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Research Paper

Out-of-plane equilibrium points in the Elliptic restricted three-body problem with triaxial-radiating primaries surrounded by a belt

JagadishSingh[1]and Ndaman Isah[2]

1. *Department of Mathematics, Faculty of Physical Sciences, Ahmadu Bello University, Zaria, Nigeria.*
2. *Department of Mathematics, Faculty of Science, Niger State College of Education, Minna,Nigeria.*

Abstract
This paper studies the motion of an infinitesimal particle near the out-of-plane equilibrium points in the elliptic restricted three body problem when the primaries are triaxial rigid bodies, sources of radiation and surrounded by a belt. Itis observed that there exist two out-of-plane equilibria which lie in the ξζ- plane in symmetrical positions with respect tothe orbital plane.The parameters involved in the system affect theirpositions.The position changes with an increase intriaxiality,radiation and belt.We found that for the binary system the effect of triaxialty and the belt moves the out-of-plane equilibrium points in opposite directions.The position and linear stability of the out-of-plane equilibrium points are investigated numerically using first, arbitrary values for theparameters and then forthe two binary systems (Xi-Bootis and Kruger 60) and they are found to be unstable in each case.

Keywords:*Triaxiality;Radiation;Elliptic restricted three body problem;Stability;gravitationalpoenial from the belt;binarysystems;out-of-planeequilibrium points;Spacedymamics;CelestialMechanics,Langragian Triangular equilibrium points*

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I. Introduction

One of the most important problem in celestial mechanics, is the three-body problem. It has been studied in many scientific researches, especially in the field ofastrodynamics and astrophysics.Reknowned mathematicians and scientists have produced interesting and significant results in an attempt to understand and predict the motion of natural bodies.

The restricted three-body problem is a configuration involving two massive bodies called the primaries and a particle of negligible mass,called the third body (infinestimalparticle,testparicle).It describes the motion of theinfinestimal particle in the vicinity of the primaries which move in circular or elliptic orbits around their common centre of mass due to their mutual gravitational attraction.It possesses three collinear points $L_{1,2,3}$ and two triangular points $L_{4,5}$. They lie on the orbital planeof motionof the primaries. The latter are stable, while the former areunstable.The restricted three-body problem is called ellipticrestricted three-body problem (ER3BP)if the primaries move in elliptic orbit around their common centre of mass and cicularrestricted three-body problem (CR3BP) if the primaries move in circular orbits around their common centre of mass.There are several communications in both ER3BP and CR3BP.In the classicalCR3BP only gravitational forces influence the motion of the particle.Thephotogravitational R3BPproblem arises when one of the participating bodies or both are intense emitters of radiation. It is inadequate to consider only the gravitational force in some solar or stellar dynamic problems. For instance, gravity is not the only dominant force present when a star collides with a particle, but also the repulsive forces of radiation pressure(Radsviesky,1950).Thefore, the potential function of the CR3BP was amended so as to admit to other pertubing forces such as radiation,tiaxiality,oblateness and so on. These have enable several researchers to propose different models under different characterisations.For instance (Narayan,et.al;2015;Danby,1964;Capdvilla,2018;Umar and Hussain, 2016; Singh and Umar2012a) have carried out detailed investigation in CR3BP or ER3BP on the existence of the collinear and non-collinear (triangular)equilibirium points and the stability of motion aroundthese points in ξη-plane and it was found to exist and the triangular equilibirium pointsare conditionally stable,when $0<\mu<\mu_c$ and unstable for $\mu_c \leq \mu \leq \frac{1}{2}$, where μ_c is the critical massratio; while the colliear equilibrium points are unstable.The existence of out-of-plane equilibrium points (OPEPs) was first pointed out by Radviesky (1950,1953)when studying the case of sun planet-particle and Galaxy-kernel-sun-particle and found the two equilibrium points $L_{6,7}$ on the $\xi\zeta$- plane to be symmetrical with respect to the $\xi\eta$-plane.Since thenseveral authors (Daset.al. 2009;Doukos and Markellos 2006; Singh 2012;Singh and Umar, 2013a;Singh and Vicent 2016) based theirstudies on the Radviesky Model under different characterisations in CR3BP or ER3BP.

On the other hand (Shankara et. al. 2011;Singh andAmuda 2015;Chakraborty and Narayan 2018;Zotos 2018)have studied the out-of-plane points in the CR3BP or ER3BP using different models under the influence of radiation pressure or Pr-drag or oblateness or in combination of one of these forces and they found the OPEPs to be unstable. The basins of attraction around the OPEPs in the Copenhagen R3BP was determined by (Zotos, 2018) using a multivariate version of the Newton-Raphson interactive method around the OPEPs when the primaries are oblate.(Doukos and Markellos,2006) obtained OPEPs analytically and then numerically by approximation with power series expansion about the smaller primary,when one of the primaries is oblate and the other radiating and when one or two of the primaries are oblate andproved that theOPEPs exist, but they are unstable.Four additional OPEPs were obtained as result of the oblateness of the primaries.Authors like (Singh and Umar 2013a, Hussain and Umar 2019,Charkraborty and Narayan, 2018) extend these results into the ER3BP,when one or the two primaries are oblate with or without radiation pressure and found that OPEPs exist but are unstable.A generalized out-of-plane model studied (Hussainand Umar, 2019)in which the primary is oblate and the secondary is triaxial and radiating in the ER3BP, shows that theOPEPs ($L_{6,7}$) are affected by the oblateness of the primary, radiation pressure and triaxiality of the secondary, semi-major axis and eccentricity.Also,(Singh and Umar,2013a) found that the position and stability of out-of-plane points are greatly affected by oblateness and radiation pressure of the primaries and the eccentricity of theorbits.Our work is a modified form of (Singh and Umar,2013a) with radiating-triaxial primaries and a potential of the belt in the framework of ER3BP.This work to the best of our knowledge does not yet exist in the literature.The OPEPs has not yet been extensively researched,hence works devoted to it arefew.Only recently,(Vicent,2022) presented a paper on OPEPs where the primaries are radiating with effective Poynting-Robertson drag force with small perturbation in corolis and centrifugal forces and obtained four OPEPs ($L_{6,7,8,9}$) out of which two $L_{6,7}$ are stable in the absence of P-R drag.

Interest in binary systems has increased, in the last decade,this is in part because many extra solar planetary systemsrevealed the presence of belts of dust particles that are regarded as the young analogues of Kuiper belt.(Aumman et al.,1984) and(Jiang andYeh,2003)suggest the position of the disc relative to the planets when they studied the effects of belts on planetary orbits and conclude that the planets might prefer to stay near the inner part instead of outer part of the belt. Later the R3BP was modified in their paper (Jiang and Yeh,2004)to include the effect of additional gravitational force from the belt on the infinitesimal mass, which results in the formation of new libration points.

