z^2-3}{2z}\right)^4}\cdot\frac{dz}{iz}$$

$$= \oint \frac{z^6+1}{2z^3\frac{(10z-3z^2-3)^4}{16z^4}}\cdot\frac{dz}{iz} = \frac{8}{i}\oint\frac{z^6+1}{(10z-3z^2-3)^4}$$

The poles of

$$-3z^2+10z-3$$

are

$$z = \frac{1}{3} or, z = 3$$

$z = \frac{1}{3}$ is a pole inside the circle

$$\therefore \operatorname{Res}\left(F, \frac{1}{3}\right) = \frac{1}{(4-1)!} \frac{d^3}{dz^3}\left\{\left(z-\frac{1}{3}\right)^4 \cdot \frac{z^6+1}{(-3z+1)^4 (z-3)^4}\right\}$$

$$= \frac{1}{3!} \frac{d^3}{dz^3}\left\{\left(\frac{3z-1}{3}\right)^4 \cdot \frac{z^6+1}{-(3z-1)^4 (z-3)^4}\right\}$$

$$= -\frac{1}{6} \cdot \frac{1}{81} \frac{d^3}{dz^3}\left(\frac{z^6+1}{(z-3)^4}\right)$$

$$= \frac{(z-3)^4 (6z^5) - \{(z^6+1)4(z-3)^3\}}{(z-3)^8} = \frac{(z-3)^3 \{6z^5(z-3) - 4(z^6+1)\}}{(z-3)^8}$$

$$= \frac{(z-3)^3 \{6z^6 - 18z^5 - 4z^6 - 4\}}{(z-3)^8} = \frac{2z^6 - 18z^5 - 4}{(z-3)^5}$$

$$\therefore \frac{1}{6} \cdot \frac{1}{81} \frac{d^2}{dz^2}\left(\frac{2z^6 - 18z^5 - 4}{(z-3)^5}\right)$$

$$= \frac{1}{486} \frac{d}{dz}\left\{\frac{(z-3)^5 (12z^5 - 90z^4) - (2z^6 - 18z^5 - 4)5(z-3)^4}{(z-3)^{10}}\right\}$$

$$= \frac{1}{486} \frac{d}{dz}\left\{\frac{(z-3)^4 [(z-3)(12z^5 - 90z^4) - 5(2z^6 - 18z^5 - 4)]}{(z-3)^{10}}\right\}$$

$$= \frac{1}{486} \frac{d}{dz}\left\{\frac{(z-3)^4 [12z^6 - 90z^5 - 36z^5 + 270z^4 - 10z^6 + 90z^5 + 20]}{(z-3)^{10}}\right\}$$

$$= \frac{1}{486} \frac{d}{dz}\left\{\frac{2z^6 - 36z^5 + 270z^4 + 20}{(z-3)^6}\right\}$$

$$= \frac{1}{486}\left\{\frac{(z-3)^6 (12z^5 - 180z^4 + 1080z^3) - 6(z-3)^5 (2z^6 - 36z^5 + 270z^4 + 20)}{(z-3)^{12}}\right\}$$

$$= \frac{1}{486}\left\{\frac{(z-3)^5 (z-3)(12z^5 - 180z^4 + 1080z^3) - 6(2z^6 - 36z^5 + 270z^4 + 20)}{(z-3)^{12}}\right\}$$

$$= \frac{1}{486}\left\{\frac{12z^6 - 180z^5 + 1080z^4 - 36z^5 + 540z^4 - 3240z^3 - 12z^6 + 216z^5 - 1620z^4 - 120}{(z-3)^7}\right\}$$

$$= \frac{1}{486}\left\{\frac{1620z^4 - 3240z^3 - 120}{(z-3)^7}\right\}$$

$$= -\frac{1}{486}\left\{\frac{-3240\left(\frac{1}{3}\right)^3 - 120}{\left(\frac{1}{3} - 3\right)^7}\right\} = -\frac{1}{486}\left\{\frac{-\frac{3240}{27} - 120}{\left(-\frac{8}{3}\right)^7}\right\} = -\frac{1}{486}\left\{-240 \times \left(\frac{-3^7}{-8^7}\right)\right\}$$

$$= \frac{1}{486}\left\{\frac{240\times 3^7}{8^7}\right\} = \frac{240\times 2187}{486\times 2097152}$$

$$\therefore \frac{8}{i}\oint \frac{z^2+1}{\left(-3z^2+10z-3\right)^4}dz = \frac{8}{i}\times 2\pi i\times \frac{240\times 2187}{486\times 2097152}$$

$$= \frac{8\pi\times 2160}{2097152} = \frac{2160\pi}{262144}$$

$$= \frac{135\pi}{16{,}384}$$

Problem 6: Evaluate $\int_0^{2\pi} \frac{d\theta}{5-3\cos^4\theta}$

Solution 6:

