Quest Journals
Journal of Research in Applied Mathematics
Volume 9 ~ Issue 3 (2023) pp: 61-73
ISSN(Online) : 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Research Paper

Ninth order and An Optimal Class of Eighth-Order Iterative Methods for Solving Nonlinear Equations

M.A. Aqel[1], I.A. Al-Subaihi[2*]
[1]*(Department of Artificial Intelligence, College of Computer and Cyber Sciences, University of Prince Mugrin, Saudi Arabia)*
[2]*(Department of General Studies, University of Prince Mugrin, Saudi Arabia)*
Corresponding Author: I.A. Al-Subaihi

ABSTRACT : *In this paper, we present two novel iterative methods for solving nonlinear equations. These methods were developed after a discovery during the process of proving a family of ninth order convergence iterative methods that were proposed in a previous paper published in 2020. The discovery showed that the previously proposed family of methods actually had a third order of convergence, therefore alternative methods were needed. The first novel method presented in this paper is a modified version of the third order method of the previous paper, which has been altered to have a ninth order of convergence with an efficiency index of 1.4422. This modification was achieved by applying specific changes in the second step of the method. Moreover, further development was conducted which led to this paper's second novel method which is an optimal eighth order class with five iterative methods that has an efficiency index of 1.6817. It was developed by approximating two derivatives followed by applying weighted functions. Several numerical comparisons were established to test the performance and efficacy of the proposed methods.*

KEYWORDS: *Optimal eighth-order, Ninth order, Nonlinear equations, Iterative Methods, Numerical Analysis, Order of Convergence, Efficiency index.*

Received 14 Mar., 2023; Revised 27 Mar., 2023; Accepted 29 Mar., 2023

I. INTRODUCTION

One of the oldest and most raised problems in Numerical Analysis is solving nonlinear functions. These problems can be applied especially in these fields: Applied Mathematics, Artificial Intelligence, and Engineering. Therefore, the demand for iterative methods to solve nonlinear equations was crucial.

$$f(x) = 0, \tag{1}$$

Where f: I ⊆ R → R for an open interval I [1-12].

Newton's method (NM) is one of the oldest and most famous methods to solve nonlinear equations which is identified as the following

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{2}$$

Newton's method is a method that converges quadratically. Using Kung-Traub conjecture [1], we can calculate if the method is optimal by 2^{d-1} where d is the number of function evaluations per iteration. Moreover, the efficiency index is determined by $r^{\frac{1}{d}}$ where r is order of convergence. Therefore, Newton's method is an optimal method and its efficiency index is $2^{\frac{1}{2}} \approx 1.4142$. Mylapalli et al (MSK) [2] used Newton's Method as a first step and is given by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - \frac{2f(x_n)}{f'(x_n) + \sqrt{(2f'(y_n) - f'(x_n))f'(x_n)}},$$

$$x_{n+1} = z_n - \frac{2f(z_n)}{f'(z_n)}\left(\frac{1}{1 + \sqrt{1 - 2\mu_n}}\right), \tag{3}$$

where

$$\mu_n = \frac{f(z_n)f''(z_n)}{[f'(z_n)]^2}, \tag{4}$$

and

$$f''(z_n) = \frac{2}{y_n - z_n}\left[3\frac{f(y_n) - f(z_n)}{y_n - z_n} - 2f'(z_n) - f'(y_n)\right]. \tag{5}$$

This method (3) has a third order of convergence, three functional evaluations, and three first derivatives. Therefore, it is not optimal. In addition, its efficiency index is $3^{\frac{1}{6}} \approx 1.2009$.

II. CONSTRUCTION OF NEW CLASS OF METHODS

Our initial aim is to modify (3) to construct a new family of ninth order of convergence. This modification was established by substituting y_n by x_n in the second step. The resulting family is presented in the following (6).

Changing y_n to x_n in the second step yields the novel ninth order of convergence method (MSM):

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = x_n - \frac{2f(x_n)}{f'(x_n) + \sqrt{(2f'(y_n) - f'(x_n))f'(x_n)}},$$

$$x_{n+1} = z_n - \frac{2f(z_n)}{f'(z_n)}\left(\frac{1}{1 + \sqrt{1 - 2\mu_n}}\right). \tag{6}$$

Where μ_n and $f''(z_n)$ are defined in (4) and (5), respectively.