The studies conducted on belt focus more on motion of the particle around triangular equilibrium points very few articles are available in OPEPs.The model by (Singh andTaura,2014a)focus on the CR3BP when the two primaries are oblate spheroids and radiating with the gravitational potential from a belt. They obtained in addition to the usual five libration points two new collinear points as a result of the potential from the belt. The influence of the belt and non-sphericity of the primaries on the infinitesimal mass was studied by(Singh andTaura,2014c).They did analytic and numerical treatment of motion of a dust grain particle around triangular equilibrium points when the bigger primary is triaxial and the smaller one an oblate spheroid with a potential from the belt.They found that triangular points are stable for $0<\mu<\mu_c$ and unstable for $\mu_c \leq \mu \leq \frac{1}{2}$, where μ_c is the critical mass ratio. It was also observed that the potential from the belt increase the range of stability.

In another study by (Singh andAmuda,2019) where the more massive primary is a triaxial body and less massive one an oblate spheroid emitting radiation enclosed by a circumbinary disc (belt) in the presence of Pr- drag force it was proved thatthe potential from the belt is a stabilizing force as it can change an unstable condition to a stable one even when the mass parameter exceeds the critical mass value $(\mu > \mu_c)$.

In this paper we investigate the effect of trixiality,radiation pressure and the potentialof the belt on a test particle around the OPEPs in the framework of ER3BP.

This paper is organized in 6 sections. The first section is introduction, the equations of motion are described in section 2, locations of equilibrium points can be found in section 3, while section 4 contains thelinear stability analysis of the out-of-plane equilibrium points usingnumerical applications,section 5 is discussion and finally section 6 is conclusion.

II. Equation of Motion

The equation of motion of an infinitesimal particle in the ER3BP when the primaries are triaxial and radiating, with a gravitational potential from the belt, in a dimensionless rotating coordinate system (ξ, η, ζ)following (Singh and Umar,2013a)are as follows:

$\xi'' - 2\eta' = \Omega_\xi$

$\eta'' + 2\xi' = \Omega_\eta$

$\zeta'' = \Omega_\zeta$ (1)

$$\Omega = (1-e^2)^{-1/2}\left[\frac{1}{2}(\xi^2+\eta^2) + \frac{1}{n^2}\left\{\frac{(1-\mu)q_1}{r_1} + \frac{(1-\mu)(2\sigma_1-\sigma_2)q_1}{2r_1^3} - \frac{3(1-\mu)(\sigma_1-\sigma_2)q_1\eta^2}{2r_1^5} - \frac{3(1-\mu)\sigma_1 q_1\zeta^2}{2r_1^5} + \frac{\mu q_2}{r_2} +\right.\right.$$

μ2σ3−σ4q22r23−3μσ3−σ4q2η22r25− 3μσ3q2ζ22r25+ Mbr2+c+ζ2+d2212 (2)

$r_1^2 = (\xi+\mu)^2 + \eta^2 + \zeta^2$

$r_2^2 = = (\xi+\mu-1)^2 + \eta^2 + \zeta^2$ (3)

$$n^2 = \frac{1}{a}\left[1 + \frac{3}{2}e^2 + \frac{3}{2}(2\sigma_1-\sigma_2) + \frac{3}{2}(2\sigma_3-\sigma_4) + \frac{2M_b r_c}{\left[r_c^2+T^2\right]^{3/2}}\right] \quad (4)$$

The effect of the gravitational potential of the belt is expressed using a model that explains a flattened potential and which best describes the gravitational potential within a system given by(Miyamoto and Nagai, 1975) as:

$$V(r,\zeta) = \frac{M_b}{\sqrt{r^2+\left(c+\sqrt{\zeta^2+d^2}\right)}}$$

(5)

r is the radial distance of the infinitesimal mass and is given by $r^2 = \xi^2+\zeta^2$, where c and d are the parameters which determine the density profile of the belt(Miyamoto and Nagai, 1975) and (Kushvah,2008)$r_c$is the distance of any out-of-plane point from the origin and T is their sum,r_1 and r_2 are distances of the bigger and smaller primaries from the infinitesimal particle,respectively.q_1 and q_2 are their mass reduction factor (radiation factor),while (σ_1, σ_2) and (σ_3, σ_4) denote their triaxiality, respectively. n is the mean motion, a and e are the semi major axis and the eccentricity of the elliptic orbis respectively.

III. Location of out-of-plane equilibrium points

The equilibrium points are the solutions of the system of equations$\Omega_\xi = \Omega_\eta = \Omega_\zeta = 0$

$$\Omega_\xi = \left[\xi - \frac{1}{n^2}\left\{\frac{(1-\mu)(\xi+\mu)q_1}{r_1^3} + \frac{3(1-\mu)(\xi+\mu)(2\sigma_1-\sigma_2)q_1}{2r_1^5} - \frac{15(1-\mu)(\xi+\mu)\sigma_1}{2r_1^7}q_1\eta^2 - \frac{15(1-\mu)(\xi+\mu)\sigma_1 q_1\zeta^2}{2r_1^7} + \frac{\mu(\xi+\mu-1)q_2}{r_2^3} +\right.\right.$$

3μξ+μ−12σ3−σ4q22r25−15μξ+μ−1σ32r27q2η2− 15μξ+μ−1σ3q2ζ22r27+Mb ξξ2+c+ζ2+d223/2=0 (5)

$$\Omega_\eta = (1-e^2)^{-1/2}\,\eta\left[1 - \frac{1}{n^2}\left\{\frac{(1-\mu)q_1}{r_1^3} + \frac{3(1-\mu)(2\sigma_1-\sigma_2)q_1}{2r_1^5} + \frac{3(1-\mu)(\sigma_1-\sigma_2)}{r_1^5}q_1 - \frac{15(1-\mu)(\sigma_1-\sigma_2)}{2r_1^7}q_1\eta^2 -\right.\right.$$

151−μσ1q1ζ22r17+ μq2r23+ 3μ2σ3−σ4q22r25+3μσ3−σ4r25q2−15μσ3−σ42r27q2η2− 15μ σ3q2ζ22r27+Mbξ2+c+ζ2+d223/2 = 0

(6)

$$\Omega_\zeta = (1-e^2)^{-1/2}\Bigg[-\frac{\zeta}{n^2}\Bigg\{\frac{(1-\mu)q_1}{r_1{}^3}+\frac{3(1-\mu)(2\sigma_1-\sigma_2)}{2r_1{}^5}q_1+\frac{3(1-\mu)\sigma_1}{r_1{}^5}q_1-\frac{15(1-\mu)(\sigma_1-\sigma_2)}{2r_1{}^7}q_1\eta^2-\frac{15(1-\mu)\sigma_1 q_1\zeta^2}{2r_1{}^7}+$$

