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Research Paper

A Study on Mensuration with Special Reference to Ellipse and Ellipsoid

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ABSTRACT: *In this article, a special type of limit relating to a real valued function* $f(r_1, r_2, \cdots, r_n)$ *will be studied which is defined as*

$$\lim_{\varepsilon \to 0} \frac{f(r_1 + \varepsilon,\ r_2 + \varepsilon, \cdots,\ r_n + \varepsilon) - f(r_1, r_2, \cdots, r_n)}{\varepsilon}$$

It will be called as marginal limit of $f(r_1, r_2, \cdots, r_n)$ *with respect to* $r_1, r_2, \cdots, r_n$ *and it will be written as* $f_*(r_1, r_2, \cdots, r_n)$. *Some applications of marginal limit will be given to find or approximate the perimeter of a simple closed curve e.g. circle, ellipse etc. from the area enclosed by that curve in* $\boldsymbol{R}^2$ *and to find or approximate the surface area of a simple closed surface e.g. sphere, ellipsoid, right elliptic cylinder etc. from the volume enclosed by that surface in* $\boldsymbol{R}^3$. *Substitutes of Ramanujan's formulas for finding the perimeter of an ellipse will be given. Two suitable substitutes which are*

$$(i)\quad A_e(a,b,c) \approx \pi\Big[24(ab+ac+bc) - \big((31ab+19ac+18bc)(18ab+31ac+19bc)(19ab+18ac+31bc)\big)^{1/3}\Big]$$

$$(ii)\quad A_e(a,b,c) \approx \pi\Big[13(ab+ac+bc) - \big((17ab+11ac+7bc)(7ab+17ac+11bc)(11ab+7ac+17bc)\big)^{1/3}\Big]$$

of Knud Thomsen's formula for approximating the surface area $A_e(a,b,c)$ *of an ellipsoid with semi-axes of lengths* a, b, c *will be given. General formula for finding the length of an elliptic arc will be given. Ellipsoidal sector will also be studied.*

KEYWORDS: *Area, Length, Marginal limit, Perimeter, Volume*

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I. INTRODUCTION

Area of a circle with radius r in $\boldsymbol{R}^2$ is πr^2, volume of a sphere with radius r in $\boldsymbol{R}^3$ is $\frac{4}{3}\pi r^3$ and volume of a circular cylinder with radius r and height h in $\boldsymbol{R}^3$ is $\pi r^2 h$, From this, we find that $\frac{d}{dr}(\pi r^2) = 2\pi r =$ perimeter of the circle with radius r, $\frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2 =$ surface area of the sphere with radius r and $\frac{d}{dr}(\pi r^2 h) = 2\pi r h$, which is the curved surface area of the circular cylinder with radius r.These examples show that there exists a relation between the perimeter of a simple closed curve and area of the region enclosed by it in $\boldsymbol{R}^2$; area of a simple closed surface and volume of the region enclosed by it in $\boldsymbol{R}^3$.

Dorff and Hall in [2], attempted to establish the equations $\frac{dA}{dr} = P$ and $\frac{dV}{dr} = A$, where P, A, V and r represent perimeter, area, volume and radius. Cohen [1], studies the ratio of volume of inscribed sphere to polyhedron. Emert and Nelson [3] describe the relation of area and volume of polyhedral and polytopes. Miller [6] studies on the topic differentiating area and volume to get perimeter and area. Struss [8] describes the relation between volume and surface area. Tong [9] studies the relation between area and perimeter, volume and surface area.

In this paper, a method of marginal limit has been given which is the analogue of the derivative of a function of one variable and it is applicable to the function of one or more than one variables. It will be used to evaluate or approximate the perimeter from area and area from volume if area and volume are the function of one or more than one variables (radii). In case a function of one variable, marginal limit with respect to single variable is equal to the derivative with respect to that single variable of that function.

For approximating the perimeter of the ellipse with semi-axes of lengths a, b; in 1609, Kepler gave the formula $2\pi\sqrt{ab}$ which is a lower bound for the perimeter of ellipse. In 1773, Euler gave the formula $\pi\sqrt{2(a^2+b^2)}$ which is an upper bound for the perimeter of an ellipse. In 1914, Ramanujan gave two formulas, (i) $\pi\left[3(a+b)-\sqrt{(3a+b)(a+3b)}\right]$(ii) $\pi(a+b)\left[1+\frac{3\hbar}{10+\sqrt{4-3\hbar}}\right], \hbar=\frac{(a-b)^2}{(a+b)^2}$, which give almost accurate value. In 1917, Hudson gave the formula $\frac{\pi(a+b)}{4}\left[3(1+L)+\frac{1}{1-L}\right]$, where $L=\frac{\hbar}{4}=\frac{1}{4}\left(\frac{a-b}{a+b}\right)^2$. In 2000, Maertens gave the formula which is called 'YNOT' formula as $P_e(a,b)\approx 4(a^y+b^y)^{1/y}$, where $y=\frac{\text{In}(2)}{\text{In}(\pi/2)}$. A formula $\frac{\pi}{2}\sqrt{2(3a^2+3b^2+2ab)}$ was also given. In this paper, substitutes of Ramanujan's formulas will be given.

For approximating the surface area of the ellipsoid with semi-axes of lengths a, b, c; in 2004, Knud Thomsen gave the formula $4\pi\left(\frac{a^p b^p+a^p c^p+b^p c^p}{3}\right)^{1/p}$, $p\approx 1.6075$. In this paper, substitutes of Knud Thomsen formula will be given.

II. MARGINAL LIMIT

Let $f(r_1, r_2, \dots, r_n)$ be a real valued function of variables $r_1, r_2, \dots, r_n$ then marginal limit of this function f with respect to $r_1, r_2, \dots, r_n$ is denoted by $f_*(r_1, r_2, \dots, r_n)$ and it is defined as follows

$$f_*(r_1, r_2, \dots, r_n)=\lim_{\varepsilon\to 0}\frac{f(r_1+\varepsilon, r_2+\varepsilon, \dots, r_n+\varepsilon)-f(r_1, r_2, \dots, r_n)}{\varepsilon}$$

Marginal limit of a function of single variable with respect to that variable is equal to the derivative of that function with respect to that variable.

Illustration: (i) If $f(a,b)=kab$, where k is a constant and a, b are variables, then

$$f_*(a,b)=\lim_{\varepsilon\to 0}\frac{f(a+\varepsilon, b+\varepsilon)-f(a,b)}{\varepsilon}=\lim_{\varepsilon\to 0}\frac{k(a+\varepsilon)(b+\varepsilon)-kab}{\varepsilon}$$

$$=\lim_{\varepsilon\to 0}\frac{kab+(ka+kb)\varepsilon+\varepsilon^2-ab}{\varepsilon}=\lim_{\varepsilon\to 0}\frac{(ka+kb)\varepsilon}{\varepsilon}=ka+kb=k(a+b)$$

Therefore marginal limit of $f(a,b)=kab$ with respect to a, b is $f_*(a,b)=k(a+b)$;

(ii) If $f(a,b,c)=kabc$, where k is a constant and a, b, c are variables, then

$$f_*(a,b,c)=k(ab+bc+ca)$$

(iii) If $f(a)=ka^n$, where k is a constant and a is a variable, then

$$f_*(a)=\lim_{\varepsilon\to 0}\frac{f(a+\varepsilon)-f(a)}{\varepsilon}=\lim_{\varepsilon\to 0}\frac{k(a+\varepsilon)^n-ka^n}{\varepsilon}=kna^{n-1}=\frac{df}{da}=f'(a)$$

Here $f(x)=kx^n$, therefore $\frac{df}{da}=f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a}\frac{kx^n-ka^n}{x-a}=kna^{n-1}$

This shows that marginal limit of a function with respect to single variable is equal to the derivative of that function with respect to that variable.

III. RESULTS RELATING TO MARGINAL LIMIT

Theorem-1: if $f(r_1, r_2, \dots, r_n)$ is the area of a region enclosed by a simple closed continuous curve in $\boldsymbol{R}^2$, where $r_1\geq r_2\geq\dots\geq r_n>0$ are the distinct radii of the region; i.e., $r_1, r_2, \dots, r_n$ are distinct distances measured from the centre of the region to some distinct points lying on its enclosing curve. Then marginal limit of the area of the region is less than or equal to the length (perimeter) of its enclosing curve; i.e. $\lim_{\varepsilon\to 0}\frac{f(r_1+\varepsilon, r_2+\varepsilon, \dots, r_n+\varepsilon)-f(r_1, r_2, \dots, r_n)}{\varepsilon}=f_*(r_1, r_2, \dots, r_n)\leq$ perimeter of the curve enclosing the area $f(r_1, r_2, \dots, r_n)$.

