Quest Journals
Journal of Research in Applied Mathematics
Volume 9 ~ Issue 5 (2023) pp: 34-40
ISSN(Online) : 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Research Paper

Integrated semi groups and some nonlinear fractional integral equations with respect to functions

Mahmoud M El-Borai
Department of Mathematics and Computer Science
Faculty of Science, Alexandria University, Egypt

Hussoin.A.H. Salem
Faculty of Science, Alexandria University, Egypt

A. Tarek S.A.
Faculty of Science, Alexandria University, Egypt

Hazar M. Kh, Al-Khatib
Faculty of Science, Al-Baath University - Syria

Abstract
Some classes of nonlinear fractional abstract integral equations with respect to functions are studied.
Closed operators with α- integrated semi groups are considered.
The existence and uniqueness of solutions of the considered equations are established.
MSC: Subject classifications: 34A12-34A40-26A33-33A06-47D26-47D60--43G20

Keywords: Fractional differential equations with respect to functions, Integro partial differential equations, Closed operators, Integrated semi groups.

***R**eceived 17 May, 2023; **R**evised 28 May, 2023; **A**ccepted 31 May, 2023*

I. Introduction

Consider the following abstract fractional integral equations with respect to functions:

$$u(t) = u_0 + \frac{1}{\Gamma(\beta)}\int_0^t [\psi(t)-\psi(s)]^{\beta-1}\frac{d\psi(s)}{ds}Au(s)ds + \frac{1}{\Gamma(\beta)}\int_0^t [\psi(t)-\psi(s)]^{\beta-1}\frac{d\psi(s)}{ds}f(s,L(s)u(s))ds$$

Where $0<\beta\le 1$, A is a linear closed operator defined on a dense set S in a Banach space E, and ψ is a real bijective functions, which has continuous derivative $\frac{d\psi(s)}{ds}$ on a finite closed interval $J=[0,T]$, $\psi(0)=0$, $\psi(\mathrm{t})\ge 0$, $\frac{d\psi(t)}{dt}>0$ on J , $\Gamma(\cdot)$
Is the gamma function, $L(t)u(t) = (B_1(t)u(t).\dots\dots.B_r(t)u(t))$, $B_1(t).\dots\dots\dots.B_r(t)$ are families of linear closed operators, defined on dense sets $S_1\dots\dots.S_r \supset S\ repectively\ in\ E.$ f is a given abstract function defined on $J\times E^r to\ E$.
It is assumed that A generates α- times integrated semi groups $\{Q(t): t\in J\}$,$0<\alpha\le 1$ such that $\{Q(t): t\in J\}$
Is a family of linear bounded operators on $E\ to\ E$, with the following properties :
(i) $Q(t)$ is strongly continuous on J.
(ii) The operator $(\lambda I - A)^{-1}$ exists and:
$(\lambda I - A)^{-1} = \lambda^{\alpha}\int_0^{\infty} e^{-\lambda t}Q(t)dt$ (1.2)
for all $\lambda>\lambda_0>0$, the interval (λ_0,∞) is contained in the resolvent of A , $Q(t)h\ \in S, for\ every\ t>0, h\ \in$

E .

(iii) $Q(t)h = \frac{t^{\alpha}}{\Gamma(\alpha+1)}\ h + \int_0^t A\,Q(s)\,h\,ds\ \ for\ every\ \ h \in E$. (1.3)

In section 2, we shell study the linear case.

In section 3, we solve the equation (1.1).

In section 4, we give an example.

The applications can be founded in the theory of elasticity and also in the quantum mechanics, see [1-3].

II. The linear case

Let us study now equation (1.1), when f depends only on t.

In other words, let us try to solve the following equation:

$$u(t) = u_0 + \frac{1}{\Gamma(\beta)}\int_0^t [\psi(t) - (s)\psi]^{\beta-1}\ \frac{d\psi(s)}{ds} Au(s)ds$$

$$+\frac{1}{\Gamma(\beta)}\int_0^t [\psi(t) - (s)\psi]^{\beta-1}\ \frac{d\psi(s)}{ds} f(s)ds\ , \qquad (2,1)$$

Suppose that the abstract derivative $\frac{df}{dt}$ Where f is a given abstract continuous function on J, with values in E. exists and continuous on J .