Furthermore, we propose our development of MSM (6) by presenting an optimal eighth order of convergence class with five iterative methods achieved by the following steps:

Firstly, we approximate $f'(y_n)$ and $f'(z_n)$ [3] to reduce the function evaluation number per iteration.

$$Q(x) = f'(y_n) = 2f[x_n, y_n] - f'(x_n), \tag{7}$$

where

$$f[x_n, y_n] = \frac{f(y_n) - f(x_n)}{y_n - x_n}, \tag{8}$$

$$P(x) = f'(z_n) = f[x_n, z_n] + f[y_n, z_n] - f[x_n, y_n], \tag{9}$$

where

$$f[x_n, z_n] = \frac{f(z_n) - f(x_n)}{z_n - x_n}, \tag{10}$$

$$f[y_n, z_n] = \frac{f(z_n) - f(y_n)}{z_n - y_n}. \tag{11}$$

Substituting by $P(x)$ and $Q(x)$ defined by (7) and (9) in $f''(z_n)$ it will be redefined as follows

$$W(x) = f''(z_n) = \frac{2}{y_n - z_n}[3\,f[y_n, z_n] - 2P(x) - Q(x)], \tag{12}$$

Moreover, substituting with $W(x)$ and $P(x)$ defined by (9) and (12), μ_n equals

$$\mu_n = \frac{f(z_n)W(x)}{P(x)^2}. \tag{13}$$

Subsequently substituting equations $(7), (9), and$ (12) into (3) will obtain a seventh-order method presented below

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = x_n - \frac{2f(x_n)}{f'(x_n) + \sqrt{(2Q(x) - f'(x_n))f'(x_n)}},$$

$$x_{n+1} = z_n - \frac{2f(z_n)}{P(x)}\left(\frac{1}{1 + \sqrt{1 - 2\mu_n}}\right). \tag{14}$$

The modified approach obtained in (14) has a seventh order of convergence. To modify it to an optimal eighth order we multiply by weighted functions $H(m_1), K(m_2)$, and , $B(m_3)$ where $m_1 = \frac{f(y_n)}{f(x_n)}$, $m_2 = \frac{f(z_n)}{f(y_n)}$, and $m_3 = \frac{f(z_n)}{f(x_n)}$. We obtain the optimal class of iterative methods with an eighth order of convergence as follows

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = x_n - \frac{2f(x_n)}{f'(x_n) + \sqrt{(2Q(x) - f'(x_n))f'(x_n)}},$$

$$x_{n+1} = z_n - \frac{2f(z_n)}{P(x)}\left(\frac{1}{1 + \sqrt{1 - 2\mu_n}}\right)\left(H(m_1)\ K(m_2)\,B(m_3)\right). \tag{15}$$

III. THE CONVERGENCE ANALYSIS AND CONCRETE ITERATIVE METHODS

We have used Maple (2022) in this section to demonstrate and analyze the methods' convergence order. What follows is an analysis of the convergence of the novel ninth order method (MSM), and the optimal eighth order method presented in (6), (15) respectively.

Theorem 1: Consider α be a simple root of a sufficiently differentiable function in I, an open interval f: I ⊆ R → R (1). if x_0 is relatively close to α then the class of iterative methods (6) has ninth order convergence.

Proof: Let $e_n = x_n - \alpha$ be the error in the nth iteration where $f(\alpha) = 0$, Expanding the Taylor expansion of $f(x)$ about α produces the subsequent result

$$f(x) = f'(\alpha)(e + c_2e^2 + c_3e^3 + c_4e^4 + c_5e^5 + c_6e^6 + c_7e^7 + c_8e^8 + c_9e^9 + O(e^{10}). \tag{16}$$

Where $c_i = \frac{f^{(i)}(\alpha)}{i!\,f'(\alpha)}$, i = 2,3 ....

$$f'(x) = f'(\alpha)(1 + 2c_2e + 3c_3e^2 + 4c_4e^3 + 5c_5e^4 + 6c_6e^5 + 7c_7e^6 + 8c_8e^7 + 9c_9e^8 + 10c_{10}e^9 + O(e^{10})). \quad (17)$$

Dividing (16) over (17), will result the following

$$\frac{f(x)}{f'(x)} = e - c_2e^2 + (2c_2^2 - 2c_3)e^3 + \cdots + (128c_2^8 - 704c_2^6c_3 + \cdots - 8c_9)e^9 + O(e^{10}). \quad (18)$$

Subsequently, substituting (18) into the initial step of (15) and (6) will cause the following

$$y_n = \alpha + c_2e^2 + (-2c_2^2 + 2c_3)e^3 + \cdots + (-128c_2^8 + 704c_2^6c_3 + \ldots + 8c_9)e^9 + O(e^{10}). \quad (19)$$

From (19), $f(y_n)$ about α by using Taylor Expansion yields

$$f(y_n) = f'(\alpha)[c_2e^2 + \cdots + (-320c_2^8 + 1424c_2^6c_3 + \cdots + 8c_9)e^9 + O(e^{10})]. \quad (20)$$

$$f'(y_n) = f'(\alpha)[1\_ + 2c_2^2e^2 + \cdots + (-256c_2^9 + 832c_2^7c_3 + \cdots + 16c_2c_9)e^9 + O(e^{10})]. \quad (21)$$

