Quest Journals
Journal of Research in Applied Mathematics
Volume 9 ~ Issue 4 (2023) pp: 04-09
ISSN(Online) : 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Research Paper

Resolving the Mathematical Paradox of Seismic Wave Velocities

[1]Dagogo Tamunonengiyeofori, [1]Balogun O. Ayomide and [1]Muzan, I. Andrew
[1](Department of Physics, University of Port Harcourt, Choba, Rivers State, Nigeria)
Corresponding Author: ***Balogun O. Ayomide***

ABSTRACT: *In the study of seismic wave propagation, there appears to be some inconsistency between theory and practice. Experimental results show a relationship between densities and seismic wave velocities that is the inverse of that given by the mathematical expressions for seismic wave velocities. Whereas the theoretical expressions for seismic velocities show that they vary inversely with the square root of densities, experimentally, seismic velocities are seen to directly vary with densities in some sort. To resolve this seeming paradox, we considered that elastic moduli must have some fundamental relationships with density in the form;* $M = k\rho^r + \gamma\rho^{r-1} + \cdots + \rho$*. Such relationships, particularly for* λ *and* μ *have been established here, and further used to re-write the expressions for seismic velocities that are devoid of this paradox, by utilising theoretical and empirical expressions that have been derived by previous workers. We note that the perception and quantification of fundamental physical properties such as mass and volume is more intuitive than those of elastic constants and moduli. Thus, representing other physical quantities with reference to them could simplify some problems. This formulation is vital as it reconciles experimental results with theory and allows for more intuitive description of seismic wave propagation phenomena. More importantly, this formulation suggests that by measuring the density of a material it may be possible to predict its elastic moduli and other parameters of interest, thus taking away the need for very sophisticated and expensive experiments.*
KEYWORDS: *Mathematical paradox, Elastic moduli, Seismic wave velocities, Fundamental physical properties*

Received 01 Apr., 2023; Revised 10 Apr., 2023; Accepted 12 Apr., 2023

I. INTRODUCTION

Physics and Mathematics are very closely and seamlessly interwoven so that there are no contradictions in principles. Considering the relationship between them, mathematics may be described as the language in which physics is communicated and historically it's doubtful there is a record of a disagreement between any of the theories of Physics when considered mathematically. However, in the study of seismic wave propagation, there appears to be a departure from this elaborate consistency between Physics and mathematics that has been enjoyed for so long.

Experimentally considering seismic wave propagation velocities, it may generally be thought of as a rule of thumb that seismic wave velocities increase with increasing density: Expressed mathematically, it will be said that there is a direct proportionality relationship between seismic velocity and density, i.e; $V \propto \rho$.

However, this statement is seen to be contradictory, with a glance at the mathematical expressions for seismic wave velocities given as:

$$V_p = \sqrt{\frac{\lambda+2\mu}{\rho}} \text{ and } V_s = \sqrt{\frac{\mu}{\rho}} \qquad 1$$

for compressional and shear velocities respectively.

These show that the relationship between density and velocity is an inverse proportionality relationship: $V \propto 1/\rho$

In practice, despite the inverse proportionality expressed by the mathematical equations for seismic velocities, almost all known rocks exhibit approximately direct proportionality: velocity increasing with increasing density, except mainly for salt, which departs from the normal having a high velocity for its low density in comparison to other rocks (Note: Even for salts, velocity is seen to increase with increase in density). Figure 1, shows plots of density against velocity for different rock types, while Table 1 gives typical matrix velocity and density values for some rock types.

Table 1: Typical matrix velocity and density values of some rocks

Rock	Density ρ (gm/cc)	P-wave velocity Vp (km/s)
Shale	2.52	3.4
Sandstone	2.65	5.5
Limestone	2.71	6.4
Dolomite	2.87	7.0
Anhydrite	2.96	6.1
Salt	2.16	4.6