Using the transformation $z = e^{i\theta}, d\theta = \frac{dz}{iz}, \cos\theta = \frac{1}{2}\left(z+z^{-1}\right)$ we move from the real contour integral as follows:

$$\int_0^{2\pi} \frac{d\theta}{5-3\cos^4\theta} = \oint \frac{dz}{iz\left(5-3\left(\frac{z^2+1}{2z}\right)^4\right)}$$

$$= \int \frac{dz}{iz\left(\frac{80z^4-3\left(z^2+1\right)^4}{16z^3}\right)}$$

$$= -i\int \frac{16z^3\,dz}{80z^4-3\left(z^8+4z^6+6z^4+4z^2+1\right)}$$

$$= -i\int \frac{16z^3\,dz}{80z^4-3z^8-12z^6-18z^4-12z^2-3}$$

Solving the polynomial $-3z^8-12z^6+62z^4-12z^2-3=0$ using polynomial calculator gives the roots; $\pm 0.596, \pm 1.675, \pm 0.377i, and, \pm 2.649i$ out of which 0.377 and 0.596 are inside the circle while the rest are lying outside the unit circle.

The residue at these points are :

Res(z,-0.596)=

$$\lim_{as z\to -0\cdot 596}\left[(z+0\cdot 596)\frac{16z^3}{(z+1\cdot 675)(z-1\cdot 675)(z+0\cdot 596)(z-0\cdot 596)(z-0\cdot 377i)(z+0\cdot 377i)(z-2\cdot 649i)(z+2\cdot 649i)}\right]$$

$$= \frac{16(-0\cdot 596)^3}{(-2\cdot 450)(-1\cdot 192)(0\cdot 497)(7\cdot 372)}$$

$$= \frac{-3\cdot 3873}{10\cdot 7000} = -0\cdot 3166$$

Similarly, Res(z, 0.596) = -0.3166

Res(z, -0.377i)=

$$\lim_{z\to -0\cdot 377i}\left[(z+0\cdot 377i)\frac{16z^3}{(z+1\cdot 675)(z-1\cdot 675)(z+0\cdot 596)(z-0\cdot 596)(z+0\cdot 377i)(z-0\cdot 377i)(z+2\cdot 649i)(z-2\cdot 649i)}\right]$$

$$= \frac{16(-0\cdot 377i)^3}{(-0\cdot 377i+1\cdot 675)(-0\cdot 377i-1\cdot 675)(-0\cdot 377i+0\cdot 596)(-0\cdot 377i-0\cdot 596)(-0\cdot 377i-0.377i)(-0.377i\pm 2.649i)}$$

$$= \frac{0.8573i}{(-2.9478)(-0.4973)(6.8751)(-0.754i)}$$

$$= \frac{0.8573i}{-7.5992i} = -0.1128$$

Similarly, Res(z, 0.377i) = 0.1128

Therefore

$$-i\oint \frac{16z^3}{-3z^8-12z^6+62z^4-12z^2-3}dz = -i.2\pi i(0.1128-0.1128-0.3166-0.3166)$$

$$= 2\pi(0.1128-0.7460)$$

$$= -0.6332\pi$$

Problem 7: Evaluate $\int_0^{2\pi}\frac{d\theta}{(2+\cos\theta)^3}$

Solution 7:

Using the transformation $z = e^{i\theta}, d\theta = \frac{dz}{iz}$ and $\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$ we move from the real integral to contour integral as shown

$$\int_0^{2\pi}\frac{d\theta}{(2+\cos\theta)^3} = \oint_c \frac{dz}{iz\left(2+\frac{z^2+1}{2z}\right)^3}$$

$$= -i\oint \frac{dz}{z\frac{(4z+z^2+1)^3}{8z^3}} = -i\oint \frac{8z^2\,dz}{(4z+z^2+1)^3}$$

$$= -8i\oint \frac{z^2\,dz}{(z^2+4z+1)^3}$$

Solving the equation $z^2+4z+1=0$ to obtain the singularities of the denominator we have

$$z^2+4z=-1$$
$$z^2+4z+2^2=-1+2^2=3$$
$$(z+2)^2=3$$
$$z+2=\pm\sqrt{3}, z=-2+\sqrt{3}, or -2-\sqrt{3}$$

The pole $-2+\sqrt{3}$ of order three lies inside the unit circle while $-2-\sqrt{3}$ is outside the circle. The residue at this pole is given by

$$\operatorname{Res}(z,-2+\sqrt{3})=\lim_{z\to-2+\sqrt{3}}\left\{\frac{1}{(n-1)!}\frac{d^{n-1}z}{dz^{n-1}}\left(\left(z-(-2+\sqrt{3})\right)^n\cdot f(-2+\sqrt{3})\right)\right\}, where, n=3$$

$$=\lim_{z\to-2+\sqrt{3}}\left\{\frac{1}{2}\frac{d^2z}{dz^2}\left[\left(z-(-2+\sqrt{3})\right)^3\cdot\frac{z^2}{\left(z-(-2+\sqrt{3})\right)^3\left(z+(-2-\sqrt{3})\right)^3}\right]\right\}$$

$$=\left\{\frac{1}{2}\lim_{z\to-2+\sqrt{3}}\frac{d^2z}{dz^2}\left[\frac{z^2}{\left(z-2+\sqrt{3}\right)^3}\right]\right\}$$

