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Research Paper

Monte Carlo Analysis of Uncertainty of Heat Distribution in a Copper Material

Samson Isobo[1], Chinedu Nwaigwe[2], Azubuike Weli[2]
1Department of Mathematics, Bayelsa State College of Education, Bayelsa State, Nigeria,
2Department of Mathematics, Rivers State University, Port Harcourt, Nigeria
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Abstract

It is known that thermal conductivity of copper exhibits some randomness. This means that the resulting temperature distribution would also exhibit some element of randomness. It is therefore necessary to investigate the effect of the randomness of the thermal conductivity on the temperature profile. Hence, in this work, we consider a stochastic heat equation in which the thermal conductivity is a random parameter. From the literature, we found that thermal conductivity of copper is 401 watts per meter per kelvin ($wm^{-1}k^{-1}$) hence, we assume it to be a normally distributed random variable with mean 401 and variance 0.02 and drew samples from this data.The solution of the heat equation at each sample point of the thermal conductivity of the copper material was obtained through the method of separation of variables, while the Monte Carlo method was used to obtain the stochastic mean and standard deviation of the random heat equation. The algorithm was implemented in python programming language. The results showed that at constant time, an increase in sample size brings the maximum stochastic mean closer to the maximum temperature profile while, for a given amount of sample size, an increase in time leads to an increase in the deviation of the maximum stochastic mean from the maximum temperature profile.

keywords: Monte Carlo, Uncertainty quantification, heat equation.

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I. Introduction

Temperature prediction is very important in many sectors of our economy for both prevention and maintenance of certain materials and equipments. The heat equation is used to describe the evolution of temperature of a given material or medium in time and space along side parameters such as thermal conductivity.
Different materials have different thermal conductivity values [6] thus, in this study we have narrowed our scope to use the thermal conductivity of copper to arrive at specific result.

Copper is a metallic substance with very high thermal conductivity and conducts heat and electricity very well. Based on it's importance in human existence, lots of studies have being conducted on the various forms of copper [9, 13]. Though, copper exist with some amount of impurities, it is said to be one of the metals that exist naturally in a usable form [7].

The heat equation which is a partial differential equation in space and time is subject to parametric uncertainty due to the thermal conductivity parameter in it. In the heat equation, thermal conductivity of material is an input parameter that must be determined for a complete analytic solution. The means by which we try to get the value for the thermal conductivity of any given material cannot be void of some assumptions thus, our input parameter becomes an uncertain input parameter which turns our model to a stochastic model.

In this paper, we focused on a non-intrusive Monte Carlo simulation method for the quantification of uncertainty in the heat equation with thermal conductivity of copper as a random input parameter and temperature as our quantity of interest (Q.I).

Results from simulations and experiments all have inherent uncertainties in their input parameters [11]. Uncertainty quantification (UQ) tries to solve the problem of how uncertain results from computational models or experiments can be. Results of uncertainty quantification are mostly statistical because uncertainties are basically probabilistic in nature [10]. Uncertainties exist in two forms; epistemic uncertainties which occur as a result of lack of knowledge of the system and aleatory uncertainties which occur as a result of natural randomness of the system [15]. Uncertainty quantification methods have being successfully

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applied in different disciplines of science and engineering, see [12, 3, 2] and references in them for more details. In this paper, we focus on the application of UQ in a well known partial differential equation model using a non-intrusive Monte Carlo method.

Monte Carlo method is one of the most used non-intrusive methods of uncertainty quantification due to its simple mode of implementation with realistic results [4]. Uncertainty quantification using Monte Carlo method has been applied by Murugan nd Ganguli [14] in the study of the performance of helicopter with interest on the thrust and power coefficients, in which they discovered that about $20-25\%$ power was needed by an helicopter for axial climb. Seven aeroelastic parameters including rotor radius and rotor angular velocity were considered as random variables. A method to quantify uncertainty in cloud computing that parallelizes the Monte Carlo method in the cloud was presented by [4]. Data from a random forest machine learning model was used by [8] where they employed the Monte Carlo method to quantify uncertainties propagated from nitrogen use efficiency prediction (NUE) using two NUE indicators. Their report showed the input parameters whose randomness affected predictions of NUE in various models considered. The uncertainty associated with temperature in an experimental heat transfer model using Monte Carlo method with temperature as the random parameter was studied by [5]. They found out that the values for temperature follow the Gaussian probability distribution. The probability density function of the solution of the randomized heat equation was the interest of [1], they considered a random diffusion coefficient, random boundary condition and an initial condition that is given as a stochastic process in the heat equation. The method applied for their problem was a polynomial approximation method.

To the best of our knowledge, the problem of quantifying uncertainty of the random heat equation with a random thermal conductivity parameter using Monte Carlo method has not being done. Moreover, the particular initial boundary condition of $sin(\pi x)$ is seen as trivial as such has not been used for this purpose.