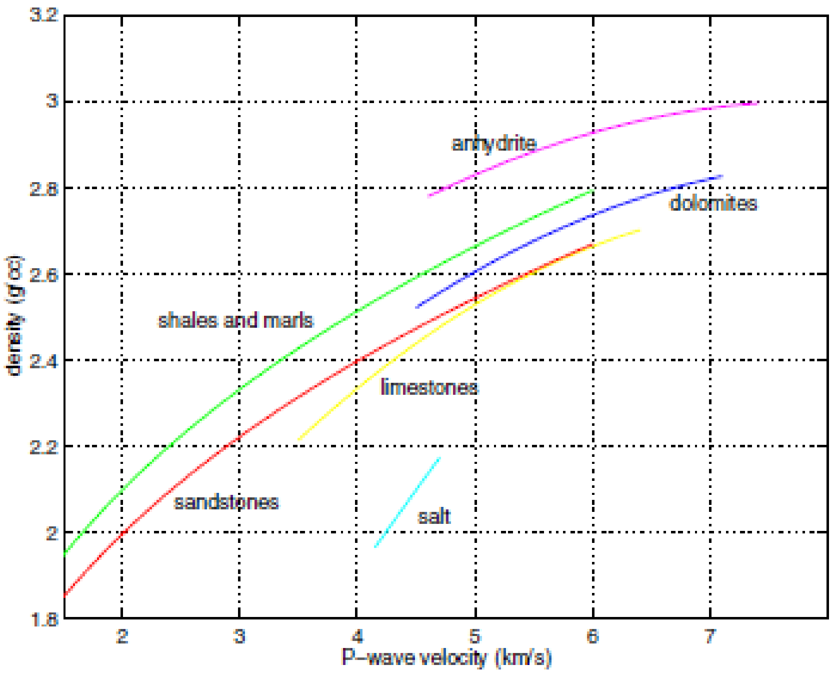

Figure 1: Relationship between compressional wave velocities and densities in different rocks [1]

Although there is an exception or a departure from the direct/inverse proportionality paradox of seismic wave velocities in the case of salt, considering the general situation, it has been explained that seismic velocities are more dependent on the elastic constants bulk and shear moduli than on the densities of the rocks. However, going forward with this thought, we consider that since these elastic moduli may not be thought of as being fundamental physically, that is to say; they generally are dependent on other physical properties, which can be related to more intuitively, it may be reasonable to consider their representation in terms of these more physically fundamental measures or properties and by extension perhaps, upon use in the seismic velocity equations, the paradox may be resolved.

In this endeavour we try to demonstrate that the seismic velocity equations can be represented in a mathematically intuitive form or a form that readily shows consistency with mathematical proportionality concepts, using relationships that obviously exist between elastic moduli and other more easily measurable physical properties of rocks such as mass and volume through density. And possibly provoke research interest into establishing standard relationships between the elastic moduli and physical properties especially density.

II. FORMULATION

In general it may be considered that the elastic moduli have some fundamental relationship with the mass of a material, through density, so that they may be represented by some linear or exponential expression in density, having the form:

$$M = k\rho^{r} + \gamma\rho^{r-1} + \cdots + \rho \qquad 2$$

Thus, we can proceed to derive this expression relating the elastic moduli of a material to its density as follows:

Given the compressional and shear wave velocities as:

$$V_p = \sqrt{\frac{\lambda+2\mu}{\rho}} \text{ and } V_s = \sqrt{\frac{\mu}{\rho}} \qquad 3$$

we can obtain a relationship between the compressional and shear velocities experimentally as is the usual custom, so using an expression in the form of Castagna's shear wave velocity estimation model, we can write thus

$$V_s = aV_p + C \qquad 4$$

from this expression, we can rewrite the shear velocity equation as

$$aV_p + C = \sqrt{\frac{\mu}{\rho}}$$

$$\sqrt{\mu} = \sqrt{\rho}\left(aV_p + C\right)$$

$$\therefore \mu = \rho\left(aV_p + C\right)^2 = \rho\left(a^2V_p^2 + 2aV_pC + C^2\right) \qquad 5$$

Substituting this expression into the primary velocity equation, equation (3a) we have

$$V_p = \sqrt{\frac{\lambda+2\{\rho\left(a^2V_p^2+2aV_pC+C^2\right)\}}{\rho}} \qquad 6$$

$$V_p^2\rho = \lambda + 2\rho(a^2V_p^2 + 2aV_pC + C^2)$$

$$= \lambda + 2\rho a^2V_p^2 + 4\rho aV_pC + 2\rho C^2$$

$$\therefore \lambda = V_p^2\rho - 2\rho a^2V_p^2 - 4\rho aV_pC - 2\rho C^2$$

$$\lambda = (\rho - 2\rho a^2)V_p^2 - 4\rho aV_pC - 2\rho C^2 \qquad 7$$

Taking the relationship between compressional velocity and density, proposed by [2] , given as;