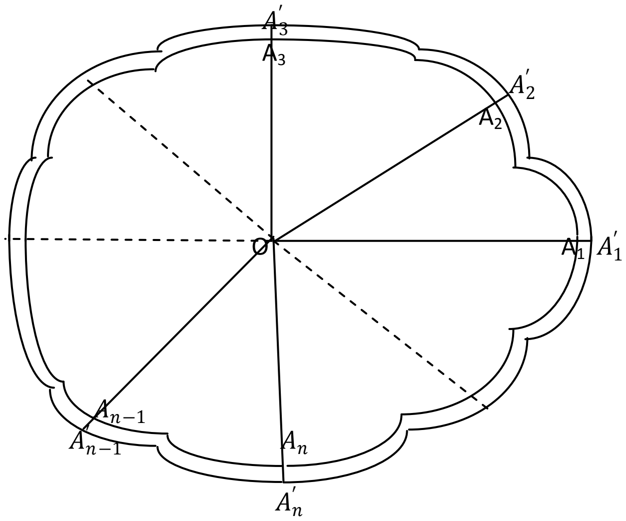

Figure-1 (a path of mean breadth $\varepsilon \to 0$ formed along the given simple closed curve)

Proof: As shown in Figure-1, let O be the centre of the region of area $A = f(r_1, r_2, \dots, r_n)$ and $OA_1 = r_1, OA_2 = r_2, \dots, OA_n = r_n$ are the radii of the region enclosed by a simple closed continuous curve, where $A_1, A_2, \cdots, A_n$ are the points on this enclosing curve. Let P is the perimeter of this curve.

Let each of radii $OA_1 = r_1, OA_2 = r_2, \dots, OA_n = r_n$ are given increment $\varepsilon \to 0$ so that they become $OA'_1 = r_1 + \varepsilon, OA'_2 = r_2 + \varepsilon, \cdots, OA'_n = r_n + \varepsilon$ such that points $O, A_1, A'_1; O, A_2, A'_2; \dots; O, A_n, A'_n$ are collinear. Therefore $A' = f(r_1 + \varepsilon, r_2 + \varepsilon, \dots, r_n + \varepsilon)$ is the area of the region with centre O and radii $OA'_1 = r_1 + \varepsilon, OA'_2 = r_2 + \varepsilon, \cdots, OA'_n = r_n + \varepsilon$. Let P' is the perimeter of the curve enclosing the region with centre O and area $A' = f(r_1 + \varepsilon, r_2 + \varepsilon, \dots, r_n + \varepsilon)$, where $P' \to P$ as $\varepsilon \to 0$.

Then $f(r_1 + \varepsilon, r_2 + \varepsilon, \dots, r_n + \varepsilon) - f(r_1, r_2, \dots, r_n)$ = area of the path formed on increasing the radii $OA_1 = r_1, OA_2 = r_2, \dots, OA_n = r_n$ by ε.

Let ε_m be the mean breadth of the path formed on increasing the radii $OA_1 = r_1, OA_2 = r_2, \dots, OA_n = r_n$ by ε. Then $P'\varepsilon_m$ = area of the path formed on increasing the radii $OA_1 = r_1, OA_2 = r_2, \dots, OA_n = r_n$ by ε, and $\varepsilon \geq \varepsilon_m$

$\Rightarrow \quad P'\varepsilon_m = f(r_1 + \varepsilon, r_2 + \varepsilon, \dots, r_n + \varepsilon) - f(r_1, r_2, \dots, r_n)$, and $\varepsilon \geq \varepsilon_m$

Now $\varepsilon \geq \varepsilon_m \Rightarrow P'\varepsilon \geq P'\varepsilon_m = f(r_1 + \varepsilon, r_2 + \varepsilon, \dots, r_n + \varepsilon) - f(r_1, r_2, \dots, r_n)$

$$\Rightarrow \qquad P' \geq \frac{f(r_1 + \varepsilon, r_2 + \varepsilon, \dots, r_n + \varepsilon) - f(r_1, r_2, \dots, r_n)}{\varepsilon}$$

$$\Rightarrow \qquad \lim_{\varepsilon \to 0} P' \geq \lim_{\varepsilon \to 0} \frac{f(r_1 + \varepsilon, r_2 + \varepsilon, \dots, r_n + \varepsilon) - f(r_1, r_2, \dots, r_n)}{\varepsilon}$$

Now $\quad P' \to P$ as $\varepsilon \to 0$; i.e., $\quad \lim_{\varepsilon \to 0} P' = P,$

$\therefore \qquad P \geq f_*(r_1, r_2, \dots, r_n)$

i.e. $\lim_{\varepsilon \to 0} \frac{f(r_1+\varepsilon, r_2+\varepsilon, \dots, r_n+\varepsilon) - f(r_1, r_2, \dots, r_n)}{\varepsilon} = f_*(r_1, r_2, \dots, r_n) \leq$ perimeter of the curve enclosing the area $f(r_1, r_2, \dots, r_n)$.

Illustration (i) A rectangle of length a and breadth b represents a simple closed continuous curve enclosing area $ab = 4\left(\frac{a}{2}\right)\left(\frac{b}{2}\right) = 4r_1 r_2 = f(r_1, r_2)$, where $r_1 = \frac{a}{2}, r_2 = \frac{b}{2} \Rightarrow a = 2r_1, b = 2r_2$.

$$\text{Therefore, perimeter of the rectangle} = f_*(r_1, r_2) = \lim_{\varepsilon \to 0} \frac{f(r_1 + \varepsilon, r_2 + \varepsilon) - f(r_1, r_2)}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{4(r_1 + \varepsilon)(r_2 + \varepsilon) - 4r_1 r_2}{\varepsilon} = 4(r_1 + r_2) = 4\left(\frac{a}{2} + \frac{b}{2}\right) = 2(a + b)$$

(ii) A circle with radius r represents a simple closed continuous curve enclosing area $\pi r^2 = f(r)$

$$\text{Therefore, perimeter of the circle} = f_*(r) = f'(r) = \lim_{\varepsilon \to 0} \frac{f(r + \varepsilon) - f(r)}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{(r + \varepsilon)^2 - r^2}{\varepsilon} = 2\pi r \quad \text{(see [2])}$$

Theorem-2: if $f(r_1, r_2, \dots, r_n)$ is the volume of a region enclosed by a simple closed continuous surface in $\boldsymbol{R}^3$, where $r_1 \geq r_2 \geq \cdots \geq r_n > 0$ are the distinct radii of the region; i.e., $r_1, r_2, \dots, r_n$ are distinct distances measured from the centre of the region to some distinct points lying on its enclosing surface. Then marginal limit of the volume of the region is less than or equal to the area of its enclosing surface; i.e.

$\lim_{\varepsilon\to 0}\frac{f(r_1+\varepsilon,r_2+\varepsilon,\dots,r_n+\varepsilon)-f(r_1,r_2,\dots,r_n)}{\varepsilon}=f_*(r_1,r_2,\dots,r_n)\leq$ area of the surface enclosing the volume $f(r_1,r_2,\dots,r_n)$.

Proof: As shown in Figure-2, let O be the centre of the region of volume $V=f(r_1,r_2,\dots,r_n)$ and $OA_1=r_1, OA_2=r_2,\dots,OA_n=r_n$ are the radii of the region enclosed by a simple closed continuous surface, where $A_1,A_2,\cdots,A_n$ are the points on the enclosing surface . Let A be the area of the surface enclosing this region.

Let each of radii $OA_1=r_1, OA_2=r_2,\dots,OA_n=r_n$ are given increment $\varepsilon\to 0$ so that they become $OA'_1=r_1+\varepsilon, OA'_2=r_2+\varepsilon,\cdots,OA'_n=r_n+\varepsilon$ such that points $O,A_1,A'_1;O,A_2,A'_2;\dots;O,A_n,A'_n$ are collinear. Therefore $V'=f(r_1+\varepsilon,r_2+\varepsilon,\dots,r_n+\varepsilon)$ is the volume of the region with centre O and radii $OA'_1=r_1+\varepsilon, OA'_2=r_2+\varepsilon,\cdots,OA'_n=r_n+\varepsilon$. Let A' be the area of the surface enclosing the region with centre O and volume $V'=f(r_1+\varepsilon,r_2+\varepsilon,\dots,r_n+\varepsilon)$, where $A'\to A$ as $\varepsilon\to 0$.