The rest of the paper is organised as follows; section 2 deals with the formulation of the stochastic heat equation, method of solution in section 3, results and discussions in section 4 and conclusion in section 5.

II. Problem Formulation

The heat equation, also known as the diffusion equation, can be stated as

$$\frac{\partial u}{\partial t} - \alpha^2 \frac{\partial^2 u}{\partial x^2} = 0 \tag{1}$$

where $u = u(x,t)$ is the temperature at any given time t and distance x along the material under study. The constant α^2, is the thermal conductivity of the material which is the ratio of rate of heat flow to the product of the specific heat capacity and the density of the material.

The measurement of the values of specific heat capacity, heat flow rate and material density are subject to uncertainties which result to uncertain thermal conductivity values of various materials. equation 1 now becomes

$$\frac{\partial u}{\partial t} - \zeta \frac{\partial^2 u}{\partial x^2} = 0 \tag{2}$$

where ζ is a random parameter representing thermal conductivity which follows the normal probability distribution function. To complete our problem, we consider a Dirichlet boundary condition and a simple sine function as the initial condition. Thus, the complete stochastic heat equation now becomes

$$\frac{\partial u}{\partial t} - \zeta \frac{\partial^2 u}{\partial x^2} = 0 \qquad x \in (0,1), \quad t > 0, \tag{3}$$

$$U(0,t) = U(1,t) = 0,$$
$$U(x,0) = \sin(\pi x),$$
$$\zeta \sim N(\mu, \sigma^2).$$

3 methodology

An analytic solution of equation 3 can be found by using the method of separation of variables. Due to the random input parameter ζ,we have a random output $u(x,t,\zeta)$. For the standard Monte Carlo method, we take several realizations of the random output and find moments of the set of realizations from $u(x,t,\zeta)$. For more complex problems, there exist other variants of Monte Carlo methods which converges faster than the standard Monte Carlo method these include Multilevel Monte Carlo (MLMC), multifidelity Monte Carlo (MFMC) and Multimodal Monte Carlo (MMMC)[17].

The method of separation of variables [16] provides a solution of the form

$$u(x,t) = X(x)T(t) \tag{4}$$

differentiating the solution form partially and putting back into (3) we get

$$X(x) = A\sin(\lambda x) + B\cos(\lambda x). \tag{5}$$

Applying the boundary conditions, we get

$$X(x) = B\sin(n\pi x), \qquad n = 1,2,3,... \tag{6}$$

and

$$T(t) = Ce^{-(n\pi\zeta)^2 t}, \qquad n = 1,2,3,... \tag{7}$$

putting 6 and 7 into 4 with the idea of solution from a linear combination of solutions, we have

$$u(x,t) = \sum_{n=1}^{\infty} D_n \sin(n\pi x)e^{-(n\pi)^2\zeta t} \qquad n = 1,2,3,... \tag{8}$$

the constants D_n are determined using Fourier sine series analysis. Using our initial value condition and orthogonality property of the sine function, we have D_n as

$$D_1 = 1 \tag{9}$$

and

$$D_n = 0, \quad n \geq 2 \tag{10}$$

Thus, our complete analytic solution becomes

$$u(x,t) = \sin(\pi x)e^{-\pi^2\zeta t} \tag{11}$$

which is the temperature distribution at time t and distance x.

To obtain the solution for the stochastic equation, we applied the Monte Carlo method using equation 11 for M normally distributed random samples of the thermal conductivity parameter to obtain M realizations of $u(x,t,\zeta)$ from

$$u_i(x,t,\zeta) = \sin(\pi x)e^{-\pi^2\zeta_i t} \tag{12}$$

For the Monte Carlo simulation, where $i = 1,2,...,M$.
Using the python programming language, we set our Monte Carlo algorithm as follows:
(1) Draw normally distributed random samples from $\zeta_i \sim N(401, 0.02)$,
(2) Compute $u_i(x,t,\zeta)$ for $i = 1,2,...,M$,
(3) Compute stochastic mean as $u(x,t,\zeta) = \overline{u_i} = \frac{1}{M}\sum_{i=1}^{M} u_i$,
(4) Compute stochastic standard deviation as $\sigma_{u_i} = \sqrt{\frac{1}{M}\sum_{i=1}^{M}(u_i - \overline{u})^2}$

4 Result

Equipped with the method of solution from the previous section, we gathered data for the thermal conductivity to input into our computational model. We compared the stochastic mean temperature and the deterministic mean of the temperature profile the heat equation over a unit length of copper at room temperature. Our data for the thermal conductivity of copper was obtained from survey [7] . We used 401 watts per meters per kelvin($wm^{-1}k^{-1}$) as the mean thermal conductivity value of copper with a variance of 0.02 and assumed thermal conductivity to be normally distributed with mean 401 and variance 0.02 for the Monte Carlo simulation. In order to get a good visual presentation of our solution, we scale the value the thermal conductivity of copper down by dividing it by 1000. In fig 1, we kept time constant at 1 unit and varied the random samples from 100 to 100000 and in fig 2, we kept the random samples constant as 1000 realizations and varied the time as $0.3, 0.6, 0.9, 1.2$ and 1.5. In the figures below, "sto mean" stands for stochastic mean, "C.I" stands for confidence interval and "deter mean" stands for temperature profile of the heat equation.