$$\rho = mV_p^n + g \qquad 8$$

we can write that

$$V_p = \left(\frac{\rho - g}{m}\right)^{1/n}$$

So that;

$$\lambda = (\rho - 2\rho a^2)\left(\frac{\rho-g}{m}\right)^{2/n} - 4\rho aC\left(\frac{\rho-g}{m}\right)^{1/n} - 2\rho C^2 \qquad 9$$

$$\lambda = (\rho - 2\rho a^2)\left(\frac{\rho^{2/n}}{m^{2/n}} - \frac{g^{2/n}}{m^{2/n}}\right) - 4\rho aC\left(\frac{\rho^{1/n}}{m^{1/n}} - \frac{g^{1/n}}{m^{1/n}}\right) - 2\rho C^2$$

$$= \frac{\rho^{n+2/n}}{m^{2/n}} - \frac{g^{2/n}\rho}{m^{2/n}} - \frac{2a^2\rho^{n+2/n}}{m^{2/n}} + \frac{2a^2 g^{2/n}\rho}{m^{2/n}} - \frac{4aC\rho^{n+1/n}}{m^{1/n}} + \frac{4aCg^{1/n}\rho}{m^{1/n}} - 2C^2\rho$$

$$= \frac{\rho^{n+2/n}}{m^{2/n}}(1 - 2a^2) + \frac{g^{2/n}\rho}{m^{2/n}}(2a^2 - 1) - \frac{4aC\rho^{n+1/n}}{m^{1/n}} + \frac{4aCg^{1/n}\rho}{m^{1/n}} - 2C^2\rho$$

$$\therefore \lambda = \frac{\rho^{n+2/n}}{m^{2/n}}(1 - 2a^2) - \frac{4aC\rho^{n+1/n}}{m^{1/n}} + \left(\frac{4aCg^{1/n}}{m^{1/n}} + \frac{(2a^2-1)g^{2/n}}{m^{2/n}} + 2C^2\right)\rho \qquad 10$$

Recall from equation (5) that

$$\sqrt{\mu} = \sqrt{\rho}\left(aV_p + C\right)$$

If we then apply the relationship between compressional velocity and density given by equation (8), we can write that

$$\sqrt{\mu} = \sqrt{\rho}\left(a\left(\frac{\rho-g}{m}\right)^{1/n} + C\right) \qquad 11$$

$$= \rho^{1/2}\left(\frac{a\rho^{1/n}}{m^{1/n}} - \frac{ag^{1/n}}{m^{1/n}} + C\right)$$

$$\frac{a\rho^{n+2/2n}}{m^{1/n}} - \frac{ag^{1/n}\rho^{1/2}}{m^{1/n}} + C\rho^{1/2}$$

$$\mu = \left\{\frac{a\rho^{n+2/2n}}{m^{1/n}} - \frac{ag^{1/n}\rho^{1/2}}{m^{1/n}} + C\rho^{1/2}\right\}^2$$

$$\mu = \left\{\frac{a\rho^{n+2/2n}}{m^{1/n}} - \frac{ag^{1/n}\rho^{1/2}}{m^{1/n}} + C\rho^{1/2}\right\}\left\{\frac{a\rho^{n+2/2n}}{m^{1/n}} - \frac{ag^{1/n}\rho^{1/2}}{m^{1/n}} + C\rho^{1/2}\right\}$$

$$= \frac{a^2\rho^{n+2/n}}{m^{2/n}} - \frac{a^2 g^{1/n}\rho^{n+1/n}}{m^{2/n}} + \frac{aC\rho^{n+1/n}}{m^{1/n}} - \frac{a^2 g^{1/n}\rho^{n+1/n}}{m^{2/n}} + \frac{a^2 g^{2/n}\rho}{m^{2/n}} - \frac{aCg^{1/n}\rho}{m^{1/n}} + \frac{aC\rho^{n+1/n}}{m^{1/n}} - \frac{aCg^{1/n}\rho}{m^{1/n}} + C^2\rho$$

$$\therefore \mu = \frac{a^2\rho^{n+2/n}}{m^{2/n}} + \left(\frac{2aC}{m^{1/n}} - \frac{2a^2 g^{1/n}}{m^{2/n}}\right)\rho^{n+1/n} + \left(\frac{a^2 g^{2/n}}{m^{2/n}} - \frac{2aCg^{1/n}}{m^{1/n}} + C^2\right)\rho \qquad 12$$