We shall consider the following operators:

$$\Lambda(\mathrm{t}) = \int_0^{\infty} \zeta_{\beta}(\theta)Q\left(t^{\beta}\theta\right)d\theta$$

$$\Lambda^{*}(t) = \beta t^{\beta-1}\int_0^{\infty} \theta\zeta_{\beta}(\theta)Q\left(t^{\beta}\theta\right)d\theta$$

Where $\zeta_{\beta}(t)$ is a probability density function defined on $[0,\infty]$ by

$$\zeta_{\beta}(t) = \frac{1}{\beta}t^{-1-\frac{1}{\beta}}\,\rho_{\beta}\left(t^{-\frac{1}{\beta}}\right) \qquad (2,2)$$

ρ_{β} is the one-sided stable probability density function.

The Laplace transform of these functions are given:

$$\int_0^{\infty} e^{-pt}\,\rho_{\beta}(t)dt = \exp(-p^{\beta}) \qquad (2,3)$$

$$\int_0^{\infty} e^{-pt}\,\zeta_{\beta}(t)\,(t)dt = \sum_{j=0}^{\infty}\frac{(-p)^j}{\Gamma(1+\beta j)}$$

We shall consider the following definitions:

$$\frac{d^{\alpha}f(t)}{dt^{\alpha}} = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-s)^{-\alpha}\,f(s)ds$$

$$= \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\frac{df(s)}{ds}ds + \frac{t^{-\alpha}}{\Gamma(1-\alpha)}f(0)$$

$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds \qquad 0 < \alpha < 1$$

(see [4-12].

Theorem 1.1.

The solution $u(t)$ of equation (2,1) is given by $u(t) = v(\psi(t))$,where

$$v(t) = \frac{d^{\alpha\beta}}{dt^{\alpha\beta}}\left[\Lambda(t)u_0 + \int_0^t \Lambda^*(t-\eta)f(\eta)d\eta\right]$$

Proof.

Consider the equation:

$$v(t) = u_0 + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}[Av(s) + f(\psi^{-1}(s))]ds$$

Where $\psi^{-1}$is the inverse function of ψ.

Set $s = (\psi(\tau))$, so :

$$v(t) = u_0 + \frac{1}{\Gamma(\beta)}\int_0^{\psi^{-1}(t)}(t-\psi(\tau))^{\beta-1}[Av(\psi(\tau)) + f\tau))]\frac{d\psi(\tau)}{d\tau}d\tau$$

Thus $u(t) = v(\psi(t))$.

Now let us try to find v.

Let$\tilde{v}(p)$ and $\tilde{g}(p)$ be the Laplace transforms of $v(t)$, and $g(t)$ respectively, where $g(t) = f(\psi^{-1}(t))$.

It is easy to find:

$$\tilde{v}(p) = (p^{\beta}I - A)^{-1}[p^{\beta-1}u_0 + \tilde{g}(p)].$$

Using (1,2) and (2,4), we get

$$\tilde{v}(p) = \int_0^{\infty} p^{\alpha\beta}e^{-tp^{\beta}}\, Q(t)\left[p^{\beta-1}u_0 + \tilde{g}(p)\right]dt \qquad (2,5)$$

From (2,3), we can write:

$$\exp\left(-tp^{\beta}\right) = \int_0^{\infty} e^{-p\theta t^{\frac{1}{\beta}}}\rho_{\beta}(\theta)d\theta \qquad (2,6)$$

Differentiating (2,6) with respect to p , we get

$$\exp\left(-tp^{\beta}\right) = p^{1-\beta}\beta^{-1}\int_0^{\infty} \theta t^{\frac{1}{\beta}-1}e^{-p\theta t^{\frac{1}{\beta}}}\rho_{\beta}(\theta)d\theta \qquad (2,7)$$