Then $f(r_1+\varepsilon,r_2+\varepsilon,\dots,r_n+\varepsilon)-f(r_1,r_2,\dots,r_n)$ = volume of the layer formed on increasing the radii $OA_1=r_1, OA_2=r_2,\dots,OA_n=r_n$ by ε.

Let ε_m be the mean thickness of the layer formed on increasing the radii $OA_1=r_1, OA_2=r_2,\dots,OA_n=r_n$ by ε. Then $A'\varepsilon_m$ = volume of the layer formed on increasing the radii $OA_1=r_1, OA_2=r_2,\dots,OA_n=r_n$ by ε, and $\varepsilon\geq\varepsilon_m$

$\Rightarrow\quad A'\varepsilon_m=f(r_1+\varepsilon,r_2+\varepsilon,\dots,r_n+\varepsilon)-f(r_1,r_2,\dots,r_n)$, and $\varepsilon\geq\varepsilon_m$

Now $\varepsilon\geq\varepsilon_m \Rightarrow A'\varepsilon\geq A'\varepsilon_m=f(r_1+\varepsilon,r_2+\varepsilon,\dots,r_n+\varepsilon)-f(r_1,r_2,\dots,r_n)$

$$\Rightarrow\qquad A'\geq\frac{f(r_1+\varepsilon,r_2+\varepsilon,\dots,r_n+\varepsilon)-f(r_1,r_2,\dots,r_n)}{\varepsilon}$$

$$\Rightarrow\qquad \lim_{\varepsilon\to 0}A'\geq\lim_{\varepsilon\to 0}\frac{f(r_1+\varepsilon,r_2+\varepsilon,\dots,r_n+\varepsilon)-f(r_1,r_2,\dots,r_n)}{\varepsilon}$$

Now $A'\to A$ as $\varepsilon\to 0$; i.e., $\lim_{\varepsilon\to 0}A'=A$,

$\therefore\qquad A\geq f_*(r_1,r_2,\dots,r_n)$

i.e. $\lim_{\varepsilon\to 0}\frac{f(r_1+\varepsilon,r_2+\varepsilon,\dots,r_n+\varepsilon)-f(r_1,r_2,\dots,r_n)}{\varepsilon}=f_*(r_1,r_2,\dots,r_n)\leq$ area of the surface enclosing the volume $f(r_1,r_2,\dots,r_n)$.

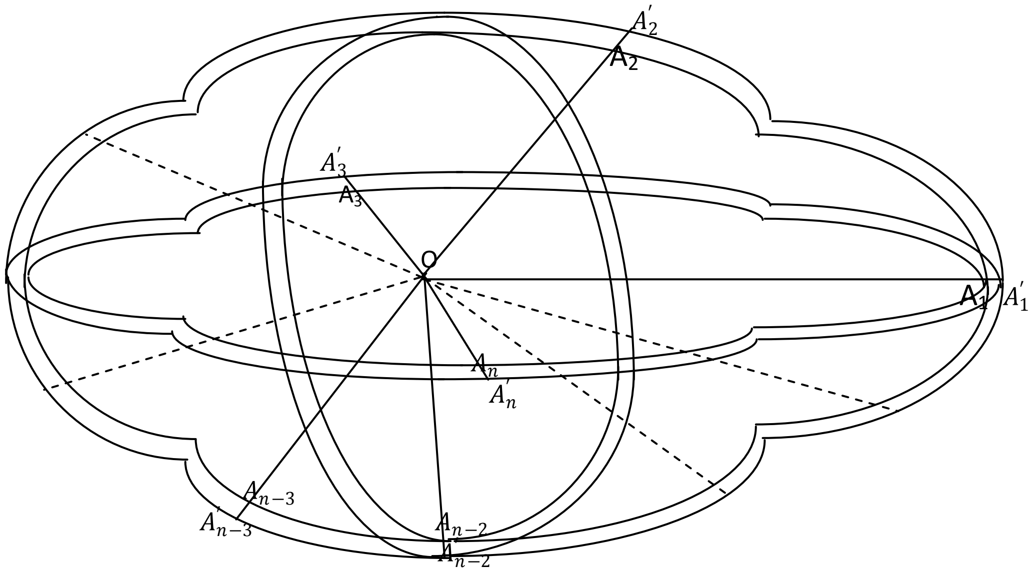

Figure-2 (a layer of mean thickness $\varepsilon\to 0$ formed along the given simple closed surface)

Illustration (i) A cuboid of length a, breadth b and height c represents a simple closed continuous surface enclosing volume

$abc=8\left(\frac{a}{2}\right)\left(\frac{b}{2}\right)\left(\frac{c}{2}\right)=8r_1r_2r_3=f(r_1,r_2,r_3)$, where $r_1=\frac{a}{2}, r_2=\frac{b}{2}, r_3=\frac{c}{2}\Rightarrow a=2r_1, b=2r_2, c=2r_3$.

$$\text{Therefore, surface area of the cuboid}=f_*(r_1,r_2,r_3)=\lim_{\varepsilon\to 0}\frac{f(r_1+\varepsilon,r_2+\varepsilon,r_3+\varepsilon)-f(r_1,r_2,r_3)}{\varepsilon}$$

$$=\lim_{\varepsilon\to 0}\frac{8(r_1+\varepsilon)(r_2+\varepsilon)(r_3+\varepsilon)-8r_1r_2r_3}{\varepsilon}=8(r_1r_2+r_1r_3+r_2r_3)$$

$$= 8\left(\frac{ab}{4} + \frac{ac}{4} + \frac{bc}{4}\right) = 2(ab + ac + bc)$$

(ii) A sphere with radius r represents a simple closed continuous surface enclosing volume $\frac{4}{3}\pi r^3 = f(r)$

Therefore, surface area of the sphere $= f_*(r) = f'(r) = \lim_{\varepsilon \to 0} \frac{f(r+\varepsilon) - f(r)}{\varepsilon}$

$$= \lim_{\varepsilon \to 0} \frac{\frac{4}{3}\pi(r+\varepsilon)^3 - \frac{4}{3}\pi r^3}{\varepsilon} = 4\pi r^2 \quad \text{(also see [2])}$$

(iii) A right circular cylinder with radius r and height h represents a simple closed continuous surface enclosing volume $\pi r^2 h = 2(\pi r^2)\left(\frac{h}{2}\right) = 2\pi r^2 r_1 = f(r, r_1)$, where $r_1 = \frac{h}{2} \Rightarrow h = 2r_1$.

Therefore, complete surface area of the right circular cylinder

$$= f_*(r, r_1) = \lim_{\varepsilon \to 0} \frac{f(r+\varepsilon, r_1+\varepsilon) - f(r, r_1)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{2\pi(r+\varepsilon)^2(r_1+\varepsilon) - 2\pi r^2 r_1}{\varepsilon}$$

$$= 2\pi(r^2 + 2rr_1) = 2\pi(r^2 + rh) = 2\pi r(r+h)$$

IV. APPROXIMATING PERIMETER OF AN ELLIPSE

4.1 Approximation of the perimeter of the Ellipse by Marginal Limit

In Figure-3, an ellipse with centre O, semi major axis of length $OA = a$ and semi minor axis of length $OB = b$ has been shown, where $a \geq b > 0$. Therefore $OA = a$ and $OB = b$ are the distances of the boundary of the ellipse from its centre O in $\boldsymbol{R}^2$; i.e. $OA = a$ and $OB = b$ are the radii of the ellipse. Area of the ellipse with semi axes $OA = a$ and $OB = b$ in $\boldsymbol{R}^2$ is $\pi ab = f(a, b)$.