μq2r23+ 3μ2σ3−σ42r25q2+3μσ3r25q2−15μσ3−σ42r27q2η2−15μσ3q2ζ22r27+MbCζ2+d2−12+1ξ2+c+ζ2+d223/2 = 0 (7)

The out-of-plane equilibrium points are the solution of above equations, when

$\xi \neq 0, \quad \eta = 0 \quad and \quad \zeta \neq 0$

From (7) with $\zeta \neq 0$ we get:

$$\frac{(1-\mu)q_1}{r_1{}^3}+\frac{3(1-\mu)(2\sigma_1-\sigma_2)}{2r_1{}^5}q_1+\frac{3(1-\mu)\sigma_1}{r_1{}^5}q_1-\frac{15(1-\mu)\sigma_1 q_1\zeta^2}{2r_1{}^7}+\frac{\mu q_2}{r_2{}^3}+\frac{3\mu(2\sigma_3-\sigma_4)}{2r_2{}^5}q_2+\frac{3\mu\sigma_3}{r_2{}^5}q_2-\frac{15\mu\sigma_3 q_2\zeta^2}{2r_2{}^7}$$

$$+\frac{M_b\left[C(\zeta^2+d^2)^{-1/2}+1\right]}{\left[\xi^2+\left(c+\sqrt{\zeta^2+d^2}\right)^2\right]^{3/2}}=0 \tag{8}$$

Let $Q_1 = (1-\mu)q_1$ and $Q_2 = \mu q_2$, then (8) becomes:

$$\frac{Q_1}{r_1{}^3}+\frac{3Q_1(2\sigma_1-\sigma_2)}{2r_1{}^5}+\frac{3Q_1\sigma_1}{r_1{}^5}-\frac{15Q_1\sigma_1\zeta^2}{2r_1{}^7}+\frac{Q_2}{r_2{}^3}+\frac{3Q_2(2\sigma_3-\sigma_4)}{2r_2{}^5}+\frac{3Q_2\sigma_3}{r_2{}^5}-\frac{15Q_2\sigma_3\zeta^2}{2r_2{}^7}+\frac{M_b\left[C(\zeta^2+d^2)^{-1/2}+1\right]}{\left[\xi^2+\left(c+\sqrt{\zeta^2+d^2}\right)^2\right]^{3/2}}=0 \tag{9}$$

Also from Equation (5) we write:

$$n^2\xi-\frac{Q_1(\xi+\mu)}{r_1{}^3}-\frac{3Q_1(\xi+\mu)(2\sigma_1-\sigma_2)}{2r_1{}^5}+\frac{15Q_1(\xi+\mu)\sigma_1\zeta^2}{2r_1{}^7}-\frac{Q_2(\xi+\mu-1)}{r_2{}^3}-\frac{3Q_2(\xi+\mu-1)(2\sigma_3-\sigma_4)}{2r_2{}^5}+\frac{15Q_2(\xi+\mu-1)\sigma_3\zeta^2}{2r_2{}^7}-\frac{M_{b\xi}}{\left[\xi^2+\left(c+\sqrt{\zeta^2+b^2}\right)^2\right]^{3/2}} \tag{10}$$

ExpandingEquation (10) we obtained:

$$\xi\left\{1-\frac{1}{n^2}\left(\frac{Q_1}{r_1{}^3}+\frac{3Q_1(2\sigma_1-\sigma_2)}{2r_1{}^5}-\frac{15Q_1\sigma_1\zeta^2}{2r_1{}^7}+\frac{Q_2}{r_2{}^3}+\frac{3Q_2(2\sigma_3-\sigma_4)}{2r_2{}^5}-\frac{15Q_2\sigma_3\zeta^2}{2r_2{}^7}+\frac{M_b}{\left[\xi^2+\left(c+\sqrt{\zeta^2+d^2}\right)^2\right]^{3/2}}\right)\right\}-$$

$$\frac{\mu}{n^2}\left(\frac{Q_1}{r_1{}^3}-\frac{15Q_1\sigma_1\zeta^2}{2r_1{}^7}+\frac{Q_2}{r_2{}^3}-\frac{15Q_2\sigma_3\zeta^2}{2r_2{}^7}+\frac{3Q_1(2\sigma_1-\sigma_2)}{2r_1{}^5}+\frac{3Q_2(2\sigma_3-\sigma_4)}{2r_2{}^5}\right)+\frac{1}{n^2}\left(\frac{Q_2}{r_2{}^3}+\frac{3Q_2(2\sigma_3-\sigma_4)}{2r_1{}^5}-\frac{15Q_2\sigma_3\zeta^2}{2r_2{}^7}\right)=0 \tag{11}$$

From (9) we have

$$\frac{15Q_1(\sigma_1-\sigma_2)\zeta^2}{2r_1{}^7}+\frac{15Q_2(\sigma_3-\sigma_4)\zeta^2}{2r_2{}^7}=\frac{Q_1}{r_1{}^3}+\frac{3Q_1(2\sigma_1-\sigma_2)}{2r_1{}^5}+\frac{3Q_1\sigma_1}{r_1{}^5}+\frac{Q_2}{r_2{}^3}+\frac{3Q_2(2\sigma_3-\sigma_4)}{2r_2{}^5}+\frac{3Q_2\sigma_3}{r_2{}^5}+\frac{M_b\left[c(\zeta^2+d^2)^{-1/2}+1\right]}{\left[\xi^2+\left(c+\sqrt{\zeta^2+d^2}\right)^2\right]^{3/2}}$$

$$\zeta^2 = \frac{2r_1{}^7 r_2{}^7}{15Q_1(\sigma_1-\sigma_2)r_2{}^7+15Q_2(2\sigma_3-\sigma_4)r_1{}^7}\left\{\frac{Q_1}{r_1{}^3}+\frac{3Q_1(2\sigma_1-\sigma_2)}{2r_1{}^5}+\frac{Q_2}{r_2{}^3}+\frac{3Q_2(2\sigma_3-\sigma_4)}{2r_2{}^5}+\frac{3Q_1\sigma_1}{r_1{}^5}+\frac{3Q_2\sigma_3}{r_2{}^5}+\frac{M_b\left[c(\zeta^2+d^2)^{-1/2}+1\right]}{\left[\xi^2+\left(c+\sqrt{\zeta^2+d^2}\right)^2\right]^{3/2}}\right\} \tag{12}$$