From (2,5), (2,6) and (2,7) one gets:

$$\tilde{v}(p) = p^{\alpha\beta}\int_0^{\infty} e^{-pt}\left[\int_0^{\infty} \rho_{\beta}(\theta)Q\left(\frac{t^{\beta}}{\theta^{\beta}}\right)u_0\, d\theta\right]\mathrm{dt}$$
$$+ p^{\alpha\beta}\int_0^{\infty} e^{-pt}\left[\int_0^{\infty} \beta\theta^{-\beta}t^{\beta-1}\rho_{\beta}(\theta)Q\left(\frac{t^{\beta}}{\theta^{\beta}}\right)\tilde{g}(p)d\theta\right]\mathrm{dt} \qquad (2,8)$$

From (2,2) and (2,8), one gets:

$$\tilde{v}(p) = p^{\alpha\beta}\int_0^\infty e^{-pt}\left[\int_0^\infty \zeta_\beta(\theta)Q(t^\beta\theta)\,u_0\,d\theta\right]\mathrm{dp} + p^{\alpha\beta}\int_0^\infty e^{-pt}\left[\int_0^\infty \theta\beta t^{\beta-1}\zeta_\beta(\theta)Q(t^\beta\theta)\,\tilde{g}(p)d\theta\right]\mathrm{d}p \quad (2,9).$$

According to properties of the Laplace transform of fractional derivatives and noticing that $Q(0)$ is the zero dement in E, one gets from (2,9):

$$v(t) = \frac{d^{\alpha\beta}}{dt^{\alpha\beta}}\left[\Lambda(t)u_0 + \int_0^t \Lambda^*(t-\eta)g(\eta)d\eta\right] \qquad (2,10)$$

Hence the required result.

Noticing that:

$$\frac{d^{\alpha\beta}}{dt^{\alpha\beta}}\, t^{\alpha\beta} = \Gamma(\alpha\beta+1), \int_0^\infty \theta^\alpha\zeta_\beta(\theta)d\theta = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha\beta+1)}$$

And using (1,3), (2,10), we get:

$$v(t) = u_0 + \frac{d^{\alpha\beta}}{dt^{\alpha\beta}}\int_o^\infty\int_0^{\theta t^\beta} \zeta_\beta(s)AQ(s)u_0 ds d\theta + \frac{d^{\alpha\beta}}{dt^{\alpha\beta}}\int_0^t \Lambda^*(t-\eta)g(\eta)d\eta \qquad (2,11)$$

It is easy to rewrite formula (2,11) in the following form:

$$v(t) = u_0 + \frac{d^{\alpha\beta}}{dt^{\alpha\beta}}\int_0^\infty\int_0^{\theta t^\beta} \zeta_\beta AQ(s)u_0 ds + \int_0^t \Lambda^*(t-\eta)\frac{d^{\alpha\beta}}{dt^{\alpha\beta}}g_0(\eta)d\eta + \frac{d^{\alpha\beta}}{dt^{\alpha\beta}}\int_0^t \Lambda^*(t-\eta)g(\eta)d\eta$$

Where $\mathrm{g}_0(s) = g(s) - g(0)$ (2,12)

Also, we can rewrite formula (2,12), in the following form

$$v(t) = u_0 + \frac{d^{\alpha\beta}}{dt^{\alpha\beta}}\int_0^t \Lambda^*(t-\eta)[g(0)+Au_0]d\eta + \int_0^t \Lambda^*(t-\eta)\frac{d^{\alpha\beta}}{dt^{\alpha\beta}}g_0(\eta)d\eta \;\; where\; u_0 \in S. \quad (2,13)$$

If $(0) = u(0) = \tilde{0}$, $\tilde{0}$ *is the zero in* E , we can write

$$v(t) = \int_0^t \Lambda^*\,(t-\eta)\frac{d^{\alpha\beta}}{d\eta^{\alpha\beta}}g(\eta)d\eta$$

In this case, we get:

$$u(t) = v(\psi(t)) = \int_0^{\psi(t)} \Lambda(\psi(t)-\eta)\frac{d^{\alpha\beta}}{d\eta^{\alpha\beta}}g(\eta)d\eta. \qquad (2.14)$$