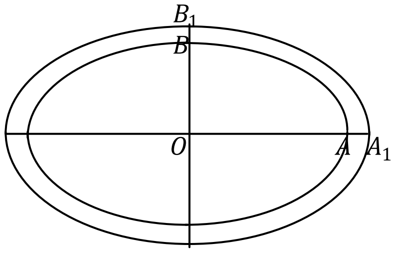

Figure-3 (a path of mean breadth $\varepsilon \to 0$ formed along the boundary of the ellipse)

If semi axes OA and OB are given increment $AA_1 = BB_1 = \varepsilon \to 0$, then we get a concentric ellipse with centre O and semi axes of lengths $OA_1 = OA + AA_1 = a + \varepsilon$, $OB_1 = OB + BB_1 = b + \varepsilon$

Then area of ellipse with semi axes $OA_1 = a + \varepsilon, OB_1 = b + \varepsilon$ in $\boldsymbol{R}^2$ is $\pi(a+\varepsilon)(b+\varepsilon) = f(a+\varepsilon, b+\varepsilon)$

Let $P_e(a, b)$ denotes the perimeter of the ellipse with semi major axis of length a and semi minor axis of length b. Then $P_e(a, b) \geq f_*(a, b) = \lim_{\varepsilon \to 0} \frac{\text{(area of ellipse with semi axes } OA_1, OB_1) - \text{(area of ellipse with semi axes } OA, OB)}{\varepsilon}$

$$= \lim_{\varepsilon \to 0} \frac{f(a+\varepsilon, b+\varepsilon) - f(a, b)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{\pi(a+\varepsilon)(b+\varepsilon) - \pi ab}{\varepsilon}$$

$$= \pi \lim_{\varepsilon \to 0} \frac{ab + a\varepsilon + b\varepsilon + \varepsilon^2 - ab}{\varepsilon} = \pi(a+b)$$

i.e. $P_e(a, b) \geq \pi(a+b)$, if the condition $b \leq a < 2b$, then $P_e(a, b) \cong \pi(a+b)$

4.2 Another Form of Ramanujan's Formula for finding Perimeter $P_e(a, b)$ of Ellipse

We have $P_e(a, b) \geq \pi(a+b)$. For establishing $P_e(a, b) \approx m\pi(a+b)$, we find the approximate value of m.

By Ramanujan's $P_e(a, b) \approx \pi\left[3(a+b) - \sqrt{(3a+b)(a+3b)}\right]$, then $m = 3 - \frac{\sqrt{(3a+b)(a+3b)}}{a+b}$

Ramanujan's formula can be changed into the following form

$$P_e(a, b) \approx \pi(a+b)\left(3 - \sqrt{4 - \hbar}\right), \quad \hbar = \frac{(a-b)^2}{(a+b)^2}$$

$$\because \quad \pi\left[3(a+b) - \sqrt{(3a+b)(a+3b)}\right] = \pi(a+b)\left(3 - \sqrt{\frac{3(a+b)^2 + 4ab}{(a+b)^2}}\right)$$

$$= \pi(a+b)\left(3 - \sqrt{\frac{3(a+b)^2 + (a+b)^2 - (a-b)^2}{(a+b)^2}}\right) = \pi(a+b)\left(3 - \sqrt{\frac{4(a+b)^2 - (a-b)^2}{(a+b)^2}}\right)$$

$$= \pi(a+b)\left(3-\sqrt{4-\frac{(a-b)^2}{(a+b)^2}}\right) = \pi(a+b)\left(3-\sqrt{4-\hbar}\right)$$

$$\therefore\ P_e(a,b) \approx \pi(a+b)\left(3-\sqrt{4-\hbar}\right) \ \Rightarrow \quad m = 3-\sqrt{4-\hbar}$$

4.3 Substitutes for Ramanujan's Formula

Following are the substitutes of the Ramanujan's Formula

(i) $P_e(a,b) \approx \pi\left[32(a+b)-\sqrt{(27a+35b)(35a+27b)}\right]$

or $P_e(a,b) \approx \pi(a+b)\left(32-\sqrt{961-16\hbar}\right)$

$$\because\ \pi\left[32(a+b)-\sqrt{(27a+35b)(35a+27b)}\right] = \pi(a+b)\left(32-\sqrt{\frac{945(a+b)^2+64ab}{(a+b)^2}}\right)$$

$$= \pi(a+b)\left(32-\sqrt{\frac{945(a+b)^2+16(a+b)^2-16(a-b)^2}{(a+b)^2}}\right)$$

$$= \pi(a+b)\left(32-\sqrt{\frac{961(a+b)^2-16(a-b)^2}{(a+b)^2}}\right) = \pi(a+b)\left(32-\sqrt{961-16\frac{(a-b)^2}{(a+b)^2}}\right)$$

$$= \pi(a+b)\left(32-\sqrt{961-16\hbar}\right)$$

(ii) $P_e(a,b) \approx \pi\left[9(a+b)-\sqrt{(10a+6b)(6a+10b)}\right]$

or $P_e(a,b) \approx \pi(a+b)\left(9-\sqrt{64-4\hbar}\right)$

(iii) $P_e(a,b) \approx 0.5\pi\left[7(a+b)-\sqrt{(4a+b)(a+4b)}-\sqrt{(3a+2b)(2a+3b)}\right]$

or $P_e(a,b) \approx 0.25\pi(a+b)\left(14-\sqrt{25-9\hbar}-\sqrt{25-\hbar}\right)$

Also we can find the approximate value of $P_e(a,b)$ by the following result

$$P_e(a,b) \cong 4\sqrt{a^2+b^2}+\frac{ab}{a+b}$$

For finding $P_e(a,b)$ more accurately, we can use the already available formula in integral calculus which is

$$P_e(a,b) = 2a\pi\left[1-\left(\frac{1}{2}\right)^2\frac{e^2}{1}-\left(\frac{1\cdot3}{2\cdot4}\right)^2\frac{e^4}{3}-\left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2\frac{e^6}{5}-\left(\frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}\right)^2\frac{e^8}{7}-\left(\frac{1\cdot3\cdot5\cdot7\cdot9}{2\cdot4\cdot6\cdot8\cdot10}\right)^2\frac{e^{10}}{9}-\cdots\right]$$

$$= 2a\pi\left(1-\frac{1}{4}e^2-\frac{3}{64}e^4-\frac{5}{256}e^6-\frac{25\cdot7}{256\cdot64}e^8-\frac{49\cdot9}{1024\cdot64}e^{10}-\frac{441\cdot11}{16384\cdot64}e^{12}-\cdots\right)$$

$$= 2a\pi\left(1-\frac{1}{2^2}e^2-\frac{3}{2^6}e^4-\frac{5}{2^8}e^6-\frac{175}{2^{14}}e^8-\frac{441}{2^{16}}e^{10}-\frac{4851}{2^{20}}e^{12}-\frac{14157}{2^{22}}e^{14}-\frac{2760615}{2^{30}}e^{16}\right.$$
$$\left.-\frac{8690825}{2^{32}}e^{18}-\frac{112285459}{2^{36}}e^{20}-\frac{48841\times7581}{2^{38}}e^{22}-\frac{48841\times406847}{2^{44}}e^{24}-\cdots\right)$$

$$= 2a\pi\left(1-\frac{k}{2^2l}-\frac{3k^2}{2^6l^2}-\frac{5k^3}{2^8l^3}-\frac{175k^4}{2^{14}l^4}-\frac{441k^5}{2^{16}l^5}-\frac{4851k^6}{2^{20}l^6}-\frac{14157k^7}{2^{22}l^7}-\frac{2760615k^8}{2^{30}l^8}-\frac{8690825k^9}{2^{32}l^9}\right.$$
$$\left.-\frac{112285459k^{10}}{2^{36}l^{10}}-\frac{48841\times7581k^{11}}{2^{38}l^{11}}-\frac{48841\times406847k^{12}}{2^{44}l^{12}}-\cdots\right)$$

where $e = \frac{\sqrt{a^2-b^2}}{a}, k = a^2-b^2, l = a^2,\ a \geq b.$

V. SURFACE AREA OF AN ELLIPSOID

5.1 Approximation of Surface Area of an Ellipsoid from its Volume by Marginal Limit

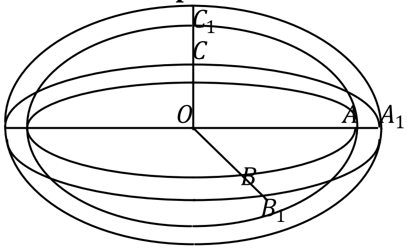

Figure-4 (Surface area of ellipsoid by limit)

In Figure-4, an ellipsoid with centre O, semi axes of lengths $OA = a, OB = b, OC = c$ has been shown, where $a \geq b \geq c > 0$. Therefore $OA = a$, $OB = b$, $OC = c$ are the radii of ellipsoid with centre O in space $\boldsymbol{R}^3$. Volume of the ellipsoid with semi axes $OA = a$, $OB = b$, $OC = c$ in space R^3 is $\frac{4}{3}\pi abc = f(a, b, c)$. If semi axes OA, OB, OC are given increment $AA_1 = BB_1 = CC_1 = \varepsilon \to 0$, then we get a concentric ellipsoid with centre O and semi axes of lengths

$$OA_1 = OA + AA_1 = a + \varepsilon, OB_1 = OB + BB_1 = b + \varepsilon, OC_1 = OC + CC_1 = c + \varepsilon$$

Then volume of the ellipsoid with semi axes $OA_1 = a + \varepsilon, OB_1 = b + \varepsilon, OC_1 = c + \varepsilon$ is

$$\frac{4}{3}\pi(a + \varepsilon)(b + \varepsilon)(c + \varepsilon) = f(a + \varepsilon, b + \varepsilon, c + \varepsilon)$$

Let $A_e(a, b, c)$ denotes the surface area of an ellipsoid.