Substituting Equation (9) into Equation (11) and solvingwe obtained:

$$\xi\left\{1-\frac{1}{n^2}\left(-\frac{3Q_1\sigma_1}{{r_1}^5}-\frac{3Q_2\sigma_3}{{r_2}^5}+\frac{M_b}{\left[\xi^2+\left(c+\sqrt{\zeta^2+d^2}\right)^2\right]^{3/2}}-\frac{M_b\left[c+(\zeta^2+d^2)^{-1/2}+1\right]}{\left[\xi^2+\left(c+\sqrt{\zeta^2+d^2}\right)^2\right]^{\frac{3}{2}}}\right)\right\}$$

$$-\frac{\mu}{n^2}\left(-\frac{3Q_1\sigma_1}{{r_1}^5}-\frac{3Q_2\sigma_3}{{r_2}^5}-\frac{M_b\left[c+(\zeta^2+d^2)^{-1/2}+1\right]}{\left[\xi^2+\left(c+\sqrt{\zeta^2+d^2}\right)^2\right]^{3/2}}\right)$$

$$+\frac{1}{n^2}\left(-\frac{Q_1}{{r_1}^3}-\frac{3Q_1(2\sigma_1-\sigma_2)}{2{r_1}^5}-\frac{3Q_1\sigma_1}{{r_1}^5}-\frac{3Q_2\sigma_3}{{r_2}^5}+\frac{15Q_1\sigma_1\zeta^2}{2{r_1}^7}-\frac{M_b\left[c+(\zeta^2+d^2)^{-1/2}+1\right]}{\left[\xi^2+\left(c+\sqrt{\zeta^2+d^2}\right)^2\right]^{3/2}}\right)=0$$

i.e.
$$\xi=\frac{\frac{Q_1}{{r_1}^3}+\frac{3Q_1(2\sigma_1-\sigma_2)}{2{r_1}^5}+\frac{3Q_1\sigma_1(1-\mu)}{{r_1}^5}+\frac{3Q_2Q_1\sigma_3}{{r_2}^5}-\frac{15Q_1\sigma_1\zeta^2}{2{r_1}^7}+\frac{M_b\,Q_1\left[c+(\zeta^2+d^2)^{-1/2}+1\right]}{\left[\xi^2+\left(c+\sqrt{\zeta^2+d^2}\right)^2\right]^{3/2}}}{n^2+\frac{3Q_1\sigma_1}{{r_1}^5}+\frac{3Q_2\sigma_3}{{r_2}^5}+\frac{M_b}{\left[\xi^2+\left(c+\sqrt{\zeta^2+d^2}\right)^2\right]^{3/2}}++\frac{M_b\left[c\,(\zeta^2+d^2)^{-1/2}+1\right]}{\left[\xi^2+\left(c+\sqrt{\zeta^2+d^2}\right)^2\right]^{3/2}}}$$

$$\xi=\frac{(1-\mu)\left\{\frac{1}{{r_1}^3}+\frac{3(2\sigma_1-\sigma_2)}{2{r_1}^5}+\frac{3Q_1\sigma_1}{{r_1}^5}+\frac{3Q_2\sigma_3}{{r_2}^5}-\frac{15\sigma_1\zeta^2}{2{r_1}^7}+\frac{M_b\left[c+(\zeta^2+d^2)^{-1/2}+1\right]}{\left[\xi^2+\left(c+\sqrt{\zeta^2+d^2}\right)^2\right]^{3/2}}\right\}}{n^2+\frac{3Q_1\sigma_1}{{r_1}^5}+\frac{3Q_2\sigma_3}{{r_2}^5}+\frac{M_b}{\left[\xi^2+\left(c+\sqrt{\zeta^2+d^2}\right)^2\right]^{3/2}}++\frac{M_b\left[c\,(\zeta^2+d^2)^{-1/2}+1\right]}{\left[\xi^2+\left(c+\sqrt{\zeta^2+d^2}\right)^2\right]^{3/2}}} \quad (13)$$

We use the initial approximation $\xi_o = (1-\mu)$ and $\zeta_o = \sqrt{3(2\sigma_3-\sigma_4)}$ to obtain the positions of out-of-plane points $L_{6,7}$numerically with the aid of the software package mathematica 10.4 in the form of power series to third order term in ($2\sigma_3$- σ_4) from (12) and (13) as: (see Duokos and Markellos2006;Singh and Umar 2013a):

$$\xi_o=\frac{1}{2a\mu\,q_2}\left\{\left[(-1+\mu)-3\sqrt{3}(-1+\mu)(2+3e^2-2aq_1)\right]\right.$$
$$\left.+\left[(2\sigma_2-\sigma_1)(3-3aq_1)(2\sigma_3-\sigma_4)^{3/2}\right]\right\}$$
$$-\frac{1}{4a\mu\,q_2}\left\{\left[9\sqrt{3}\ (-1+\mu)\ (2+3a\,(2+15(2\sigma_2-\sigma_1)q_1)\ (2\sigma_3-\sigma_4)^{5/2}\right]-[27\ (-1+\mu)(2+3e^2-\right.$$
2aq1+2σ1−σ23−3aq1(−2−3e2+2σ2−σ1−3+6a
$$\left.(-1+\mu)q_1)))(4(a^2\mu^2{q_2}^2)^{-1}(2\sigma_3-\sigma_4)^3+\ 0\ (2\sigma_3-\sigma_4)^{7/2}\right]\right\} \quad (14)$$

$$\zeta_o=\sqrt{3}\sqrt{(2\sigma_3-\sigma_4)}-\frac{9(-1+\mu)(2+9(2\sigma_1-\sigma_2)q_1}{10\mu q_2}(2\sigma_3-\sigma_4)^2$$
$$+\frac{81(-1+\mu)}{20\mu q_2}(2+25)(2\sigma_1-\sigma_2)q_1(2\sigma_3-\sigma_4)^3-0\ (2\sigma_3-\sigma_4)^{7/2} \quad (15)$$

Theequilibrium points $(\xi_0, 0, \pm\zeta_0)$ given by equations (14) and (15) are called the out-of-plane equilibrium points and are denoted by L_6 and L_7 respectively.