By substituting the equations (16), (17), and (21) into (6), the resultant equation will be

$$z_n = \alpha + \frac{1}{2}c_3e^3 + \left(c_4 - \frac{3c_2c_3}{2}\right)e^4 + \ldots + \left(78c_2^3c_3c_4 + \ldots + \frac{7}{2}c_9\right)e^9 + O(e^{10}). \quad (22)$$

Taylor expansion of $f(z_n)$ about α from (22) is

$$f(z_n) = f'(\alpha)[\frac{c_3}{2}e^3 + \ldots + \left(92c_2^3c_3c_4 + \ldots + \frac{7c_9}{2} - \frac{181c_2^2c_3c_5}{2}\right)e^9 + O(e^{10})]. \quad (23)$$

$$f'(z_n) = f'(\alpha)[1 + c_2c_3e^3 + (2c_2c_4 - 3c_2^2c_3)e^4 + \cdots + (-167c_2^3c_3c_5 - 394c_2^2c_3^2c_4 + \ldots + 156c_2^4c_3c_4)e^9 + O(e^{10})]. \quad (24)$$

From (19), (20), (22), and (23) we conclude

$$f[y_n, z_n] = f'(\alpha)\left[1 + c_2^2e^2 + \cdots + \left(-32c_2^7 + \frac{175c_2^2c_3c_4}{2} + \ldots + \frac{17c_2c_7}{2}\right)e^7 + O(e^8)\right]. \quad (25)$$

Substituting by (19), (21), (22), (24), and (25) into (6)

$$f''(z_n) = f'(\alpha)[2c_2 + 3c_3^2e^3 + \ldots + (8c_2^3c_4 + 18c_2^2c_3^2 + \ldots - 18c_2c_3c_4)e^5 + O(e^6)]. \quad (26)$$

Substituting by (23), (24), and (26) into (6)

$$\mu_n = c_2c_3e^3 + (-3c_2^2c_3 + 2c_2c_4)e^4 + \cdots + (-69c_2^3c_3c_4 + \cdots - 18c_3^4)e^8 + O(e^9). \quad (27)$$

By substituting (22), (23), (24), (27) into (6)

$$x_{n+1} = \alpha - \frac{1}{8}c_3^4e^9 + O(e^{10}). \quad (28)$$

Theorem 2: Consider α be a simple root of a sufficiently differentiable function in I, an open interval f: I ⊆ R → R (1). if x_0 is relatively close to α then the class of iterative methods (15) has an optimal eighth order convergence once the conditions below are fulfilled.

$$H(0) = 1,\ H'(0) = H''(0) = H'''(0) = 0,\ \left|H^{(4)}(0)\right| < \infty,$$
$$K(0) = 1, K'(0) = 0, |K''(0)| < \infty, B(0) = B'(0) = 1.$$

Proof: Let $e_n = x_n - \alpha$ be the error in the nth iteration where $f(\alpha) = 0$, Expanding the Taylor expansion of $f(x)$ about α produces the subsequent result

By using (16) – (20) we continue the proof as follows

From (16), (19), and (20) we obtain

$$f[x_n, y_n] = f'(\alpha)[1 + c_2 e + (c_2^2 + c_3)e^2 + \cdots + (112c_2^5 c_3 + \cdots - 30c_2 c_3 c_5)e^7 + O(e^8)]. \tag{29}$$

By substituting (17), (29) in (14)

$$Q(x) = f'(\alpha)[1 + (2c_2^2 - c_3)e^2 + \cdots + (-512c_2^6 c_3 + 368c_2^5 c_4 + \cdots - 80c_2 c_4 c_5)e^8 + O(e^9)]. \tag{30}$$

By substituting the equations (16), (17), and (30) into (15), the resultant equation will be

$$z_n = \alpha - c_2 c_3 e^4 + \left(2{c_2}^2 c_3 - 2c_2 c_4 - 2c_3^2\right)e^5 + \cdots + \left(-16c_2^5 c_3 + \cdots - 17c_4 c_5\right)e^8 + O(e^9). \tag{31}$$

Taylor expansion of $f(z_n)$ about α from (31) is

$$f(z_n) = f'(\alpha)[-c_2 c_3 e^4 + (2c_2^2 c_3 - 2c_2 c_4 - 2c_3^2)e^5 + (-16c_2^5 c_3 + \cdots - 17c_4 c_5)e^8 + O(e^9)]. \tag{32}$$

From (16), (31), and (32) we obtain

$$f[x_n, z_n] = f'(\alpha)[1 + c_2 e + c_3 e^2 + \cdots + (8c_2^5 c_3 + \cdots - 14c_2 c_3 c_5)e^7 + O(e^8)]. \tag{33}$$

