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Review Paper

The General Solution of $\frac{du}{dx} = x - u^2$

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ABSTRACT: In this paper, we find the general solution of $\frac{du}{dx} = x - u^2$ by determining the infinitesimal generator of a one-parameter Lie group which leaves the equation invariant. Employing the group invariant as the new independent variable, the equation is transformed into one which is solved by a quadrature.
KEYWORDS: ODE, Symmetry Group, Lie Group, Symmetry Algebra, Infinitesimal Generator.

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I. INTRODUCTION

The general solution of a first-order ODE may possibly be found if one can determine a symmetry group, i.e., a one-parameter Lie group the action of which on $\mathbb{R}^2$ leaves the equation invariant [1], [2]. The method has been successfully applied to several particular equations [3]. If the equation is expressed in the form

$$\frac{du}{dx} = F(x, u), \tag{1}$$

then the symmetry algebra, i.e., the Lie algebra of the symmetry group, is generated by the vector field

$$\mathbf{v} = \xi(x, u)\frac{\partial}{\partial x} + \phi(x, u)\frac{\partial}{\partial u}, \tag{2}$$

the *infinitesimal generator* of the group. The components ξ and ϕ of $\mathbf{v}$ must satisfy the *underdetermined* first-order PDE

$$\phi_x + (\phi_u - \xi_x)F - \xi_u F^2 = \xi F_x + \phi F_u. \tag{3}$$

Any pair (ξ, ϕ) which satisfies Equation (3) yields a symmetry vector (2). If

$$F(x, u)\xi(x, u) - \phi(x, u) \not\equiv 0, \tag{4}$$

then Equation (1) can be solved by either one of two methods. The first is to note that, in view of Equation (3), if Equation (1) is expressed as

$$F(x, u) - \frac{du}{dx} = 0, \tag{5}$$

then

$$I = \frac{1}{F(x,u)\xi(x,u) - \phi(x,u)}$$

is an integrating factor which makes Equation (5) exact. It is then solved by computing a potential function. The second is to transform into new coordinates (y, w) in terms of which Equation (1) becomes $\frac{dw}{dy} = G(y)$ and may then be integrated. The new coordinates are given by $y = \eta(x, u)$ and $w = \zeta(x, u)$, where the functions η and ζ are determined by the equations

$$\mathbf{v}(\eta) = \xi\frac{\partial \eta}{\partial x} + \phi\frac{\partial \eta}{\partial u} = 0 \quad \text{and} \quad \mathbf{v}(\zeta) = \xi\frac{\partial \zeta}{\partial x} + \phi\frac{\partial \zeta}{\partial u} = 1. \tag{6}$$

The *group invariant* η is determined by the solution $\eta(x, u)$ =constant of the equation

$$\frac{dx}{\xi(x,u)} = \frac{du}{\phi(x,u)}. \tag{7}$$

If Equation (4) is violated, then this method also fails because Equation (7) becomes equivalent to Equation (1).

We shall apply this method to solve the *nonlinear* equation

$$\frac{du}{dx} = x - u^2. \tag{8}$$

In addition to the foregoing, the procedure will involve solving an initial-value problem for a third-order *linear* ODE. Equation (8) is of interest as it cannot be solved by any of the standard elementary methods.

II. A SYMMETRY VECTOR FOR EQUATION (8)

For Equation (8), $F(x, u) = x - u^2$ and Equation (3) becomes

$$\phi_x + (\phi_u - \xi_x)(x - u^2) - \xi_u(x - u^2)^2 = \xi - 2u\phi. \tag{9}$$

As the general solution of (9) is not required and may be too difficult to determine, we seek a simple solution of the form

$$\xi(x, u) = a(x) \quad \text{and} \quad \phi(x, u) = b(x) + c(x)u. \tag{10}$$

Then, denoting derivatives with respect to x by primes, Equation (9) is satisfied if

$$\left\{ \begin{array}{c} c + a' = 0 \\ c' + 2b = 0 \\ b' + xc - xa' - a = 0 \end{array} \right\},$$

with the solutions $c = -a'$, $b = \frac{a''}{2}$, and

$$a''' - 4xa' - 2a = 0. \tag{11}$$

Since 0 is an ordinary point of Equation (11), the general solution takes the form