III. Nonlieaner integral equations

Consider the following equation:

$$u(t) = \frac{1}{\Gamma(\beta)}\int_0^t [\psi(t)-\psi(s)]^{\beta-1}\frac{d\psi(s)}{ds}Au(s)ds + \frac{1}{\Gamma(\beta)}\int_0^t [\psi(t)-\psi(s)]^{\beta-1}\frac{d\psi(s)}{ds}f^*(s,L(s)u(s))ds \quad (3,1)$$

Where $f^*(t, L(t)u(t)) = \frac{1}{\Gamma(\alpha\beta)}\int_0^t (t-s)^{\alpha\beta-1} f(s, L(s)u(s)ds.$

It is assumed that f is uniformly Holdre continuous in $t \in J$,that is:

$\|f(t_2,v)-f(t_1,v)\| \le K(t_1-t_2)^c, for\ all\ t_2 > t_1,\ t_1,t_2 \in J$ (3,2)

Where $v \in E^r\ and\ K, c\ are\ positive\ constsnts, c \le 1, \|\cdot\|\ is\ the\ norm\ in\ E.$

It is assumed also that the Lipchitz condition

$$\|f(t,v^*)-f(t,v)\| \le K\sum_{i=1}^{i=r}\|v_i^*-v_i\| \qquad (3,3)$$

Is satisfied for all $v, v^* \in E$, $(v=(v_1,\cdots,v_r),\ v^*=(v_1^*,\cdots,v_r^*))$,

Where K is a positive constant.

About the operators $B_1(\mathrm{t}),\cdots,B_r(t)$, we assumed that functions $B_1(t)h,\ldots..,B_r(t)h$,are uniformly Holder continuous in J for $h \in \bigcap_{i=1}^{r} S_i$.

It is assumed also that

$$\|B(t_2)Q(t_1)h\| \le \frac{K}{t_1^{\gamma}}\|h\| \qquad (3,4)$$

Where K is positive constant, $0<\gamma<1, t_2 \in J, t_1 \in (0,T]$.

We notice that the solution $u(t)$ of equation (8,1) can be represented by

$$u(t)=v(\psi(t)) \qquad (3,5)$$

Where

$$v(t)=\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}Av(s)ds+\frac{1}{\Gamma(\beta)}\int_0^t (\mathrm{t}-\mathrm{s})^{\beta-1}f^*\left(\psi^{-1}(s),L(\psi^{-1}(s))v(\psi^{-1}(s))\right)ds \qquad (3,6)$$

Set $V(t)=f(t,L(t)u(t))$, we can write , by using formula (2,14), the following representation:

$$v(t)=\int_0^t \Lambda^*(t-\eta)V(\psi^{-1}(\eta))d\eta$$

Thus

$$u(t)=v(\psi(t))=\int_0^t \Lambda^*\left(\psi(t)-\psi(\eta)\right)\frac{d\psi(\eta)}{d\eta}V(\eta)d\eta \qquad (3,7)$$

Let $C_E(J)$ be the set of all abstract continuous functions u on J , with values in E.

We define a distance function $d(u,v)$ by

$$d(u,v)=\max_{t\in J}\left[e^{-\lambda t}\|u(t)-v(t)\|\right]$$

Where λ is a positive number . It is clear that $(C_E(J), d(u,v))$ is a complete metric space.

We shall solve equation (3.7).

Theorem 3.1. If $u_1,u_2 \in C_E(J)$ are two solutions of equation (3,7), then $u_1(t)$=$u_2(t)$, for all $t \in J$.

Proof.

According to the conditions (3,3), (3,4), and gets

$$||V_1(t)-V_2(t)|| \le$$

$$\leq K \int_0^t [\psi(t) - \psi(\eta)]^{\delta-1} \|V_1(\eta) - V_2(\eta)\| \, d\eta$$

Where $\delta = \beta(1 - ȣ)$,K is a positive constant , $V_i(t) = f(t,L(t)u_i)$, $i = 1,2$.