Substituting (25),(33), and (29) in (15)

$$P(x) = f'(\alpha)[1 - c_2 c_3 e^3 + (-c_2 c_7 - 7c_3 c_6 + \ldots . -8c_2 c_3 c_5)e^7 + O(e^8)]. \tag{34}$$

By using (19), (25), (30), (31), and (34) in (15) we define $W(x)$ *as*

$$W(x) = f'(\alpha)[2c_2 + \frac{2c_3}{c_2} + (4c_3 + \cdots - \frac{4c_3^2}{c_2^2})e + \cdots + (16c_6 + \ldots + 8c_2^3 c_3)e^4 + \cdots + O(e^6)]. \tag{35}$$

Substituting (32), and (34), and (35) into (15) would result

$$\mu_n = (-2c_2^2 c_3 - 2c_3^2)e^4 + (4c_3 c_2^3 + \ldots - 8c_3 c_4)e^5 + \cdots + (-32c_2^6 c_3 + \cdots - 18c_5^2)e^8 + O(e^9) \tag{36}$$

In a view of (31), (32), (34), and (36) into (15) we acquire

$$x_{n+1} = \alpha + c_2^2 c_3^2 e^7 + (-4c_2^3 c_3^2 + 3c_2^2 c_3 c_4 + 3c_2 c_3^3)e^8 + O(e^9). \tag{37}$$

Since the method (14) has a seventh order of convergence, we transform it into an eighth order of convergence by multiplying in weighted functions. We expand $H(m_1), K(m_2)$ and $B(m_3)$ by Taylor expressions as shows.

$$H(m_1) = H(0) + H'(0)m_1 + \frac{1}{2}H''(0)m_1^2 + H'''(0)\frac{m_1^3}{3!} + H^{(4)}(0)\frac{m_1^4}{4!} + \cdots + O(m_1)^9. \tag{38}$$

$$K(m_2) = K(0) + K'(0)m_2 + \frac{1}{2}K''(0)m_2^2 + K'''(0)\frac{m_2^3}{3!} + K^{(4)}(0)\frac{m_2^4}{4!} + \cdots + O(m_2)^9. \tag{39}$$

$$B(m_3) = B(0) + B'(0)m_3 + \frac{1}{2}B''(0)m_3^2 + B'''(0)\frac{m_3^3}{3!} + B^{(4)}(0)\frac{m_3^4}{4!} + \cdots + O(m_3)^9. \tag{40}$$

Finally, using (37) – (40) and applying its conditions $H(0)=1$, $H'(0) = H''(0) = H'''(0) = 0$, $\left|H^{(4)}(0)\right| < \infty$, $K(0)=1$, $K'(0) = 0$, $\left|K''(0)\right| < \infty$, $B(0) = B'(0) = 1$

from $e_{n+1} = x_{n+1} - \alpha$, we have the error expression

$$e_{n+1} = \left(\frac{1}{2}c_2c_3^3K''(0) - c_2c_3^3 + c_2^3c_3^2 - c_2^2c_3c_4 + \frac{1}{24}c_2^5c_3H^{(4)}(0)\right)e^8 + O(e^9). \tag{41}$$

We now conclude the proof of the method defined by (15) which proves that the method has an optimal eighth order of convergence.

The Iterative Methods

In this section, we present iterative methods for weighted functions that satisfy its conditions to obtain an optimal eighth order family.

Method 1 (MSM1): Let

$H(m_1) = a\,m_1^f + 1, \qquad a, f \in R \text{ and } f \geq 4$

$K(m_2) = t\,m_2^q + 2\,m_2^r + 1, \qquad t, q, r \in R \text{ and } q, r > 1$

$B(m_3) = m_3 + 1.$

We can observe that the functions $H(m_1)$, $K(m_2)$ and $B(m_3)$ satisfy the conditions of the theorem which results in an eighth order family (MSM1)

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = x_n - \frac{2f(x_n)}{f'(x_n) + \sqrt{(2Q(x) - f'(x_n))f'(x_n)}},$$

$$x_{n+1} = z_n - \frac{2f(z_n)}{P(x)}.\left(\frac{1}{1+\sqrt{1-2\mu_n}}\right)\left((a\,m_1^f + 1)\left(t\,m_2^q + 2\,m_2^r + 1\right)(m_3 + 1)\right). \tag{42}$$

Method 2 (MSM2): Let

$H(m_1) = \cos(m_1) + \frac{m_1^2}{2}$

$K(m_2) = \cos(m_2) + g\,m_2^l, \qquad g, l \in R \text{ and } l > 1$

$B(m_3) = \cos(m_3) + m_3 + s\,m_3^j. \qquad s, j \in R \text{ and } j > 1$

It can be noticed that the functions satisfy the conditions of the theorem which produces a new family that has an eighth order (MSM2)