IV. Linear stability of out-of-plane equilibrium points

The stability or instability of these equilibrium points are determined by the eigen-values of the characteristic equation (16). If all the characteristic roots(λ_i (i=1,2,3,4,5,6)) are pure imaginary roots or complex roots with negative real parts the equilibrium point will be stable otherwise it will be unstable.
The characteristic equation of the system near any one of the out-of-plane points can be written as:

$$\lambda^6 + \left(4 - \Omega^0{}_{\xi\xi} - \Omega^0{}_{\eta\eta} - \Omega^0{}_{\zeta\zeta}\right)\lambda^4 + \left(\Omega^0{}_{\eta\eta}\Omega^0{}_{\zeta\zeta} + \Omega^0{}_{\xi\xi}\Omega^0{}_{\zeta\zeta} + \Omega^0{}_{\xi\xi}\Omega^0{}_{\eta\eta} - 4\Omega^0{}_{\zeta\zeta} - (\Omega^0{}_{\xi\zeta})^2\right)\lambda^2 - \left(\Omega^0{}_{\xi\xi}\Omega^0{}_{\eta\eta}\Omega^0{}_{\zeta\zeta} - (\Omega^0{}_{\xi\zeta})^2\Omega^0{}_{\eta\eta}\right) = 0 \qquad (16)$$

The superscript O denotes that the partial derivatives are evaluated atthe out-of-plane point (ξ_o, o, ζ_o) where we have:

$$\Omega^0{}_{\xi\xi} = (1-e^2)^{-1/2}\left[1 + \frac{1}{n^2}\left\{\frac{3Q_1(\xi_o+\mu)^2}{r_{10}{}^3} - \frac{Q_1}{r_{10}{}^3} + \frac{15Q_1(\xi_o+\mu)^2(2\sigma_1-\sigma_2)}{2r_{10}{}^7} - \frac{3Q_1}{2r_{10}{}^5} + \frac{105Q_1(\xi_o+\mu)^2\sigma_1\zeta_o{}^2}{2r_{10}{}^9} - \frac{15Q_1\sigma_1\zeta_o{}^2}{2r_{10}{}^7} + \right.\right.$$

3Q2ξo+μ−12r205−Q2r205+15Q2ξo+μ−122σ3−σ42r207−3Q22r205−105Q2ξo+μ−12σ3ζo22r209+ 15Q2σ3ζo22r207+ 3Mbξ02ξ02+c+ζ02+d225/2− Mbξ02+c+ζ02+d223/2(17)

$$\Omega^0{}_{\eta\eta} = (1-e^2)^{-1/2}\left[1 - \frac{1}{n^2}\left\{\frac{Q_1}{r_{10}{}^3} + \frac{3Q_1(2\sigma_1-\sigma_2)}{2r_{10}{}^5} - \frac{15Q_1\sigma_1\zeta_o{}^2}{2r_{10}{}^7} + \frac{Q_2}{r_{20}{}^3} + \frac{3Q_2(2\sigma_3-\sigma_4)}{2r_{20}{}^5} - \frac{15Q_2\sigma_3\zeta_o{}^2}{2r_{20}{}^7} - \frac{3M_b\eta_0{}^2}{\left[\xi_0{}^2+\left(c+\sqrt{\zeta_o{}^2+d^2}\right)^2\right]^{\frac{5}{2}}} + \right.\right.$$

Mbξ02+c+ζo2+d223/2 (18)

$$\Omega^0{}_{\zeta\zeta} = (1-e^2)^{-1/2}\left[\frac{1}{n^2}\left\{-\frac{Q_1}{r_{10}{}^3} + \frac{3Q_1\zeta_o{}^2}{r_{10}{}^5} - \frac{3Q_1(2\sigma_1-\sigma_2)}{2r_{10}{}^5} + \frac{15Q_1(2\sigma_1-\sigma_2)\zeta_o{}^2}{2r_{10}{}^7} - \frac{3Q_1\sigma_1}{r_{10}{}^5} + \frac{15Q_1\sigma_1\zeta_o{}^2}{r_{10}{}^7} + \frac{45Q_1\sigma_1\zeta_o{}^2}{2r_{10}{}^7} - \right.\right.$$

*105Q1σ1ζo42r109−Q2r203+3Q2ζo2r205−
3Q22σ3−σ42r205+15Q22σ3−σ4ζo22r207−3Q2σ3r205+15Q2σ3ζo22r207+45Q2σ3ζo22r207−105Q 2σ3ζo42r209−Mbcζ02+d2−12+1ξ02+c+ζ02+d2232+Mbc2ζ02ζ02+d2−32ξ02+c+ζ02+d2232+3M bζ02cζ02+d2−12 +12ξ02+c+ζ02+d2252*
(19)

$$\Omega^0{}_{\xi\zeta} = (1-e^2)^{-1/2}\left[\frac{3\zeta_0}{n^2}\left\{\frac{Q_1(\xi_o+\mu)}{r_{10}{}^5} + \frac{5Q_1(\xi_o+\mu)(2\sigma_1-\sigma_2)}{2r_{10}{}^7} - \frac{35Q_1(\xi_o+\mu)\sigma_1\zeta_o{}^2}{2r_{10}{}^9} - \frac{15Q_1\sigma_1\zeta_o{}^2}{r_{10}{}^7} + \frac{Q_2(\xi_o+\mu-1)}{r_{20}{}^5} + \frac{5Q_2(\xi_o+\mu-1)(2\sigma_3-\sigma_4)}{2r_{20}{}^7} - \frac{35Q_2(\xi_o+\mu-1)\sigma_3\zeta_o{}^2}{2r_{20}{}^9} + \frac{15Q_2\sigma_3\zeta_o{}^2}{r_{20}{}^7} + \frac{3M_b\xi_0{}^2\left[c\left(\zeta_0{}^2+d^2\right)^{-1/2}+1\right]}{\left[\xi_0{}^2+\left(c+\sqrt{\zeta_0{}^2+d^2}\right)^2\right]^{5/2}} \right\}\right] \qquad (20)$$

V. Numerical Application

We present the effect of triaxiality,belt andradiaionpressure on the locations (Eqns.14 and 15) and stability (Eqns.16-20) of OPEPsusing arbitrary valuesIn Table 1- 4 ,while in Table 6-9the effects on thebinary system (xi-Bootis andKruger 60) are shown.The results inTable 6-9 were obtained by substitutingthe values of theorbital parametres (fixed) of the binary system (xi-Bootis andKruger 60) and the varied values of triaxiality and radiation into (Eqns.14 and 15) and (Eqns.16-20) for thelocationsand stability respectively.