According to the properties of the function ψ and the mean value theorem, we can find a positive constant K such that

$$\|V_1(t) - V_2(t)\| \leq K \int_0^t (t-\eta)^{\delta-1} \|V_1(\eta) - V_2(\eta)\| d\eta$$

It is easy to see that

$$\|V_1(t) - V_2(t)\| \leq K\lambda^{1-\delta} d(V_1.V_2) \int_0^{t-\frac{1}{\lambda}} e^{\lambda s}\, ds + K d(V_1.V_2) \int_0^t e^{\lambda s}(t-s)^{\delta-1} ds$$

Thus

$$d(V_1.V_2) \leq K \left(\frac{1}{\lambda}\right)^{\delta} (1 + \frac{1}{\delta}) d(V_1.V_2)$$

Choosing λ sufficiently large such that $ȣ = K\left(\frac{1}{\lambda}\right)^{\delta}\left(1 + \frac{1}{\delta}\right) < 1$,
We get $d(V_1.V_2) = 0$. Hence the required result.
Theorem 3.2. Equation (3,7) has a unique solution $u \in C_E(J)$.
Proof. Set
$\mathrm{V_k(t) = f(t.\, L(t) u_k(t))}$
Thus
$d(V_{k+1}, V_k) \leq \nu d(V_k, V_{k-1})$
By induction, we get
$d(V_{k+1},\ \ V_k) \leq \nu^k\, d(V_1.V_2)$
where V_0 is zero approximation, which can be takes the zero element in E.Thus the sequence $\{V_R(t)\}$ uniforomaly converges in the space $C_E(J)$ to a continuous abstract function $V(t)$, which satisfies
$V(t) = f(t.L(t)u(t))$
Hence the required result. (see [13-22].
4-Example
Let $p > 1$, $0 < \alpha < \frac{p-1}{p}$. suppose that $L^p[0.1]$ is the set of all measurable functions f such that $\int_0^1 |f(x)|^p dx$ exists.
Define an operator A by:
$(Af)(x) = -\frac{d(f(x))}{dx} + \frac{\alpha}{x} f(x)$

The domain of definition S of A is the set of all absolutely continuous functions f defined on [0.1] with $f(0) = 0.\frac{df}{dx} \in L^p[0.1]$.
The considered operator A generates the integrated semi group $Q(t)$, where

$$(Q(t)f)(x) = \int_0^t x^{\alpha}(x-s)^{-\alpha} f(x-s) H(x-s) ds.\ x \in [0.1]$$

H Is the Heaviside function.
Consider the following equations

$$u(x.t) = \varphi(x) + \frac{1}{\Gamma(\beta)}\int_0^t [\psi(t) - \psi(s)]^{\beta-1} \frac{d\psi(s)}{ds}\left\{-\frac{\partial u(x.s)}{\partial x} + \frac{\alpha}{x} u(x.s)\right\} ds + \frac{1}{\Gamma(\beta)}\int_0^t [\psi(t) - \psi(s)]^{\beta-1} \frac{d\psi(s)}{ds} f(x.s) ds,$$

$$u(x.t) = \frac{1}{\Gamma(\beta)} \int_0^t [\psi(t) - \psi(s)]^{\beta-1} \frac{d\psi(s)}{ds} \left\{-\frac{\partial u(x.s)}{\partial x} + \frac{\alpha}{x} u(x.s)\right\} ds$$

$$+ \frac{1}{\Gamma(\beta)} \int_0^t [\psi(t) - \psi(s)]^{\beta-1} \frac{d\psi(s)}{ds} f^*(x.s.u(x.s)) ds$$

$$f^*(x.s.u(x.t)) = \frac{1}{\Gamma(\alpha\beta)} \int_0^t (t-s)^{\beta-1} f(x.s.u(x.s)) ds$$

These equations can be solved as in section 3.

Reference:

[1]. W. Arendt, Resolvent positive operators and integrated semi groups, Proc. London. Math. Soc. 54 (1984), 321-349.

[2]. Mathias Hieber, Integrated semi groups and differential operators and L^p space, Math. Ann. 201, (1991), 1-16.

[3]. Lars Hormander, Estimates for translation invariant operators in L^p space, Act. Math. 104.(1060. 93-139).

[4]. Mahmoud M. El-Borai, Some probability densities and fundamental solution of fractional evolution equations, Chaos, Soltion and Fractals 14(2002),433-440.