Thus using these expressions for λ and μ from equations (10) and (12) we can write the compressional and shear wave velocity equations as

$$V_p = \sqrt{\frac{\lambda+2\mu}{\rho}} =$$

$$\sqrt{\frac{\frac{\rho^{n+2/n}}{m^{2/n}}(1-2a^2) + \frac{g^{2/n}\rho}{m^{2/n}}(2a^2-1) - \frac{4aC\rho^{n+1/n}}{m^{1/n}} + \frac{4aCg^{1/n}\rho}{m^{1/n}} - 2C^2\rho + 2\left\{\frac{a^2\rho^{n+2/n}}{m^{2/n}} + \left(\frac{2aC}{m^{1/n}} - \frac{2a^2 g^{1/n}}{m^{2/n}}\right)\rho^{n+1/n} + \left(\frac{a^2 g^{2/n}}{m^{2/n}} - \frac{2aCg^{1/n}}{m^{1/n}} + C^2\right)\rho\right\}}{\rho}}$$

$$\sqrt{\frac{\frac{\rho^{n+2/n}}{m^{2/n}} - \frac{2a^2\rho^{n+2/n}}{m^{2/n}} + \frac{2a^2 g^{2/n}\rho}{m^{2/n}} - \frac{\rho g^{2/n}}{m^{2/n}} - \frac{4aC\rho^{n+1/n}}{m^{1/n}} + \frac{4aCg^{1/n}\rho}{m^{1/n}} - 2C^2\rho + \frac{2a^2\rho^{n+2/n}}{m^{2/n}} + \frac{4aC\rho^{n+1/n}}{m^{1/n}} - \frac{4a^2 g^{1/n}\rho^{n+1/n}}{m^{2/n}} + \frac{2a^2 g^{2/n}\rho}{m^{2/n}} - \frac{4aCg^{1/n}\rho}{m^{1/n}} + 2C^2\rho}{\rho}}$$

$$\therefore V_p = \sqrt{\frac{\rho^{2/n}}{m^{2/n}} - \frac{4a^2 g^{1/n}\rho^{1/n}}{m^{2/n}} - \frac{g^{2/n}}{m^{2/n}}} \qquad 13$$

And

$$V_s = \sqrt{\frac{\mu}{\rho}} = \sqrt{\frac{\frac{a^2\rho^{n+2/n}}{m^{2/n}} + \left(\frac{2aC}{m^{1/n}} - \frac{2a^2 g^{1/n}}{m^{2/n}}\right)\rho^{n+1/n} + \left(\frac{a^2 g^{2/n}}{m^{2/n}} - \frac{2aCg^{1/n}}{m^{1/n}} + C^2\right)\rho}{\rho}}$$

$$\therefore V_s = \sqrt{\frac{a^2\rho^{2/n}}{m^{2/n}} + \left(\frac{2ac}{m^{1/n}} - \frac{2a^2 g^{1/n}}{m^{2/n}}\right)\rho^{1/n} + \frac{a^2 g^{2/n}}{m^{2/n}} - \frac{2aCg^{1/n}}{m^{1/n}} + C^2} \qquad 14$$

Where **a** and **c** are coefficients relating compressional and shear wave velocities, while **m**, **n** and **g** are coefficients relating compressional wave velocity and density in a given medium.

III. DISCUSSION

Generally, the perception of physical properties such as mass and volume feels more natural and their quantification or measurement is easier and more straightforward than those of elastic constants and moduli. In the same line of thought it may be posited that the ease of measurement and the intuitiveness enjoyed in relating with these primary physical properties may be transmitted along, if other physical quantities are considered and represented with reference to them.