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = x_n - \frac{2f(x_n)}{f'(x_n) + \sqrt{(2Q(x) - f'(x_n))f'(x_n)}},$$

$$x_{n+1} = z_n - \frac{2f(z_n)}{P(x)}\left(\frac{1}{1+\sqrt{1-2\mu_n}}\right)\left((\cos(m_1) + \frac{m_1^2}{2})\,(\cos(m_2) + g\,m_2^l)\left(\cos(m_3) + m_3 + s\,m_3^j\right)\right). \tag{43}$$

Method 3 (MSM3): Let

$H(m_1) = \sin(m_1) + e^{m_1} - \frac{m_1^2}{2} - 2m_1\,,$

$K(m_2) = m_2^w + 1\,, \qquad w \in R \text{ and } w > 1$

$B(m_3) = \sin(m_3) + 1.$

Since the functions meet the conditions for the theorem a new family of eighth order of convergence is presented (MSM3)

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = x_n - \frac{2f(x_n)}{f'(x_n) + \sqrt{(2Q(x) - f'(x_n))f'(x_n)}},$$

$$x_{n+1} = z_n - \frac{2f(z_n)}{P(x)}\left(\frac{1}{1+\sqrt{1-2\mu_n}}\right)\left((sin(m_1) + e^{m_1} - \frac{m_1^2}{2} - 2m_1)\ (\ m_2^w + 1)(sin(m_3) + 1)\right). \quad (44)$$

Method 4 (MSM4): Let

$H(m_1) = e^{m_1} - \frac{m_1^3}{6} - \frac{m_1^2}{2} - m_1,$

$K(m_2) = m_2^h e^{m_2} + 1, \qquad h \in R \ and \ h > 1$

$B(m_3) = \sin(m_3) + \cos(m_3).$

The functions certainly satisfy the conditions for the theorem which yields a new family of an eighth order (MSM4).

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = x_n - \frac{2f(x_n)}{f'(x_n) + \sqrt{(2Q(x) - f'(x_n))f'(x_n)}},$$

$$x_{n+1} = z_n - \frac{2f(z_n)}{P(x)}\left(\frac{1}{1+\sqrt{1-2\mu_n}}\right)\left((e^{m_1} - \frac{m_1^3}{6} - \frac{m_1^2}{2} - m_1)\ (\ m_2^h e^{m_2} + 1)(sin(m_3) + cos(m_3))\right). \quad (45)$$

Method 5 (MSM5): Let

$H(m_1) = k\, m_1^f\ e^{m_1} + 1, \qquad k, f \ \in R \ and \ , f \geq 4.$

$K(m_2) = y\, m_2^p + 1, \qquad y, p \in R \ and \ p > 1$

$B(m_3) = m_3 + (d\, m_3^b) + 1. \quad b, d \in R \ and \ b > 1$

As it appears, the functions certainly meet the conditions for the theorem which creates a new family of an eighth order (MSM5).

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = x_n - \frac{2f(x_n)}{f'(x_n) + \sqrt{(2Q(x) - f'(x_n))f'(x_n)}},$$

$$x_{n+1} = y_n - \frac{2f(z_n)}{P(x)}\left(\frac{1}{1+\sqrt{1-2\mu_n}}\right)\left(\ k\, m_1^f\ e^{m_1} + 1)\ (\ y\, m_2^p + 1)(m_3 + (d\, m_3^b) + 1\right). \quad (46)$$

IV. NUMERICAL EXAMPLES

We put the proposed methods MSM (6) and MSM1-MSM5 (42 - 46) to the test by evaluating them on some nonlinear examples to demonstrate their efficiency, and performance by comparing it to other eighth, and ninth order of convergence methods. For eighth order we present the following methods for comparisons.

Method proposed by Sharma et al (SAM) [4]:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - \left(3 - 2\frac{f[y_n, x_n]}{f'(x_n)}\right)\frac{f(y_n)}{f'(x_n)},$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)}\left(\frac{f'(x_n) - f[y_n, x_n] + f[z_n, y_n]}{2f[z_n, y_n] - f[z_n, x_n]}\right). \quad (47)$$

Method proposed by Abbas and Al-Subaihi (HSM) [5] :

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = x_n + (\beta - 1)\frac{f(x_n)\big(f(x_n) - f(y_n)\big)}{f'(x_n)\big(f(x_n) - 2\,f(y_n)\big)} - \beta\left(\frac{f(x_n)}{f'(x_n)} + \frac{f(y_n)\ \left(f(x_n)^3 f(y_n)^2 f(x_n) + \frac{1}{2} f(y_n)^3\right)\big(f(x_n) + f(y_n)\big)^2}{f'(x_n) f(x_n)^5}\right),$$

$$x_{n+1} = z_n - \frac{f(z_n)}{a_1 + 2a_2(z_n - x_n) + 3a_3(z_n - x_n)} \tag{48}$$