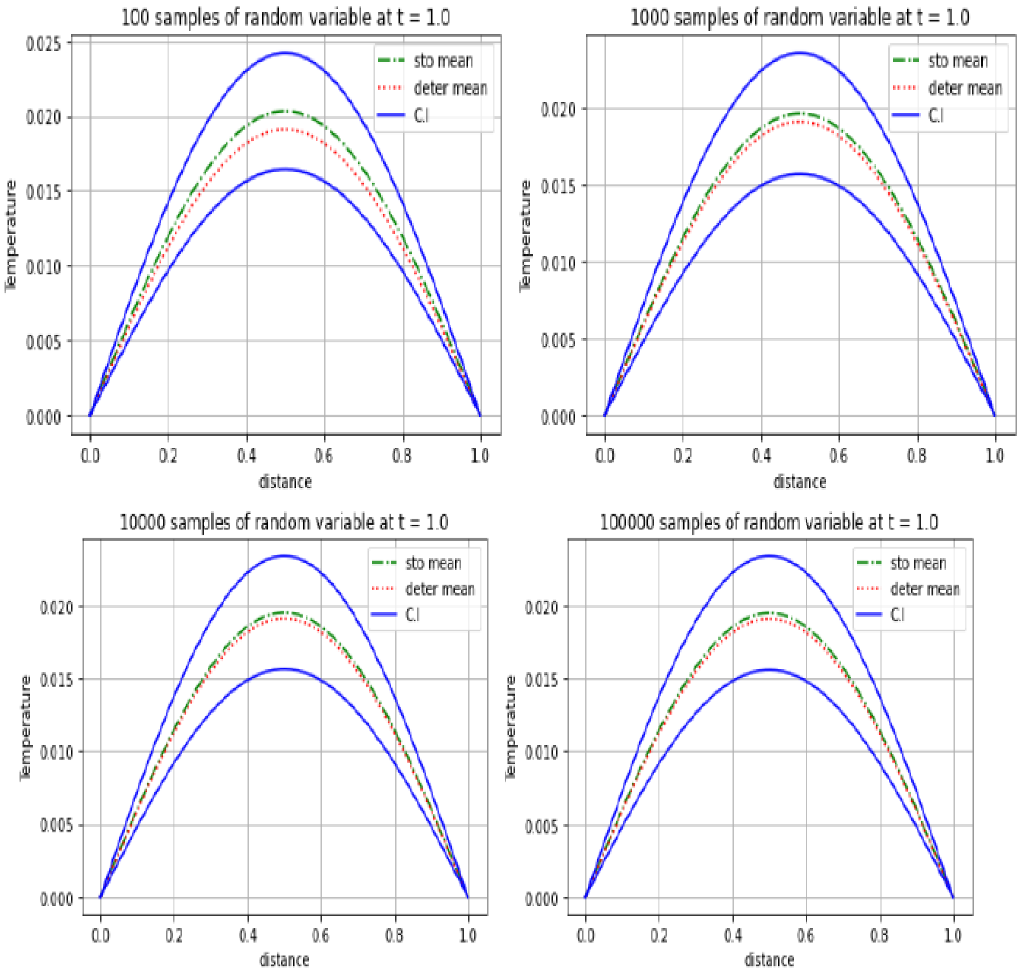

Figure 1: results time t=1 with different numbers of random samples

From figure 1, the maximum temperature, for both the temperature profile and stochastic mean distribution, remained the same for all sample sizes considered. this can be seen in all four graphs presented. Both solutions show similar temperature distribution in over 50% of the given distance of the material. The difference between the two solutions can be observed around the mid point of the distance where the stochastic solution is a little higher than the temperature profile which accounts for the uncertainties due to the uncertain input parameter. Both solutions remained within the boundaries of the 68% confidence interval of the generated data. The distance between the peaks of the boundaries of the confidence interval remained constant with increase in sample size.