The elastic moduli λ and μ have been derived as expressions in terms of density, which is a direct derivative of a fundamental physical quantity - mass. The derivations have been employed to re-write seismic wave propagation velocities. The derivation has utilized some empirical relationships that have been in use in seismic exploration, to estimate some unavailable parameters, such as the Castagna's relation, for the synthesis of shear wave velocity (Vs), from Compressional velocity (Vp) and a supposed generalized form of the Gardener's relation for the synthesis of Vp from density. We note that there are several variants of Castagna's relation, which can be employed, however, for convenience Castagna's relation [3] was used. For the density-Vp relation, utilising the original Gardener's equation [4], gave Vp as

$$V_p = \left(\frac{\rho}{m}\right)^{1/n}$$

and

$$V_s = \sqrt{\frac{a^2\rho^{2/n}}{m^{2/n}} + \frac{2aC\rho^{1/n}}{m^{1/n}} + C^2}$$

this result validates Garderner's relation and give some credibility to this formulation. The modified form of the relation [2], was employed however, as it supposedly remains valid over a wider range of densities and velocities, than does the original Gardener's equation which has been noted by other workers not to be robust in diverse settings [5, 6, 7].

Although the velocity equations now appear a little more cumbersome than they do when written in terms of the elastic constants, this does not necessarily diminish the benefit from this formulation as in the first place the seeming mathematical paradox is taken care of as the equations now appear in a form that shows consistency between experimental findings and the proportionality principle of mathematics. And in any case, the equations will reduce to simpler forms if numerical values for the coefficients a, c, m, n and g are substituted appropriately. The estimation of these constants picked from the empirical relations and employed in this formulation is critical to the accuracy of the results obtained. It will be necessary as required theoretically, that the constants turn out such that for fluids, the shear wave velocity goes to zero.

There have been discussions offering explanation that the observed increase in seismic velocity with density is only correlative and not causative, citing the situation where fluid changes in reservoirs leading to density increase does not result to an increase in shear wave velocity, but a decrease, since there is no corresponding increase in shear modulus. However, practically it is known that fluids do not propagate or influence shear waves and it has also been shown by Gassmann's fluid substitution modelling that shear wave velocity is negligibly influenced by the fluid replacement. Thus where there is a detectable change in Vs, in course of fluid replacement, it must be a result of rock formation changes that may result from pore pressure changes or other factors.

IV. CONCLUSION

The venture into finding expressions for elastic moduli in terms of more a fundamental quantity mass/density is definitely worthwhile as it reconciles experimental results with theory and allows for more intuitive description of seismic wave propagation phenomena. The derived relationships between the elastic moduli and density show that for any increase in density, the elastic moduli enjoy an increase equal to the increment value, raised to some exponent. Thus when density increases, the exponential increase in the numerator parts of the regular velocity equations outweigh the diminishing effect of the increment value in the denominator. Although the derived expressions may be argued not be in their most accurate forms, since only simple linear models of the relationship between shear wave and compressional wave velocities, and density and compressional velocity have been applied in the formulation, other models can be applied to give more experimentally co-relatable expressions. More importantly, this line of thought should provoke research into finding exact ties between these physical properties, such that by measuring the mass/density of a material it may be possible to infer or predict its elastic moduli, thus taking away the need for sophisticated and expensive experiments.

REFERENCES

[1]. Seismic waves and Earth oscillations, (2008), Lecture notes.
[2]. AL Ismail, M. On the relationship between compressional wave velocity of Saturated porous rocks and density: theory and application, Research project library archives, Massachusetts Institute of Technology, 2017
[3]. Castagna, J. P., Batzle, M. L. and Eastwood, R. L. Relationships between compressional-wave and shear-wave velocities in clastic silicate rocks: Geophysics, 1985. 50(4): p. 571-581.
[4]. Gardener, G. H.F., Gardener, L. W. and Gregory, A. R. Formation velocity and density – the diagnostic basics for stratigraphic traps. Geophysics, 1974. 39(6): p. 770-780.
[5]. Pranter, M. J., Hurley, N. F. and Davis, T. L. Anhydrite distribution within a shelf-margin carbonate reservoir: San Andres Formation, vacuum field, New Mexico, USA, Journal of Petroleum Geoscience, 2004. 10(1): p. 43-52.
[6]. Dey, A. K. and Stewart, R. R. Predicting density using Vs and Gardener's relationship, CREWES Res. Report, 1997. 9(6): p. 1-9.
[7]. Nwozor, K., Onuorah, L.O., Onyekuru, S. and Egbuachor, C. Calibration of Gardener coefficient for density-velocity relationships in Tertiary Sediments in Niger Delta Basin, Journal of Exploration Production Technology, 2017. 7(3): p. 627-635