Where $\beta = 2,\ a_1 = f'(x_n),\ a_2 = \frac{f[y_n,x_n,x_n](z_n - x_n) - f[z_n,x_n,x_n](y_n - x_n)}{z_n - y_n},\ a_3 = \frac{f[z_n,x_n,x_n] - f[y_n,x_n,x_n]}{z_n - y_n}$.
Method proposed by Liu and Wang (LWM) [6] :

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - \left(\frac{f(y_n) f(x_n)}{f'(x_n) f(x_n) - 2f'(x_n) f(y_n)}\right),$$

$$x_{n+1} = z_n - \left(\frac{f(z_n)}{f'(x_n)}\right)\left(\left(\frac{f(x_n) - f(y_n)}{f(x_n) - 2f(y_n)}\right)^2 + \left(\frac{f(z_n)}{f(y_n) - f(z_n)}\right) + \left(4\frac{f(z_n)}{f(x_n) + f(z_n)}\right)\right). \tag{49}$$

Method proposed by Al-Harbi and Al-Subaihi (TSM) [7] :

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - \frac{f(x_n) + \beta f(y_n)}{f(x_n) + (\beta - 2) f(y_n)} \frac{f(y_n)}{f'(x_n)},$$

$$x_{n+1} = z_n - \{-2\beta\, s_1^3\, e^{s_1}(\, s_2^4 + \, s_2^4 \sin(s_3) + \sin(s_3) + 1) + (\, s_2^4 + 1)(\sin(s_3) + 1)\}\left(\frac{f(z_n) f[x_n, y_n]}{f[y_n, z_n] f[x_n, z_n]}\right). \tag{50}$$

Where $s_1 = \frac{f(y_n)}{f(x_n)},\ s_2 = \frac{f(z_n)}{f(y_n)},\ s_3 = \frac{f(z_n)}{f(x_n)}, and\ \beta = 0.$

The following methods are provided for comparison with the ninth order family.
Method proposed by Al-Subaihi et al (SM) [8]

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - \frac{2f(y_n) f'(y_n)}{2\big(f'(y_n)\big)^2 - f(y_n) P_f(x_n, y_n)},$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f[z_n, y_n] + f[z_n, x_n, x_n](z_n - y_n)}. \tag{51}$$

Where (10), (11),

$$P_f(x_n, y_n) = \frac{2}{(y_n - x_n)}\left(2f'(y_n) + f'(x_n) - 3\ \frac{f(y_n) - f(x_n)}{y_n - x_n}\right),$$

$$f[z_n, x_n, x_n] = \frac{f[z_n, x_n] - f'(x_n)}{(z_n - x_n)}.$$

Method proposed by Zhong et al (ZHONG) [9]

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - \left(1 + \left(\frac{f(y_n)}{f(x_n)}\right)^2\right)\frac{f(y_n)}{f'(y_n)},$$

$$x_{n+1} = z_n - \left(1 + 2\left(\frac{f(y_n)}{f(x_n)}\right)^2 + 2\frac{f(z_n)}{f(y_n)}\right)\frac{f(z_n)}{f'(y_n)}. \qquad (52)$$

Method proposed by Muhaijir et al (MM) [10]

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - \frac{f(y_n)}{f'(y_n)}\left(1 - \frac{f'(y_n)f'(x_n)f(y_n) - f(y_n)\left(f'(x_n)\right)^2}{2f(x_n)\left(f'(y_n)\right)^2}\right),$$

$$x_{n+1} = z_n - \frac{f(z_n)}{2f[x_n, z_n] + f[y_n, z_n] - 2f[x_n, y_n] + (y_n - z_n)f[x_n, x_n, y_n]}, \qquad (53)$$

where (8), (10), (11),

$$f[x_n, x_n, y_n] = z_n - \frac{f[y_n, x_n] - f'(x_n)}{y_n - x_n}.$$

Table 1 shows the nonlinear examples used to test the methods with twenty-seven decimal digits.

Table 1. Test functions and their roots

Functions	Roots (α)
$f_1(x) = xe + \log(1 + x + x^4)$	0.0
$f_2(x) = x^3 + 4x^2 - 15$	1.631980805566063517522106446
$f_3(x) = \dfrac{\sqrt{x} - 1}{x - 3}$	9.633595562832695192406312709
$f_4(x) = \log x + \sqrt{x} - 5$	8.309432694231571795346955683

Since (MSM) only converges in the first example, two more examples were added to test the efficiency of the method.