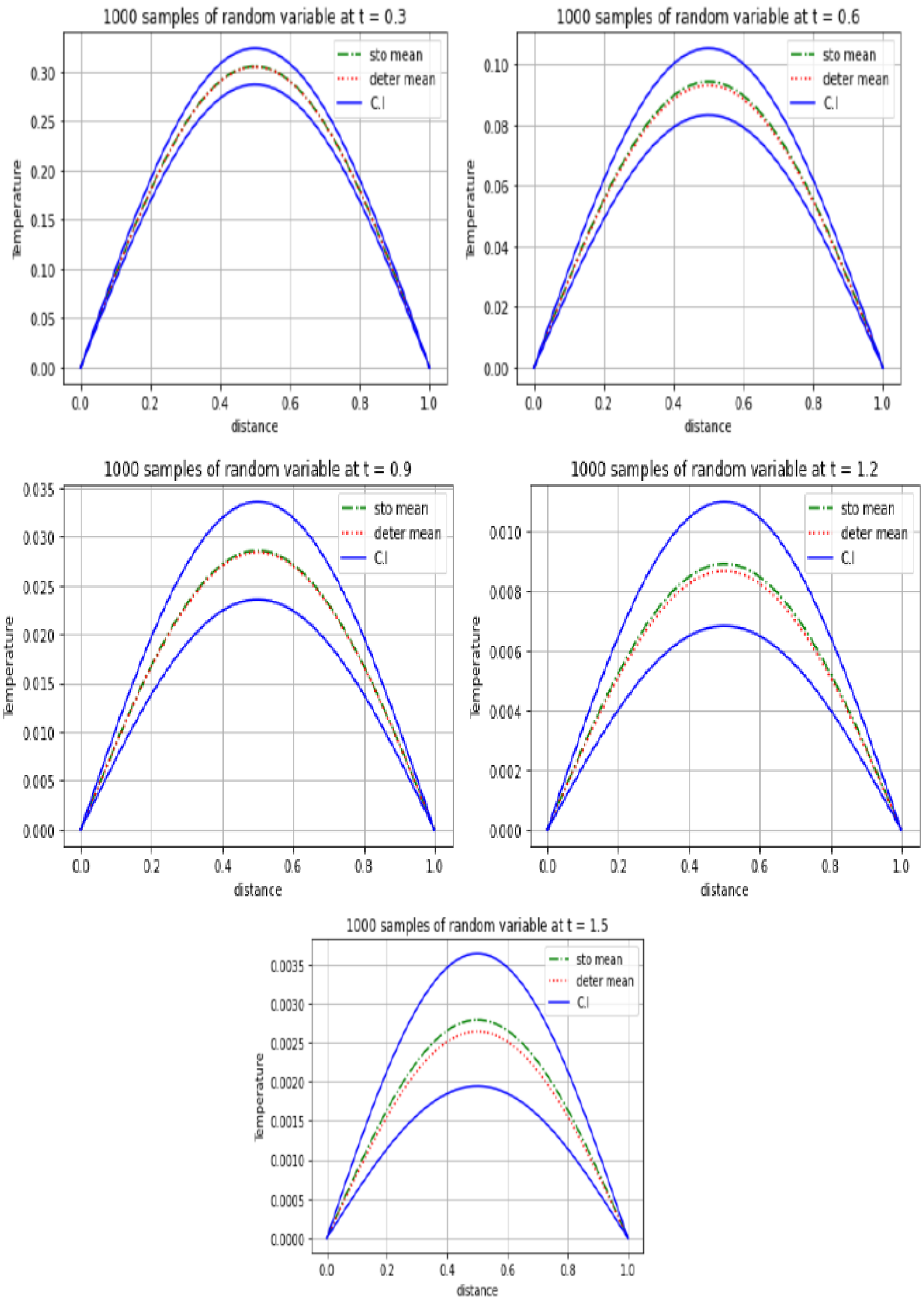

Figure 2: results for constant size of random samples at different time points

In figure 2, we used 1000 random variables to run the Monte Carlo simulation for 5 different time points. Our results showed similar temperature distribution for $t = o.3$ and $t = 0.6$ in both stochastic solution and deterministic solution. The results also show a reduction in temperature as time increases in both solutions. As time increases, we see a change in the difference between the maximum temperature of the stochastic solution and that of the temperature profile which implies an increases in the uncertainty level of the input parameter. There was also an increase in the difference between the confidence intervals as time increases while, both solutions remain within the confidence interval. This also confirms an increase in uncertainty due to increase in time.

5 Conclusion

The effect of random thermal conductivity of the heat distribution in a copper material has been investigated by applying Monte-Carlo analysis to one-dimensional heat equation which is solved by the method of separation of variables. The results show that a random thermal conductivity propagates a random Temperature that exhibits the behaviour of the initial temperature. The Monte Carlo approach also gave results that are very significant, when compared with the temperature profile, which clearly showed the effect of the uncertainty of the input parameter on the output. For more research into the accuracy of the method used for this study, one can consider other methods of uncertainty quantification like the collocation methods and compare to these results so far obtained. Quartiles and percentiles of the generated results can also be calculated to further expose the uncertainties propagated by the input parameter.

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