Table 1: The effect of Triaxiality on the location and stability of out-of-plane equilibrium points for e =0.3, a =0.87, $\mu = 0.45 q_1$ =0.9988, q_2 =0.9977, M_b = 0.01

	Triaxiality				Out-of plane points		Roots of the characteristic equation		
S/no	σ_1	σ_2	σ_3	σ_4	ξ	$\pm\zeta$	$\lambda_{1,2}$	$\lambda_{3,4}$	$\lambda_{5,6}$
1.	0.00	0.00	0.00	0.00	0.521012	0.311723	± 87.4420	120.2235	$\pm 33.9675i$
2.	0.02	0.015	0.003	0.002	0.477810	0.269001	± 87.5631	± 120.5631	$\pm 34.12654i$
3.	0.03	0.019	0.004	0.003	0.49061	0.24443	± 87.6615	± 120.9985	±34.241001i
4.	0.04	0.02	0.005	0.004	0.517438	0.219650	± 88.43423	± 121.43423	$\pm 35.35790i$
5	0.05	0.03	0.006	0.005	0.539144	0.209341	± 88.9780	± 122.110	± 35.41283

Table2 : The effect of belt on the location and stability of out-of-plane equilibrium points for e =0.3, a =0.87, $\mu = 0.45 q_1$ =0.9988, q_2 =0.9977

S/no	M_b	Out-of plane points		Roots of the characteristic equation		
		ξ	$\pm\zeta$	$\lambda_{1,2}$	$\lambda_{3,4}$	$\lambda_{5,6}$
1	0.01	0.06735	0.741593	−2.364473±0.364473i	±1.470823i	2.364473± 0.364473i
2	0.02	0.04894	0.73965	−6.243416± 0.765014i	±14.51723i	6.243416± 0.765014i
3	0.03	0.03646	0.72671	−5.459825± 0.886517i	±13.44601i	5.459825± 0.886517i
4	0.04	0.03238	0.72136	−3.556463± 0.876321i	±11.52649i	3.556463± 0.876321i
5	0.05	0.02671	0.71641	−1.524192± 0.837649i	±8.875206i	1.524192± 0.837649i

Table 3: The Effect of radiaionpressure on the location and stability of out-of-plane equilibrium points for e=0.3,a=0.87,$\mu = 0.35$,$\sigma_1 = 0.02, \sigma_2 = 0.015, \sigma_3 = 0.003, \sigma_4 = 0.004$, M_b= 0.01

S/no	Radiaion Pressure		Out-of plane points		Roots of the characteristic equation		
	q_1	q_2	ξ	$\pm\zeta$	$\lambda_{1,2}$	$\lambda_{3,4}$	$\lambda_{5,6}$
1	0.9960	0.9950	0.66735	0.412681	−2.54373± 0.543728i	±11.42462i	2.54373± 0.543728i
2	0.9964	0.9954	0.67024	0.394326	−3.24342± 0.810034i	±16.23703i	3.24342± 0.810034i
3	0.9968	0.9958	0.67646	0.343671	−6.45986± 0.886517i	±27.42462i	6.45986± 0.886517i
4	0.9972	0.9962	0.68434	0.328763	−10.5756±0.47632i	±38.57823i	10.5756± 0.47632i
5	0.9976	0.9966	0.69101	0.310641	−13.4140± 0.357649i	±45.41365i	13.4145± 0.357649i

Table 4: The Combined effect of thepertubationson the location and stability of out-of-plane equilibrium points for e =0.3,a =0.34

(a)

S/no.	Triaxiality				Radiation Factors		Belt	Mass ratio
	σ_1	σ_2	σ_3	σ_4	q_1	q_2	M_b	μ
1.	0.02	0.01	0.002	0.001	0.9980	0.9976	0.01	0.0375
2.	0.03	0.02	0.003	0.002	0.9984	0.9980	0.02	0.0380
3.	0.04	0.03	0.004	0.003	0.9988	0.9984	0.03	0.0385
4.	0.05	0.04	0.005	0.004	0.9992	0.9988	0.04	0.0390
5.	0.06	0.05	0.006	0.005	0.9996	0.9992	0.05	0.0395

(b)

out-of-plane points		The characteristic Roots		
ξ	$\pm\zeta$	$\lambda_{1,2}$	$\lambda_{3,4}$	$\lambda_{5,6}$
0.633412	0.197065	± 227.584	± 19.2802	$\pm 86.7122i$
0.633011	0.205634	± 331.385	± 90.4429	$\pm 87.2517i$
0.632785	0.218767	± 379.863	± 90.547	$\pm 88.5313i$
0.632145	0.224261	± 463.07	± 90.5989	$\pm 88.9132i$
0.631004	0.234659	± 88.9780	± 122.110	$\pm 35.41283i$

In Table 5 below we present the numerical data of the binary system xi-Bootis and Kruger 60.

Table 5: Numerical data for the Binary System

Binary system	Masses (MΘ)		Eccentricity (e)	Semi-major axis (a)	Luminosity LΘ		Spectral Types
	M_1	M_2			L_1	L_2	
Xi Bootis	0.9	0.66	0.5117	4.9044	0.49	0.061	G8/k4
Kruger 60	0.271	0.176	0.4100	2.3830	0.01	0.0034	M3/M4

Source:NASA ADS

Table 6: The effect of triaxiality on the location and stability of out-of-plane equilibrium pointsof xi-Bootis for e = 0.5117, a = 0.7304, μ = 0.4231 q_1 = 0.9988, q_2 = 0.9998.

S/no	Triaxiality				Out-of plane points		Roots of the characteristic equation		
	σ_1	σ_2	σ_3	σ_4	ξ	$\pm\zeta$	$\lambda_{1,2}$	$\lambda_{3,4}$	$\lambda_{5,6}$
1.	0.015	0.011	0.002	0.001	0.466010	0.275418	± 610.524	$-173.012\pm 184.316i$	$173.012\pm 184.316i$
2.	0.02	0.015	0.003	0.002	0.477810	0.269001	± 814.061	$-175.981\pm 182.895i$	$175.981\pm 182.895i$
3.	0.03	0.019	0.004	0.003	0.49061	0.24443	± 998.23	$-174.887\pm 180.49i$	$174.887\pm 180.49i$
4.	0.04	0.02	0.005	0.004	0.517438	0.219650	± 1627.48	$-175.13\pm 176.832i$	$175.13\pm 176.832i$
5.	0.05	0.03	0.006	0.005	0.539144	0.209341	± 1321.43	$-173.39\pm 176.972i$	$173.39\pm 176.972i$

Table 7: The effect of belt (M_b) on the location and stability of out-of-plane equilibrium pointsof xi-Bootis for e = 0.5117, a = 0.7304, μ = 0.4231 q_1 = 0.9988, q_2 = 0.9998.σ_1=0.02, σ_2=0.015,σ_3=0.003,σ_4=0.002