Then $A_e(a, b, c) \geq f_*(a, b, c)$

$$= \lim_{\varepsilon \to 0} \frac{(\text{volume of ellipsoid with semi axes } OA_1, OB_1, OC_1) - (\text{volume of ellipsoid with semi axes } OA, OB, OC)}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{f(a + \varepsilon, b + \varepsilon, c + \varepsilon) - f(a, b, c)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{\frac{4}{3}\pi(a + \varepsilon)(b + \varepsilon)(c + \varepsilon) - \frac{4}{3}\pi abc}{\varepsilon}$$

$$= \frac{4}{3}\pi \lim_{\varepsilon \to 0} \frac{abc + ab\varepsilon + bc\varepsilon + ca\varepsilon + a\varepsilon^2 + b\varepsilon^2 + c\varepsilon^2 + \varepsilon^3 - abc}{\varepsilon} = \frac{4}{3}\pi(ab + bc + ca)$$

i.e. $A_e(a, b, c) \geq \frac{4}{3}\pi(ab + ac + bc)$

If the condition $c \leq a < 2c$ is satisfied, then $A_e(a, b, c) \cong \frac{4}{3}\pi(ab + bc + ca)$

5.2 Substitutes for Knud Thomsen Formula

Knud Thomsen gave the formula $4\pi\left(\frac{a^p b^p + a^p c^p + b^p c^p}{3}\right)^{1/p}$, $p \approx 1.6075$, for approximating the surface area $A_e(a, b, c)$ of the ellipsoid given by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Similar type of the formulas for approximating the surface area $A_e(a, b, c)$ of an ellipsoid can be found by measuring $A_e(a, b, c)$ practically for at least two different ellipsoids and then we can create a formula which is a symmetric function of $a \geq b \geq c > 0$ which gives the two values which has been obtained practically of two different ellipsoids.

We can approximate the surface area of the ellipsoid by using any one of the following formulas because values given by both formulas are almost close in measure.

(i) $$A_e(a, b, c) \approx \pi \Big[24(ab + ac + bc) - \big((31ab + 19ac + 18bc)(18ab + 31ac + 19bc)(19ab + 18ac + 31bc)\big)^{1/3}\Big]$$

(ii) $$A_e(a, b, c) \approx \pi \Big[13(ab + ac + bc) - \big((17ab + 11ac + 7bc)(7ab + 17ac + 11bc)(11ab + 7ac + 17bc)\big)^{1/3}\Big]$$

These two formulas may be the suitable options for Knud Thomsen's formula for finding the surface area of ellipsoid. For example: At $a = 73, b = 47, c = 5$; formula (i) gives $A_e(a, b, c) = 22315.298655105$; formula (ii) gives $A_e(a, b, c) = 22306.087072404$, and Knud Thomsen's formula gives $A_e(a, b, c) = 22315.2140111$

5.3 Approximation of Surface Area of an Ellipsoid by using Surface area of Octahedrons

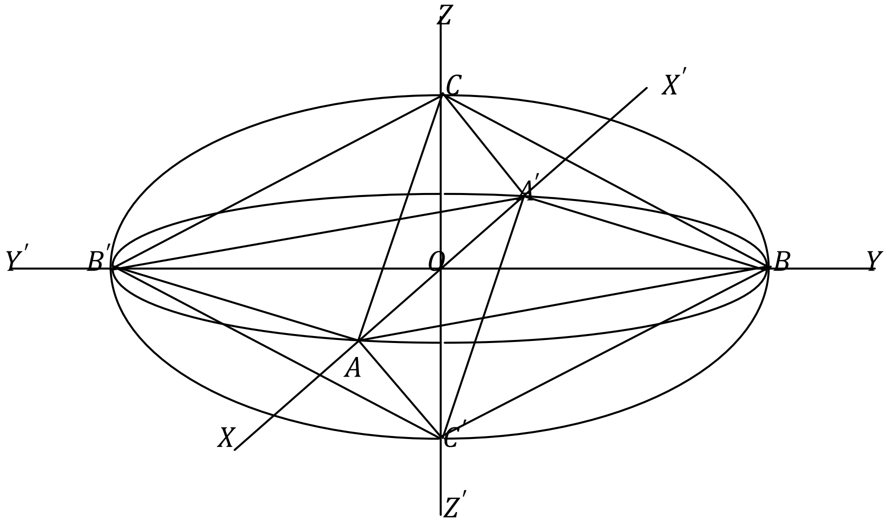

Figure-5 (octahedron associated with an ellipsoid)

Two other Forms of Heron Formula: First we find two other forms of Heron formula for finding the area of a triangle given as follows:

If u, v, w are the lengths of sides of a triangle PQR and $s = \frac{1}{2}(u+v+w)$, then area of triangle PQR is

$$\Delta = \sqrt{s(s-u)(s-v)(s-w)} = \frac{1}{4}\sqrt{[(u+v)^2 - w^2][w^2 - (u-v)^2]} = \frac{1}{4}\sqrt{(2uv)^2 - (u^2+v^2-w^2)^2}$$

Proof: By Heron formula, area of the triangle with sides of lengths u, v, w and $s = \frac{1}{2}(u+v+w)$ is

$$\Delta = \sqrt{s(s-u)(s-v)(s-w)} = \frac{1}{4}\sqrt{(u+v+w)(v+w-u)(w+u-v)(u+v-w)}$$
$$= \frac{1}{4}\sqrt{[(u+v)^2 - w^2][w^2 - (u-v)^2]} = \frac{1}{4}\sqrt{(u^2+v^2+2uv-w^2)(w^2-u^2-v^2+2uv)}$$
$$= \frac{1}{4}\sqrt{(2uv)^2 - (u^2+v^2-w^2)^2}$$

Octahedron $ABCA'B'C'$ has been shown in Figure-5. Suppose that the line $x = y = z$ determines the octahedron $A_1B_1C_1A_1'B_1'C_1'$ which has not been shown in Figure-5 such that plane faces of $A_1B_1C_1A_1'B_1'C_1'$ are parallel to the corresponding plane faces of $ABCA'B'C'$.Therefore surface area $A_e(a,b,c)$ of the ellipsoid can be approximated by 0.5(surface area of octahedron $ABCA'B'C'$ + surface area of octahedron $A_1B_1C_1A_1'B_1'C_1'$). Also in Figure-5, XOX', YOY', ZOZ' are co-ordinate axes and octahedron $ABCA'B'C'$ lie completely inside the ellipsoid represented by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Plane face ABC of octahedron $ABCA'B'C'$ makes intercepts $OA = a$, $OB = b$, $OC = c$ on X-axis, Y-axis, Z-axis respectively, so the equation of the plane ABC is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. Octahedron $ABCA'B'C'$ has 8 equal plane triangular faces each having sides of lengths $u = \sqrt{a^2+b^2}, v = \sqrt{a^2+c^2}, w = \sqrt{b^2+c^2}$. Let $s = \frac{u+v+w}{2}$, then area of each plane face of octahedron $ABCA'B'C' = \sqrt{s(s-u)(s-v)(s-w)} = \Delta$. Now by the explanation of the forms of Heron's formula given in the beginning of this section,

$$\Delta = \sqrt{s(s-u)(s-v)(s-w)} = \frac{1}{4}\sqrt{(2uv)^2 - (u^2+v^2-w^2)^2}$$
$$= \frac{1}{4}\sqrt{4(a^2+b^2)(a^2+c^2) - (a^2+b^2+a^2+c^2-b^2-c^2)^2} = \frac{1}{4}\sqrt{4(a^4+a^2b^2+a^2c^2+b^2c^2) - 4a^4}$$
$$= \frac{1}{2}\sqrt{a^2b^2+a^2c^2+b^2c^2} = \frac{1}{2}\sqrt{(ab)^2+(ac)^2+(bc)^2} = \frac{1}{2}[(ab)^2+(ac)^2+(bc)^2]^{0.5}$$