$$=\frac{1}{2}\lim_{z\to-2+\sqrt{3}}\frac{d}{dz}\left[\frac{\left(z-2+\sqrt{3}\right)^3(2z)-3z^2\left(z-2+\sqrt{3}\right)^2}{\left(z-2+\sqrt{3}\right)^6}\right]$$

$$=\frac{1}{2}\lim_{z\to-2+\sqrt{3}}\frac{d}{dz}\left[\frac{\left(z-2+\sqrt{3}\right)^2\left[\left(z-2+\sqrt{3}\right)(2z)-3z^2\right]}{\left(z-2+\sqrt{3}\right)^6}\right]=\frac{1}{2}\lim_{z\to-2+\sqrt{3}}\frac{d}{dz}\left[\frac{-z^2-z(4-2\sqrt{3})}{\left(z-2+\sqrt{3}\right)^4}\right]$$

$$=\frac{1}{2}\lim_{z\to-2+\sqrt{3}}\left[\frac{\left(z-2+\sqrt{3}\right)^4\left(-2z-(4-2\sqrt{3})\right)-4\left(z-2+\sqrt{3}\right)^3\left(-z^2-z(4-2\sqrt{3})\right)}{\left(z-2+\sqrt{3}\right)^8}\right]$$

$$=\frac{1}{2}\lim_{z\to-2+\sqrt{3}}\left[\frac{2z^2-16z-8\sqrt{3}z-8\sqrt{3}+14}{\left(z-2+\sqrt{3}\right)^5}\right]$$

$$=\frac{1}{2}\left[\frac{2\left(-2+\sqrt{3}\right)^2+16(-2+\sqrt{3})-8\sqrt{3}(-2+\sqrt{3})-8\sqrt{3}+14}{\left(-2+\sqrt{3}-2+\sqrt{3}\right)^5}\right]=\frac{1}{2}\left[\frac{16\sqrt{3}-28}{\left(-4+2\sqrt{3}\right)^5}\right]$$

$$=\frac{1}{2}\left(\frac{27.7128-28}{\left(-4+3.4641\right)^5}\right)=\frac{1}{2}\left(\frac{-0.2872}{-0.0442}\right)=3.2489$$

$$\oint_c f(z)dz = 2\pi i \sum \operatorname{Re} s$$

$$-8i\oint \frac{z^2 dz}{\left(z^2+4z+1\right)^3} = -8i \times 2\pi i \times 3.2489$$

$$= 16\pi(3.2489) = 51.98\pi$$

Discussion of Results and Conclusion

III. DISCUSSION OF RESULTS:

Problems 1, 2 and 3:
The standard method of evaluating integrals by residue theorem was applied to each of the questions. The real valued functions are converted to contour integration by the usual parameterization of the unit circle $z = e^{i\theta}$. The poles existing inside the unit circle and on the contour are identified and the residues at those points are calculated. The integrals are evaluated by multiplying the sum of the residues at the poles by $2\pi i$. These are of the usual and more common type which does not pose much challenges to solve.

Problem 4: The power two of sine in the numerator makes it a little more complicated than the previous three. However the solution is obtained through the application of the residue theorem. The higher power is responsible for the number of singular points in line with the fundamental theorem of algebra.

Problem 5: The power four in the denominator is more challenging to handle. Calculating residue at the pole of order four involves evaluating a third order derivative.

The residue theorem was then applied and the final solution followed.

Problem 6: The difference between problem five and that of six is noteworthy though their denominators are of the same degree. The parameterization of the unit circle in this case gives rise to a polynomial of degree eight(8). There is at the moment no established analytical method of solving polynomials of degree six and above. The roots of the polynomial $-3z^8 - 12z^6 + 62z^4 - 12z^2 - 3 = 0$ in the denominator was obtained using 'polynomial root calculator' accessed on the site www.mathportal.org/calculator/polynomial-roots-calculator.php". This means that the roots can only be approximated values. The final result which comes from a combination of approximated imaginary and real roots also shows some unusual behavior (carrying a negative sign).

Problem 7: The procedure involves working with the terms inside the bracket without first expanding with respect to power three. This removes much complications of higher degree polynomial in the denominator as seen in problem six.

IV. CONCLUSION:

We have demonstrated through the solved problems that the residue theorem allows us to evaluate integrals without actually physically integrating but by just knowing the residues contained inside a curve.

Our experiments show that the residue theorem is compliant to evaluating rational trigonometric functions of order three and four. However as the order of the denominator increases above two, it becomes more tasking to obtain exact roots of the resulting higher degree polynomial appearing from the parameterization of the unit circle. This may partly suggest why these higher orders are not commonly found in researched works. The greatest worry is therefore in the area of finding the singularities (poles). The higher the degree of the denominator of the trigonometric function, the less the degree of accuracy of the evaluated integral which is now a product of a series of approximations.

V. RECOMMENDATION FOR FURTHER STUDIES:

It is clear from our work that residue theorem can always be applied to evaluate integral as long as the singularities can be calculated. I wish to recommend to any student that wishes to pick my project to look into the analytic method of determining the zeros of higher degree polynomial to avoid much approximations in the final value of the integrals involving them.

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