Table 2. Test functions for (MSM) and their roots

Functions	Roots (α)
$f_5(x) = e^x \sin x + \log(x^2 + 1)$	0.0
$f_6(x) = x^3 - 10$	2.154434690031883721759293567

All computations were done by MATLAB (R2022a) software using 1000 digits floating points. The stopping criteria are

$$|x_n - \alpha| \le 10^{-300},$$
$$|f(x_n)| \le 10^{-300}.$$

Table 2 presents a comparison of methods with an optimal eighth order and ninth order in its number of iteration (IT), absolute value of the function $|f(x_n)|$ and the absolute error $|x_n - \alpha|$. Finally, the computational order of convergence approximated by the following

$$\rho = \frac{ln|(x_{n+1} - \alpha)/(x_n - \alpha)|}{ln|(x_n - \alpha)/(x_{n-1} - \alpha)|}. \quad (54)$$

Remark: the values of the weighted functions in the methods are specified as follows:

In MSM1 (42) a,t =1, q, r =2, and in both MSM1 and MSM5 (46) f =4.

In MSM2 (43) g,$s = 1$ and l, $j = 2$.

In MSM3(44) $w = 5$, and MSM4 (45) $h = 2$.

Finally, in MSM5(46) $k, y, d = 1$ and p, b =2.

Table 3. Comparison of various iterative methods.

Method	IT	$\lvert f(x_n)\rvert$	$\lvert x_n - \alpha\rvert$	COC
$f_1(x) = xe + \log(1 + x + x^4)$ x_0 =0.25				
NM	9	5.41196e-552	2.70598e-552	2
MSK	6	2.77321e-707	1.3866e-707	3
SAM	3	1.50377e-441	7.51884e-442	8
HSM	3	2.86805e-436	1.43403e-436	8
LWM	3	9.9968e-389	4.9984e-389	8
TSM3	3	4.91806e-441	2.45903e-441	8
MSM1	3	1.25839e-366	6.29193e-367	8
MSM2	3	3.45139e-400	1.72569e-400	8
MSM3	3	6.14484e-382	3.07242e-382	8
MSM4	3	7.67392e-440	3.83696e-440	8
MSM5	3	9.05287e-477	4.52644e-477	8
SM	3	6.52188e-638	3.26094e-638	9
ZHONG	3	1.19914e-660	5.99571e-661	9
MM	3	5.11027e-677	2.55514e-677	9
MSM	3	1.69323e-665	8.46615e-666	9
$f_2(x) = x^3 + 4x^2 - 15$ $x_0 = 2$				
NM	9	2.77652e-437	1.31927e-438	2
MSK	6	1.12811e-839	5.36024e-841	3

SAM	3	1.55077e-365	7.36848e-367	8
HSM	3	1.64273e-354	7.80544e-356	8
LWM	3	3.95747e-386	1.88039e-387	8
TSM3	3	1.20424e-426	5.72198e-428	8
MSM1	3	1.60084e-461	7.60639e-463	8
MSM2	3	1.14695e-483	5.44972e-485	8
MSM3	3	4.66091e-486	2.21464e-487	8
MSM4	3	1.6871e-481	8.01628e-483	8
MSM5	3	1.96352e-465	9.32968e-467	8
SM	Diverges			
ZHONG	Diverges			
MM	Diverges			
MSM	Diverges			
$f_3(x) = \frac{\sqrt{x}-1}{x-3} \quad x_0 = 15.5$				
NM	9	1.08608e-410	6.31926e-410	2
MSK	7	1.8462e-585 +1.48388e-584i	8.7004e-584	3
SAM	3	1.47857e-345	8.60297e-345	8
HSM	3	2.26327e-319	1.31686e-318	8
LWM	3	9.06532e-355	5.27459e-354	8
TSM3	Diverges			
MSM1	3	3.85341e-339	2.24208e-338	8
MSM2	3	1.79309e-330	1.0433e-329	8
MSM3	3	4.88191e-328	2.8405e-327	8
MSM4	3	3.72102e-328	2.16505e-327	8
MSM5	3	1.13073e-329	6.57908e-329	8
SM	3	7.97237e-493	4.63867e-492	9
ZHONG	Diverges			
MM	Diverges			
MSM	Diverges			
$f_4(x) = \log x + \sqrt{x} - 5 \quad x_0 = 11.9$				

NM	9	1.90095e-437	6.47023e-437	2
MSK	7	2.87502e-879	9.78565e-879	3
SAM	3	1.20235e-355	4.09242e-355	8
HSM	3	2.76523e-342	9.41198e-342	8
LWM	3	2.98533e-398	1.01611e-397	8
TSM3	3	8.43778e-493	2.87195e-492	8
MSM1	3	2.30698e-431	7.85224e-431	8
MSM2	3	6.31106e-387	2.14809e-386	8
MSM3	3	7.46997e-397	2.54254e-396	8
MSM4	3	3.76958e-389	1.28305e-388	8
MSM5	3	5.76497e-398	1.96221e-397	8
SM	Diverges			
ZHONG	Diverges			
MM	Diverges			
MSM	Diverges			

Table 4. Comparison of various iterative methods with (MSM).