$\therefore \sqrt{(ab)^2+(ac)^2+(bc)^2} = [(ab)^2+(ac)^2+(bc)^2]^{0.5} = 2\Delta$ and $(ab)^2+(ac)^2+(bc)^2 = 4\Delta^2$

So surface area of octahedron $ABCA'B'C' = 8\sqrt{s(s-u)(s-v)(s-w)} = 8\Delta = 4\sqrt{(ab)^2+(ac)^2+(bc)^2}$

Line $x = y = z$ intersects the ellipsoid given by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ in the points $\left(\pm\frac{abc}{\sqrt{(ab)^2+(ac)^2+(bc)^2}}, \pm\frac{abc}{\sqrt{(ab)^2+(ac)^2+(bc)^2}}, \pm\frac{abc}{\sqrt{(ab)^2+(ac)^2+(bc)^2}}\right)$. Let $t = \sqrt{(ab)^2+(ac)^2+(bc)^2}$. Then equation of the plane through the point $\left(\frac{abc}{t}, \frac{abc}{t}, \frac{abc}{t}\right)$ and parallel to the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=\frac{ab+ac+bc}{\sqrt{(ab)^2+(ac)^2+(bc)^2}}=\lambda$. Intercepts made by this plane on X-axis, Y-axis, Z-axis are $a_1=a\lambda$, $b_1=b\lambda$, $c_1=c\lambda$. Then $u_1=\sqrt{a_1^2+b_1^2}=\lambda\sqrt{a^2+b^2}=\lambda u$, $v_1=\sqrt{a_1^2+c_1^2}=\lambda\sqrt{a^2+c^2}=\lambda v$, $w_1=\sqrt{b_1^2+c_1^2}=\lambda\sqrt{b^2+c^2}=\lambda w$. So each triangular plane face of octahedron $A_1B_1C_1A_1^{'}B_1^{'}C_1^{'}$ has sides of lengths u_1, v_1, w_1.

Let $s_1=\frac{u_1+v_1+w_1}{2}=\frac{\lambda u+\lambda v+\lambda w}{2}=\lambda s$, then area of each plane face of octahedron $A_1B_1C_1A_1^{'}B_1^{'}C_1^{'}$

$$=\sqrt{s_1(s_1-u_1)(s_1-v_1)(s_1-w_1)}=\lambda^2\sqrt{s(s-u)(s-v)(s-w)}=\lambda^2\Delta$$

Therefore surface area of octahedron $A_1B_1C_1A_1^{'}B_1^{'}C_1^{'}=8\sqrt{s_1(s_1-u_1)(s_1-v_1)(s_1-w_1)}=8\lambda^2\Delta$

$$=8\left(\frac{ab+ac+bc}{\sqrt{(ab)^2+(ac)^2+(bc)^2}}\right)^2\Delta=8\left(\frac{ab+ac+bc}{\sqrt{(ab)^2+(ac)^2+(bc)^2}}\right)^2\left(\frac{1}{2}\sqrt{(ab)^2+(ac)^2+(bc)^2}\right)$$

$$=4\frac{(ab+ac+bc)^2}{\sqrt{(ab)^2+(ac)^2+(bc)^2}}=4\frac{(ab+ac+bc)^2}{[(ab)^2+(ac)^2+(bc)^2]^{0.5}}$$

Let $\Delta_1=m\left(8\sqrt{s(s-u)(s-v)(s-w)}+8\sqrt{s_1(s_1-u_1)(s_1-v_1)(s_1-w_1)}\right)=8m(\Delta+\lambda^2\Delta)$ $=8m\Delta(1+\lambda^2)$, where $m>0$ is so determined that $\Delta_1\cong A_e(a,b,c)$ =Surface area of ellipsoid with semi-axes of lengths a,b,c

For $m=0.5$, let $H_1=8\times0.5\times\Delta(1+\lambda^2)=4\Delta(1+\lambda^2)=4\Delta\left(1+\frac{(ab+ac+bc)^2}{(ab)^2+(ac)^2+(bc)^2}\right)$

$$=2\left([(ab)^2+(ac)^2+(bc)^2]^{0.5}+\frac{(ab+ac+bc)^2}{[(ab)^2+(ac)^2+(bc)^2]^{0.5}}\right)$$

Again, the equation of tangent plane at the point $\left(\frac{a}{\sqrt{3}},\frac{b}{\sqrt{3}},\frac{c}{\sqrt{3}}\right)$ to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=\sqrt{3}$ Suppose that this plane determines the octahedron $A_2B_2C_2A_2^{'}B_2^{'}C_2^{'}$. Let the intercepts made by this plane on X-axis, Y-axis, Z-axis respectively are $OA_2=a_2=\sqrt{3}a$, $OB_2=b_2=\sqrt{3}b$, $OC_2=c_2=\sqrt{3}c$. Now octahedron $A_2B_2C_2A_2^{'}B_2^{'}C_2^{'}$ has 8 equal plane triangular faces each having sides of lengths

$$u_2=\sqrt{a_2^2+b_2^2}=\sqrt{3}\sqrt{a^2+b^2}=\sqrt{3}u,\qquad v_2=\sqrt{a_2^2+c_2^2}=\sqrt{3}v,\qquad w_2=\sqrt{b_2^2+c_2^2}=\sqrt{3}w$$

Let $s_2=\frac{u_2+v_2+w_2}{2}=\frac{\sqrt{3}u+\sqrt{3}v+\sqrt{3}w}{2}=\sqrt{3}s$, then area of each plane face of octahedron $A_2B_2C_2A_2^{'}B_2^{'}C_2^{'}$

$$=\sqrt{s_2(s_2-u_2)(s_2-v_2)(s_2-w_2)}\quad=\sqrt{\sqrt{3}s\left(\sqrt{3}s-\sqrt{3}u\right)\left(\sqrt{3}s-\sqrt{3}u\right)\left(\sqrt{3}s-\sqrt{3}w\right)}=3\Delta$$

Therefore surface area of octahedron $A_2B_2C_2A_2^{'}B_2^{'}C_2^{'}=8\sqrt{s_2(s_2-u_2)(s_2-v_2)(s_2-w_2)}=24\Delta$

Let $\Delta_2=m\left(8\sqrt{s(s-u)(s-v)(s-w)}+8\sqrt{s_2(s_2-u_2)(s_2-v_2)(s_2-w_2)}\right)=m(8\Delta+24\Delta)=32m\Delta$, where $m>0$ is so determined that $\Delta_2\cong A_e(a,b,c)$ =Surface area of ellipsoid.

For $m=0.5$, let $H_2=32\times0.5\times\Delta=16\Delta=8\sqrt{(ab)^2+(ac)^2+(bc)^2}=8[(ab)^2+(ac)^2+(bc)^2]^{0.5}$

We notice the following facts

(i) Marginal limit $f_*(a,b,c)=\frac{4}{3}\pi(ab+ac+bc)\le A_e(a,b,c)$, equality holds only when $a=b=c$, otherwise $f_*(a,b,c)<A_e(a,b,c)$.

(ii) $A_e(a,b,c)\cong\frac{1}{2}(H_1+H_2)=[(ab)^2+(ac)^2+(bc)^2]^{0.5}\left(5+\frac{(ab+ac+bc)^2}{(ab)^2+(ac)^2+(bc)^2}\right)$

(iii) $A_e(a,b,c)\cong\frac{1}{2}(f_*(a,b,c)+H_2)=\frac{2}{3}\pi(ab+ac+bc)+4[(ab)^2+(ac)^2+(bc)^2]^{0.5}$

Accuracy increases if the condition $c\le a<2c$ is satisfied, where $a\ge b\ge c>0$.