S/no	M_b	Out-of plane points		Roots of the characteristic equation		
		ξ	$\pm\zeta$	$\lambda_{1,2}$	$\lambda_{3,4}$	$\lambda_{5,6}$
1	0.02	0.521012	0.211723	$\pm$ 47.347306	$\pm 113.5678\, i$	± 33.98450
2	0.03	0.513422	0.211965	± 47.748921	$\pm 120.5631i$	± 34.12654
3	0.04	0.51200	0.221343	± 48.256439	$\pm 120.9985i$	± 34.241001
4	0.05	0.51042	0.229867	± 48.84320	$\pm 121.43423i$	± 35.35790
5	0.06	0.50964	0.239341	$\pm$ 49.22418	$\pm 122.1101i$	± 35.4128321

Table 8: The effect of triaxiality on the location and stability of out-of-plane equilibrium pointsof Kruger 60 for$e$ = 0.4100,a = 0.5894, μ = 0.3937 q_1= 0.9992 and q_2 = 0. 9996

	Triaxiality				Out-of-plane points		Roots of the characteristic equation		
S/No	σ_1	σ_2	σ_3	σ_4	ξ	$\pm\zeta$	$\lambda_{1,2}$	$\lambda_{3,4}$	$\lambda_{5,6}$
1.	0.02	0.002	0.002	0.001	0.946710	0.241070	$-37.2193\pm 21.4739i$	$0\pm 42.9477i$	$37.2193\pm 21.47396i$
2.	0.03	0.025	0.003	0.002	0.951193	0.216110	-38.567± -22.0922i	$0\pm 44.1875i$	38.567± 22.0922i
3.	0.04	0.035	0.004	0.003	0.958414	0.213279	$-51.476\pm 28.7198i$	$0\pm 57.5122i$	$51.476\pm 28.7198i$
4.	0.05	0.045	0.005	0.004	0.959130	0.207454	$-100.461\pm 55.2628i$	$0\pm 110.803i$	$100.461\pm 55.2628i$
5.	0.06	0.055	0.006	0.005	0.960314	0.204511	$-154.48\pm 81.4207i$	$0\pm 164.244i$	$154.48\pm 81.4207i$

Table 9: The effect of belt (M_b) on the location and stability of out-of-plane equilibrium pointsof Kruger-60 for e = 0.4100,a = 0.5894, μ = 0.3937 q_1= 0.9992 and q_2 = 0. 9996.σ_1=0.02, σ_2=0.015,σ_3=0.003,σ_4=0.002

S/no	M_b	Out-of plane points		Roots of the characteristic equation		
		ξ	$\pm\zeta$	$\lambda_{1,2}$	$\lambda_{3,4}$	$\lambda_{5,6}$
1	0.01	0.321012	0.200534	± 54.223624	±19.2802	$\pm 86.7122i$
2	0.02	0.321342	0.201823	± 54.534534	±90.4429	$\pm 87.2517i$
3	0.03	0.321440	0.201944	±54.655978	±90.547	±88.5313i
4	0.04	0.321452	0.202112	±55.232720	±90.5989	$\pm 88.9132i$
5	0.05	0.3214634	0.202472	± 55.703529	±122.110	$\pm 35.41283i$