[5]. Mahmoud M. El-Borai, Semi groups and some nonlinear fractional differential equations, J. of Appl. Math. And Comp. (2004), 823-831.

[6]. Mahmoud M. El-Borai, Khaiaria El-Said El-Nadi., Mostafa , O.L., and Ahmad, H.M. Volterra equations with fractional stochastic integrals. Mathematical problems in Engineering 2004, 5(2004), 453-468.

[7]. Khaiaria El-Said El-Nadi. On some stochastic parabolic differential equations in a Hilbert space. International Journal of Stochastic Analysis 2005,2(2005), 167-173.

[8]. Mahmoud M. El-Borai, On some fractional differential equations in the Hilbert space, Journal of Discrete and Continuous Dynamical Systems, Series A, 2005, 233-241.

[9]. Mahmoud M. El-Borai, Khairia El-Said El-Nadi, On the stable probability distributions and some abstract nonlinear fractional integral equations with respect to functions, Turkish Journal of Physiotherapy and Rehabilitations; 32(3), 2022, 32858-32863

[10]. G. Jumarie, Modified Riemann-Liouville derivative and fractional Taylor series of non-differentiable functions further results, Computers and Mathematics with applications, 51 (2006), 1367-1376.

[11]. Mahmoud M. El-Borai, Khaiaria El-Said El-Nadi ., Mostafa , O.L., and Ahmad, H.M., Semi group and some fractional stochastic integral equations. International Journal Pure and Applied Mathematical Sciences 3 1(2006), 47-52.

[12]. Mahmoud M. El-Borai, A. Tarek S. A. and Al-Khatib Hazar M. Kh., On the parabolic transform and general fractional integro-partial differential equations with respect to functions, NEUROQUANTOLOGY, 2022, Volume 20, Issue 11, 7519-7524.

[13]. G-Jumarie, Fractional partial differential equations and modified Riemann- Liouville derivative new methods foe solutions, Journal of Applied Mathematics and Computations, 24 (2007), 31-48.

[14]. Khairia El-Said El-Nadi, On the stability of some stochastic integro partial differential equations. Int. J. Contemp. Math. Sci 2, 27(2007), 1317-1326.

[15]. V.E. Tarasov, fractioal Dynamics:Applications of fractional calculus in dynamics, fields and media, Spring, New York, 2011.

[16]. Mahmoud M. El-Borai, Khaiaria El-Said El-Nadi. On some fractional parabolic equations driven by fractional Gaussian noise. Special Issue SCIENCE and Mathematics with applications, International Journal of Research and Reviews in Applied Sciences6, 3(2011), 236-241 .

[17]. Mahmoud M. El-Borai, Khaiaria El-Said El-Nadi, integrated semi group and Caucy proplem for some fractional abstract differential equtions, Life Science Journal, 2013; 10(3).

[18]. Mahmoud M. El-Borai, Khaiaria El-Said El-Nadi ., Integrated semi groups and Cauchy problem for some fractional abstract differential equqtions. Life Science Journal 10, 3(2013), 793-795 .

[19]. Mahmoud M. El-Borai, Khaiaria El-Said El-Nadi, A parabolic transform and som stochastic ill-posed problem, British Journal of Mathematics and Computer Science, 9(5), 418-426, 2015.

[20]. Ahmad, H. M., and El-Borai, M.M. Hilfer fractional stochastic integro-differential equations . Applied Mathematics and Copmutation 331(2018), 182-189.

[21]. Mahmoud M. El-Borai, Khaiaria El-Said El-Nadi, H.M.Ahmad, H.M.El-Owaidy, A.S.Ghanem and R. Sakthivel, Existence and stability for fractinal parabolic integro-partial differential equations with fractional Brownian motion and nonlocal condition, Congent Mathematics and Statistics, Vol.5, 2018, Issue1.

[22]. Mahmoud M. El-Borai, Khaiaria El-Said El-Nadi, Ultimate behavior of some fractinal stochastic nonlinear parabolic system, LINGUISTICA ANTVERPIENSIA, 2021 Issu3, pp. 888-893.