An interesting result relating to the expression $a^2b^2+b^2c^2+c^2a^2=(ab)^2+(bc)^2+(ca)^2$ is as follows:

Theorem-3: If $a+b=c$, then $(ab)^2+(ac)^2+(bc)^2=(a^2+bc)^2=(b^2+ac)^2$

Proof: $(ab)^2+(bc)^2+(ca)^2=a^2b^2+b^2c^2+c^2a^2=a^2(b^2+c^2)+(bc)^2$

$$=a^2[(c-b)^2+2bc]+(bc)^2=a^2(a^2+2bc)+(bc)^2=a^4+2a^2bc+(bc)^2=(a^2+bc)^2$$

Similarly, we can prove $(ab)^2+(bc)^2+(ca)^2=(b^2+ac)^2$

VI. Approximating Surface Area of an Elliptic Cylinder from its Volume

Right elliptic cylinder is the analogue of the right circular cylinder because the circular base of right circular cylinder has one radius of length 'say' r as semi major axis and semi minor axis whereas elliptic base of right elliptic cylinder has two distinct radii of length 'say' a, b as semi major axis and semi minor axis. Therefore volume of right elliptic cylinder with elliptic base of semi major axis of length a, semi minor axis of length b and height $\hbar$ can be obtained by making the correspondence of a, b with radius r of the circular base of right circular cylinder of height $\hbar$.

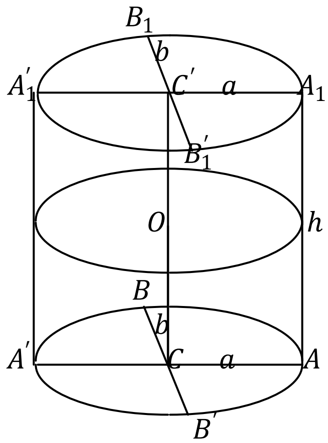

Figure-6 (right elliptic cylinder)

Volume of a right circular cylinder of base radius of length r and height $\hbar = \pi r^2 \hbar = \pi r.r.\hbar = \pi r_1.r_2.\hbar$.
Now put $r_1 = a, r_2 = b$, we get the volume of the right elliptic cylinder$= \pi ab\hbar$.

As shown in the Figure-6, centre of the elliptic cylinder is O and base radii are: $CA = a, CB = b$ and its height from the base is $CC' = \hbar$. So the volume of the cylinder with base radii of lengths a, b and height $\hbar$ is $\pi ab\hbar$. Radii of the cylinder measured from its centre O, are of the lengths $a, b, k = \frac{\hbar}{2}$, then $\hbar = 2k$. Therefore volume the cylinder becomes $2\pi abk$, let $2\pi abk = f(a, b, k)$.

Then complete surface area of the right elliptic cylinder

$$\geq f_*(a,b,k) = \lim_{\varepsilon\to 0}\frac{f(a+\varepsilon, b+\varepsilon, k+\varepsilon) - f(a,b,k)}{\varepsilon} = \lim_{\varepsilon\to 0}\frac{2\pi(a+\varepsilon)(b+\varepsilon)(k+\varepsilon) - 2\pi abk}{\varepsilon}$$

$$= 2\pi\lim_{\varepsilon\to 0}\frac{\{ab + (a+b)\varepsilon + \varepsilon^2\}(k+\varepsilon) - abk}{\varepsilon}$$

$$= 2\pi\lim_{\varepsilon\to 0}\frac{(ak+bk+ab)\varepsilon + (k+a+b)\varepsilon^2 + \varepsilon^3}{\varepsilon} = 2\pi(ak+bk+ab)$$

$$= \pi(a+b)2k + 2\pi ab = \pi(a+b)\hbar + 2\pi ab$$

i.e. complete surface area of the right elliptic cylinder $\geq \pi(a+b)\hbar + 2\pi ab$ and equality holds if $a = b$. If $b \leq a < 2b$, then complete surface area of the right elliptic cylinder $\cong \pi(a+b)\hbar + 2\pi ab$.

From this we conclude that curved surface area of the right elliptic cylinder $\geq \pi(a+b)\hbar$, equality holds if $a = b$, and its plane surface area $= 2\pi ab$. If $b \leq a < 2b$, then curved surface area of the right elliptic cylinder $\cong \pi(a+b)\hbar$.

VII. LENGTH OF AN ELLIPTIC ARC

7.1 Approximating Length of an Elliptic Arc by Marginal Limit

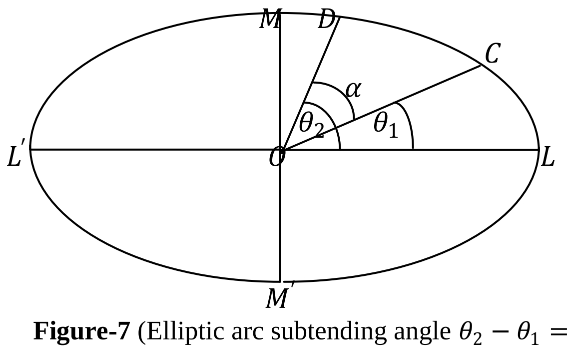

Figure-7 (Elliptic arc subtending angle $\theta_2 - \theta_1 = \alpha$ at the centre O)

In Figure-7, an ellipse with centre O, semi-major axis $OL = OL' = a$ and semi-minor axis $OM = OM' = b$, where $a \geq b > 0$, This ellipse is represented by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. We find the area of the sector COD of the ellipse, where $\angle LOC = \theta_1, \angle LOD = \theta_2, \angle COD = \theta_2 - \theta_1 = \alpha$

Let A and A_{es} denote the area of the ellipse represented by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and area of the elliptic sector COD with angle of α radians subtended at the centre O of the ellipse. Therefore $A = \pi ab$,

$$A_{es} = ab \iint r\, dr\, d\theta \text{ over the region } \{(r,\theta): 0 \leq r \leq 1, 0 \leq \theta_1 \leq \theta \leq \theta_2 \leq 2\pi\}$$

$$= ab \int_{\theta_1}^{\theta_2} \int_0^1 r\ dr\, d\theta = \frac{1}{2}(\theta_2 - \theta_1)ab = \frac{1}{2}\alpha ab$$

If $b \leq a < 2b$, and let $A_{es} = \frac{1}{2}\alpha ab = f(a,b) =$ area of the elliptic sector COD

Then, length of the arc CD of the ellipse

$$\cong f_*(a,b) = \lim_{\varepsilon \to 0} \frac{f(a+\varepsilon, b+\varepsilon) - f(a,b)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{\frac{1}{2}\alpha(a+\varepsilon)(b+\varepsilon) - \frac{1}{2}\alpha ab}{\varepsilon}$$

$$= \frac{1}{2}\alpha \lim_{\varepsilon \to 0} \frac{\{ab + (a+b)\varepsilon + \varepsilon^2\} - ab}{\varepsilon} = \frac{1}{2}\alpha \lim_{\varepsilon \to 0} \frac{(a+b)\varepsilon + \varepsilon^2}{\varepsilon} = \frac{1}{2}\alpha(a+b)$$

i.e. length of the arc $CD \cong \frac{1}{2}\alpha(a+b)$ if the condition $b \leq a < 2b$ is satisfied.