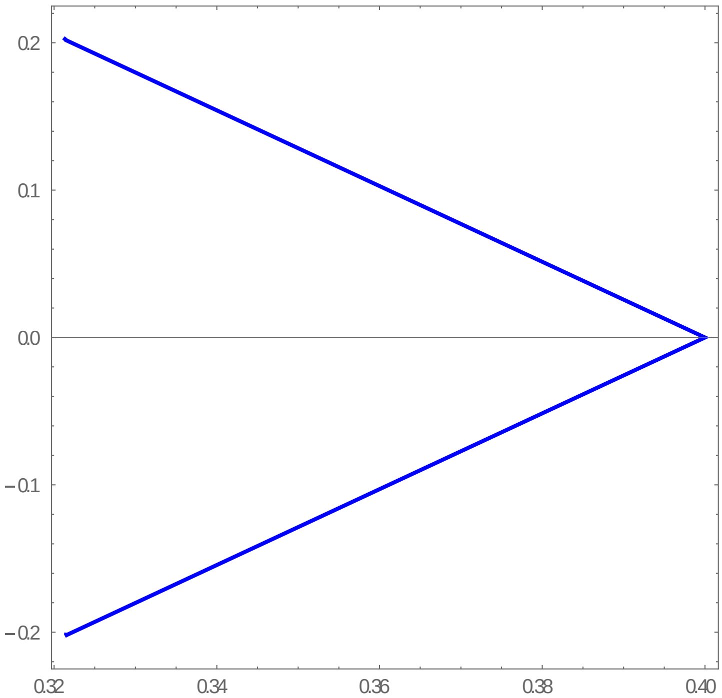

Fig.1 Graph showing the effect of triaxiality on the OPEPs of XI-Bootis

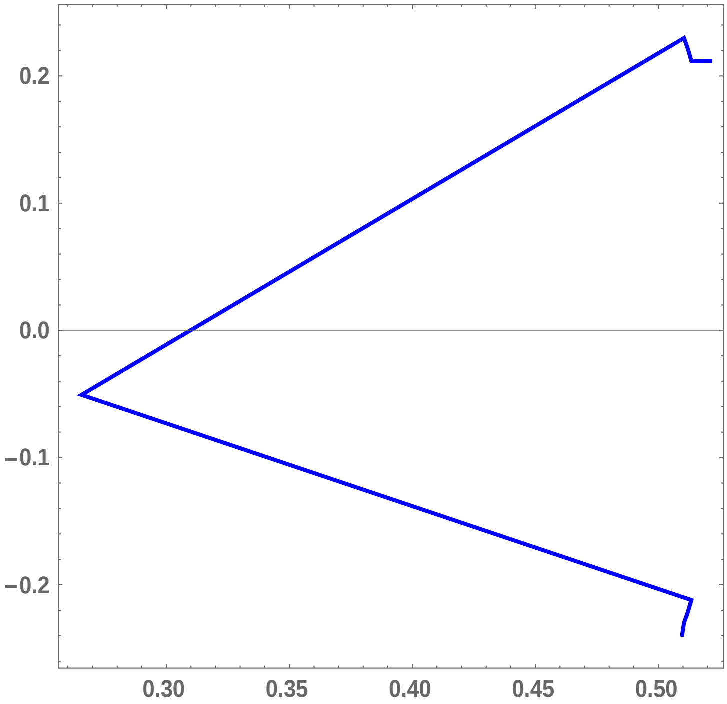

Fig.2 Graph showing the effect of the belt on the OPEPs of XI-Bootis

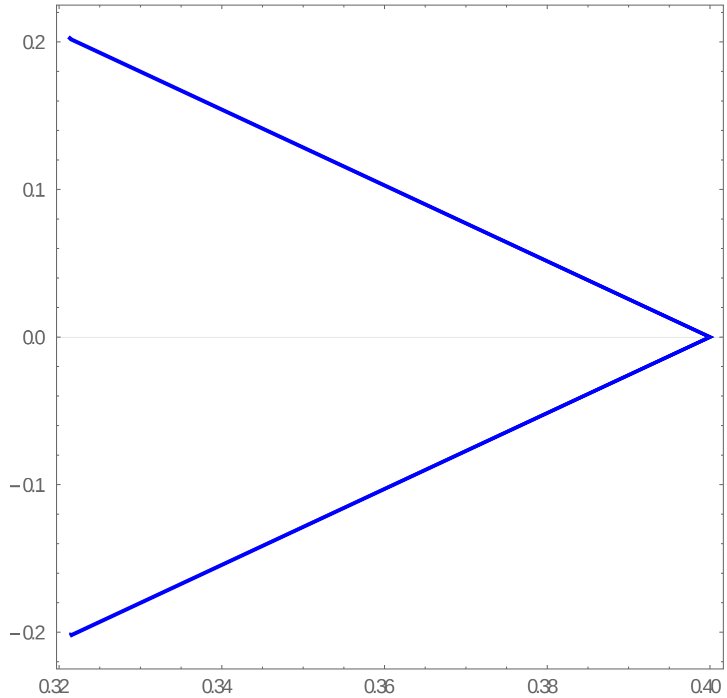

Fig.3 Graph showing the effect of triaxiality on the OPEPs of Kruger-60

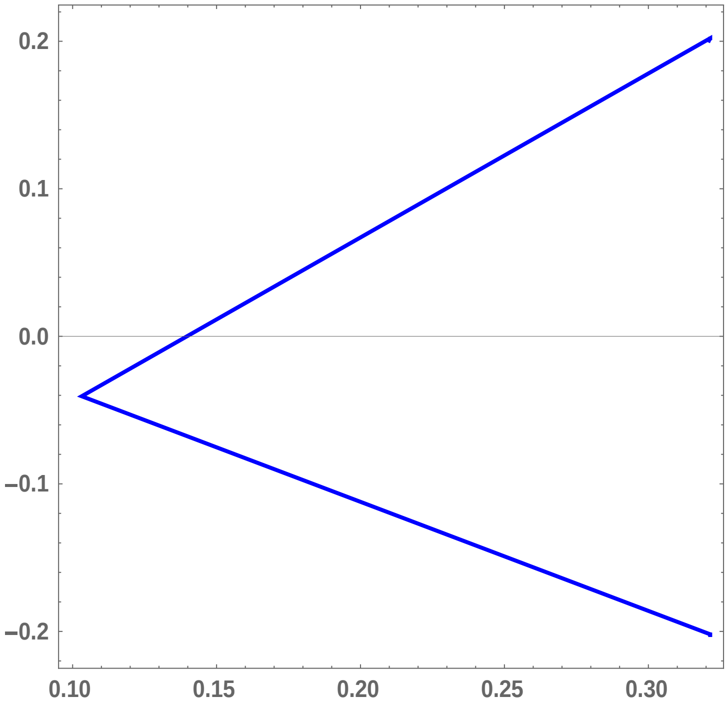

Fig.4 Graph showing the effect of the belt on the OPEPs of Kruger-60

VI. Discussion

The motion of a third body under the influence of triaxial and radiating primaries together with a circumbinary disc has been described in equation (1)-(4).The positions of out-of-plane equilibrium points aregivenin equations14 and15 and are first obtained analytically and then numerically by power series expansion about the triaxiality coefficient of the smaller primary in Equations 14 and 15 to third order term with the aid of the software MATHEMATICA 10.4.The stability of these points are obtained by solving the roots of Equation (16) numerically.The positions ofout-of- plane points and the characteristic roots obtained using arbitrary values for the parameters are shown in Tables 1-4. Generally, EPs are stable only if the six roots λ_i (i=1,2,3,4,5,6) are purely imaginary roots or complex roots with negative real parts and are unstable if λ_i (i=1,2,3,4,5,6) are complex or real roots (Szehebely,1967).

Table 1 and 2, shows that the point $L_{6,7}$ shifts towards the line joining the primaries as the effects of triaxiality and belt are being increased respectively,while in Table 3 $L_{6,7}$is seen to move away from the line joining the primaries as the radiation factors is increasing.The combined effects of all the parameters are shown in Table 4.The arbitrary values for the parameters are shown in Table 4a.Table 4b shows their effects onOPEPs and itsstability,In all cases the out-of- plane equilibrium points moves away fromthe ξ-axis whenthe values of the parameters wereincreased.The roots (λ_i (i=1,2,3,4,5,6)) in Tables1-4 are complex or real roots,hence theOPEPs are unstable.The numerical data of the binary systems (xi- bootis and kruger-60) are shown in Table 5.The effects oftriaxiality and the belt on the binary systems can be observed in Table 6-9 and Fig.1-4. These Tables and the graphs shows that increasingthe values of triaxiality and belt,while keeping the orbitalparametres of the Xi-bootis and Kruger-60 constant,results in a shift of the OPEPs.It can be seen in Table 6 that OPEPs shifts towards the ξ-axis this can be seen clearly in Fig.1,this in contrast to the effect of the belt on OPEPs ofXi-bootis in Table 7, where OPEPs shifts away from the the line joining the primaries (see also Fig.2).The OPEPs in both Tables are unstable due to nature of their roots which are complex or real roots.The effects of triaxiality and the belt on Kruger-60 is similar to their effects on Xi-bootis.The effects of triaxialitymovesthe OPEPs towards the line joining the primaries(see Table 8 and Fig. 3) ,while the effect of thebelt moves OPEPs away fromthe ξ-axis (See Table 9 and Fig. 4).Similar to what obtains in the case of xi-Bootis,the roots obtained for OPEPs of Kruger-60 are either complex or real as such OPEPs are unstable. The changes in the positions of OPEPs are as shown in the graphs (Fig.1-4) below.Thisinstability has been confirmed by (Douskos andMarkellos2006;Kushvah2008;Singh and Umar 2013a).

VII. Conclusion

We have established the existence of out of plane equilibrium points and their stability in the framework of ER3BP when the primaries are triaxial,radiating and surrounded by a belt.It is found that the positions are affected by triaxiality, radiation and thebelt.We found that for the binary system the effect of triaxialty and the belt moves OPEPs in opposite directions-while the effect of triaxiality moves OPEPs towards the ξ-axis,the belt moves OPEPs away from the ξ-axis.Our OPEPS (Equations 12 and 13) tally with that of(Singh and Umar 2013a) when $(2\sigma_1 - \sigma_2) = A_1$and $(2\sigma_3 - \sigma_4) = A_2$.

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