Method	IT	$\lvert f(x_n)\rvert$	$\lvert x_n-\alpha\rvert$	COC
$f_5(x)=e^x\sin x+\log(x^2+1)\quad x_0=0.5$				
SM	Diverges			
ZHONG	Diverges			
MM	Diverges			
MSM	3	6.86138e-621	6.86138e-621	9
$f_6(x)=x^3-10\quad x_0=4$				
SM	Diverges			
ZHONG	4	2.26827e-1007	0	9
MM	Diverges			
MSM	3	3.90753e-448	2.80617e-449	9

V. CONCLUSION

In this paper, a novel ninth order method which is a development of Mylapalli et al method [2] and an optimal class of eighth-order convergence with five iterative methods were proposed to solve nonlinear equations. The ninth order family was developed by Changing y_n to x_n in its second step and has an efficiency index of 1.4422. In addition, the optimal eighth class was obtained by developing the ninth order family (MSM) by approximating the first derivatives $f'(y_n)$, $f'(z_n)$ and finally applying weighted functions which led to the novel optimal eighth method (MSM1-MSM5). It contains two first derivatives and two function evaluations with an efficiency index equal to $8^{\frac{1}{4}} = 1.6817$. Moreover, the methods were examined by numerical examples and compared with other eighth, ninth order methods.

REFERENCES

[1]. H. T. Kung and J. F. Traub, "Optimal order of one-point and multipoint iteration," *Journal of the ACM (JACM)*, vol. 21, no. 4, pp.643–651, 1974.

[2]. M. S. Mylapalli, R. K. Palli, and R. Sri, "A ninth order iterative method for solving non-linear equations with high-efficiency index," *Advances in Mathematics: Scientific Journal*, vol. 9, no. 7, pp. 5283–5290, 2020.

[3]. M. N. Muhaijir, M. Soleh, and E. Safitri, "New modification of chebyshev's method with seventh-order convergence," *Applied Mathematical Sciences*, vol. 11, pp. 2341–2350, 2017.

[4]. J. R. Sharma and H. Arora, "An efficient family of weighted-Newton methods with optimal eighth order convergence," *Applied Mathematics Letters*, vol. 29, pp. 1–6, 2014.

[5]. H. M. Abbas and I. A. Al-Subaihi, "A New Family of Optimal Eighth-Order Iterative Method for Solving Nonlinear Equations," *Quest Journals*, vol. 8, no. 3, pp. 10–17, 2022.

[6]. L. Liu and X. Wang, "Eighth-order methods with high efficiency index for solving nonlinear equations," *Applied Mathematics and Computation*, vol. 215, no. 9, pp. 3449–3454, 2010.

[7]. T. G. Al-Harbi and I. A. Al-Subaihi, "An Optimal Class of Eighth-Order Iterative Methods Based on King's Method," *Journal of Progressive Research in Mathematics(JPRM)*, 2018, [Online]. Available: www.scitecresearch.com/journals

[8]. I. A. Al-Subaihi, T. M. Shatnawi, and H. I. Siyyam, "A ninth-order iterative method free from second derivative for solving nonlinear equations," *Int. J. Math. Anal*, vol. 5, no. 47, pp. 2337–2347, 2011.

[9]. Z. Hu, L. Guocai, and L. Tian, "An Iterative method with ninth order convergence for solving non-linear equations," *Int. J. Contemp. Math. Sciences*, vol. 6, no. 1, pp. 17–23, 2011.

[10]. M. N. Muhaijir, M. Imran, and M. D. H. Gamal, "Variants of Chebyshev Method with Ninth-Order Convergence for Solving Nonlinear Equations," *Global Journal of Pure and Applied Mathematics*, vol. 13, no. 10, pp. 7331–7338, 2017.

[11]. Z. Al-Turkmani and I. A. Al-Subaihi, "A New Family of Optimal Eighth-Order Iterative Scheme for Solving Nonlinear Equations," *Journal of Progressive Research in Mathematics*, vol. 19, no. 1, pp. 44–54, Apr. 2022.

[12]. Z.O. Abdelhadi and I. A. Al-Subaihi, "An Optimal class of Eighth-Order Iterative Methods for Solving Nonlinear Equations," *Quest Journals*, vol. 9, no. 1, pp. 28–34, 2023.