7.2 General Formula for finding the Length of Elliptic Arc

Length of an arc of an ellipse with semi axes of lengths a and b, where $a \geq b > 0$, eccentricity $e = \frac{\sqrt{a^2 - b^2}}{a}$, $b^2 = a^2(1 - e^2)$, lying between the angles θ_1 and $\theta_2 = \theta_1 + \alpha$, measured in the anticlockwise direction from the major axis of the ellipse

$$= \int_{\theta_1}^{\theta_2} \sqrt{a^2 \sin^2\theta + b^2 \cos^2\theta}\, d\theta = a \int_{\theta_1}^{\theta_2} (1 - e^2\cos^2\theta)^{\frac{1}{2}}\, d\theta$$

$$= a \int_0^{\theta_2} (1 - e^2\cos^2\theta)^{1/2}\, d\theta - a \int_0^{\theta_1} (1 - e^2\cos^2\theta)^{1/2}\, d\theta$$

For evaluating it, we find the length of an arc of an ellipse with semi axes of lengths a and b lying between the angles 0 and β, where β is measured in radians.

$$= a \int_0^{\beta} (1 - e^2\cos^2\theta)^{1/2}\, d\theta$$

$$= a \int_0^{\beta} \left[1 + \frac{1}{2}(-e^2\cos^2\theta) + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)}{2!}(-e^2\cos^2\theta)^2 + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)\left(\frac{1}{2} - 2\right)}{3!}(-e^2\cos^2\theta)^3 + \cdots\right] d\theta$$

$$= a \int_0^{\beta} \left[1 - \frac{1}{2}e^2\cos^2\theta - \frac{1}{2^2 \cdot 2!}e^4\cos^4\theta - \frac{1 \cdot 3}{2^3 \cdot 3!}e^6\cos^6\theta - \cdots - \frac{1 \cdot 3 \cdot \cdots \cdot (2n-3)}{2^n \cdot n!}e^{2n}\cos^{2n}\theta - \cdots\right] d\theta$$

$$= a \int_0^{\beta} \left(1 - \frac{1}{2}e^2\cos^2\theta - \frac{1}{8}e^4\cos^4\theta - \frac{1}{16}e^6\cos^6\theta - \frac{5}{128}e^8\cos^8\theta - \frac{7}{256}e^{10}\cos^{10}\theta - \cdots\right) d\theta$$

Also $\cos^{2n}\theta = \frac{1}{2^{2n-1}}\left[\binom{2n}{0}\cos 2n\theta + \binom{2n}{1}\cos(2n-2)\theta + \binom{2n}{2}\cos(2n-4)\theta + \cdots + \frac{1}{2}\binom{2n}{n}\right]$

$$= \frac{1}{2^{2n-1}}\left[\cos 2n\theta + 2n\cos(2n-2)\theta + \frac{2n(2n-1)}{1 \cdot 2}\cos(2n-4)\theta + \frac{2n(2n-1)(2n-2)}{1 \cdot 2 \cdot 3}\cos(2n-6)\theta + \cdots + \frac{1}{2}\binom{2n}{n}\right]$$

Therefore, general formula for finding the length of an arc of ellipse lying between the angles 0 and $\beta, n > 1$ is

$$a\left[\beta - \frac{1}{2^2 \cdot 1!}e^2\left(\frac{\sin 2\beta}{2} + \beta\right) - \frac{1}{2^5 \cdot 2!}e^4\left(\frac{\sin 4\beta}{4} + 2\sin 2\beta + 3\beta\right) - \cdots\right.$$

$$-\frac{1\cdot 3\cdot \cdots \cdot (2n-3)}{2^{3n-1}\cdot n!}e^{2n}\left(\binom{2n}{0}\frac{\sin 2n\beta}{2n}+\binom{2n}{1}\frac{\sin(2n-2)\beta}{2n-2}+\binom{2n}{2}\frac{\sin(2n-4)\beta}{2n-4}+\cdots+\beta\binom{2n}{n}\frac{1}{2}\right)-\cdots\right]$$

VIII. APPROXIMATING THE AREA OF THE BASE OF THE ELLIPSOIDAL SECTOR ON THE SURFACE OF THE ELLIPSOID BY MARGINAL LIMIT

Study of the spherical sector has been done by Harris and Stocker [4], Kern and Bland [5], Smith [7], Weisstein [10] etc. Here, ellipsoidal sector will be considered.

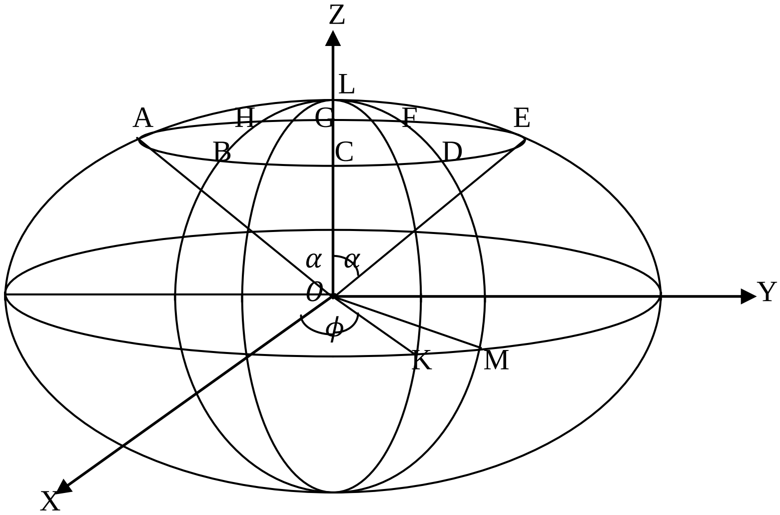

Figure-8 (Ellipsoidal Sector)

As shown in Figure-8, $OABCDEGH$ is an arbitrary ellipsoidal sector of the ellipsoid with angle $\angle AOE = 2\alpha$ subtended at the centre O of the ellipsoid, where $\angle AOL = \angle EOL = \alpha\ (0 \le \alpha \le \pi)$. Line segment OL is along z-axis. This sector is generated by one complete revolution of a line passing through the centre O of the ellipsoid and making constant angle of magnitude α with the fixed line OL passing through the centre O of the ellipsoid. An ellipsoid is also an ellipsoidal sector with angle subtended at the centre of ellipsoid is of the magnitude $2\alpha = 2\pi$ which is bisected by z-axis (OZ) at O.

Let V denotes the volume of the ellipsoid represented by the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,\ a \ge b \ge c > 0.$

$$\therefore \qquad V = abc\int_0^{2\pi}\int_0^{\pi}\int_0^{1} r^2\sin\theta\ dr\, d\theta\, d\phi = \frac{4}{3}\pi abc$$

Let V_{es} denotes the volume of the ellipsoidal sector $OABCDEFGH$;

$$\therefore \qquad V_{es} = abc\ \times \text{volume of the region}\{(r,\theta,\phi): 0 \le r \le 1, 0 \le \theta \le \alpha, 0 \le \phi \le 2\pi\}$$

$$\Rightarrow \qquad V_{es} = abc\int_0^{2\pi}\int_0^{\alpha}\int_0^{1} r^2\sin\theta\ dr\, d\theta\, d\phi = \frac{2}{3}\pi(1-\cos\alpha)abc$$

Let $V_{es} = \frac{2}{3}\pi(1-\cos\alpha)abc = f(a,b,c)$ and the condition $c \le a < 2c$ is satisfied.

Then, area of the base of the ellipsoidal sector $OABCDEFGH$ on the surface of the ellipsoid

$$\begin{aligned}\cong f_*(a,b,c) &= \lim_{\varepsilon\to 0}\frac{f(a+\varepsilon, b+\varepsilon, c+\varepsilon) - f(a,b,c)}{\varepsilon}\\ &= \lim_{\varepsilon\to 0}\frac{\frac{2}{3}\pi(1-\cos\alpha)(a+\varepsilon)(b+\varepsilon)(c+\varepsilon) - \frac{2}{3}\pi(1-\cos\alpha)abc}{\varepsilon}\\ &= \frac{2}{3}\pi(1-\cos\alpha)\lim_{\varepsilon\to 0}\frac{abc+(ab+bc+ca)\varepsilon+(a+b+c)\varepsilon^2+\varepsilon^3-abc}{\varepsilon}\\ &= \frac{2}{3}\pi(1-\cos\alpha)\lim_{\varepsilon\to 0}\frac{(ab+bc+ca)\varepsilon+(a+b+c)\varepsilon^2+\varepsilon^3}{\varepsilon} = \frac{2}{3}\pi(1-\cos\alpha)(ab+bc+ca)\end{aligned}$$

IX. CONCLUSION

Substitutes of Knud Thomsen's formula for approximating the surface area of an ellipsoid may be good options for approximating the surface area of ellipsoid. Substitutes of Ramanujan's formulas for approximating the perimeter of an ellipse can be used to approximate the perimeter of ellipse. The idea of the marginal limit may be beneficial for approximating the perimeter of an ellipse represented by the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ \ a\geq b>0$ with condition $b\leq a<2b$ and approximating the surface area of an ellipsoid represented by the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, a\geq b\geq c>0$ with condition $c\leq a<2c$. Also further advancement may be done in the method of marginal limit. We can approximate the length of an elliptic arc with appropriate accuracy by using general formula for finding the length of an elliptic arc.

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