$$a = \sum_{n=0}^{\infty} a_n x^n, \tag{12}$$

with the coefficient recursion relation

$$a_{n+3} = \frac{2(2n+1)a_n}{(n+1)(n+2)(n+3)}, \quad n \geq 0. \tag{13}$$

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For reasons which will become apparent later, we require only the particular solution which satisfies the initial conditions $a_0 = a(0) = 1$, $a_1 = a'(0) = 0$ and $a_2 = a''(0) = 0$,

$$a = 1 + \sum_{k=1}^{\infty} \frac{2^k \cdot 1 \cdot 7 \cdot 13 \cdot 19 \cdots (6k-5)}{(3k)!} x^{3k}, \tag{14}$$

with infinite radius of convergence. Thus, with a defined by (14),

$$\xi(x,u) = a(x) \quad \text{and} \quad \phi(x,u) = \frac{1}{2}a''(x) - a'(x)u, \tag{15}$$

(2) is a symmetry vector for Equation (8).

III. THE SOLUTION OF EQUATION (8)

We require functions $\eta(x,u)$ and $\zeta(x,u)$ satisfying

$$\mathbf{v}(\eta) = a\eta_x + \left(\frac{1}{2}a'' - a'u\right)\eta_u = 0 \tag{16}$$

and

$$\mathbf{v}(\zeta) = a\zeta_x + \left(\frac{1}{2}a'' - a'u\right)\zeta_u = 1. \tag{17}$$

The solutions of Equations (16) and (17) are

$$\eta(x,u) = a(x)u - \frac{1}{2}a'(x) \quad \text{and} \quad \zeta(x,u) = f(x), \tag{18}$$

respectively, where $f'(x) = \frac{1}{a(x)}$.

Let $y = \eta(x,u) = au - \frac{1}{2}a'$ and $w = \zeta(x,u) = f(x)$. Then $x = F(w)$, where F is the inverse function of f, and

$$F(f(x)) = 1 \quad \Rightarrow \quad \frac{dF}{dw}\frac{dw}{dx} = 1 \quad \Rightarrow \quad \frac{dF}{dw} = \frac{1}{f'(x)} = a.$$

In addition,

$$\frac{dx}{dy} = \frac{dF}{dw}\frac{dw}{dy} = a\frac{dw}{dy}$$

and, from $y = au - \frac{1}{2}a'$, we obtain $u = \frac{y}{a} + \frac{a'}{2a}$. Then

$$\frac{du}{dy} = \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}\frac{dx}{dy} = \frac{1}{a} + \left[\frac{-a'}{a^2}y + \frac{aa'' - (a')^2}{2a^2}\right] a\frac{dw}{dy}$$

and

$$\frac{du}{dx} = \frac{\frac{du}{dy}}{\frac{dx}{dy}} = \frac{1}{a^2\frac{dw}{dy}} - \frac{a'}{a^2}y + \frac{a''}{2a} - \frac{(a')^2}{2a^2}.$$

Equation (8) then transforms into

$$\frac{1}{a^2\frac{dw}{dy}} - \frac{a'}{a^2}y + \frac{a''}{2a} - \frac{(a')^2}{2a^2} = x - \left(\frac{y}{a} + \frac{a'}{2a}\right)^2,$$

which simplifies to

$$\frac{dw}{dy} = \frac{1}{\frac{1}{4}[2aa'' - 4xa^2 - (a')^2] - y^2}. \tag{19}$$

Note that the quantity $g(x) = 2aa'' - 4xa^2 - (a')^2$ vanishes at $x = 0$ since $a'(0) = a''(0) = 0$, and it is constant because

$$g'(x) = 2a(a''' - 4xa' - 2a) = 0$$

by Equation (11). Hence,

$$2aa'' - 4xa^2 - (a')^2 \equiv 0, \tag{20}$$

and (19) reduces to

$$\frac{dw}{dy} = -\frac{1}{y^2},$$

with the solution $w = \frac{1}{y} + k$, where k is an arbitrary constant. In terms of the original variables, we obtain

$$\int \frac{1}{a(x)}\,dx = \frac{1}{au - \frac{1}{2}a'} + k,$$

or, absorbing k into the arbitrary constant of integration,

$$u = \frac{a'}{2a} + \frac{1}{a\int \frac{1}{a(x)}\,dx}. \tag{21}$$

It can readily be verified that (21) satisfies Equation (8) by making use of the relation (20).

IV. CONCLUSION

A nonlinear ODE of the first order is solved by the Lie symmetry method. The procedure is further complicated by the components of the symmetry vector being given by the solution as an infinite series of a third-order linear equation, upon which suitable initial conditions needed to be imposed.

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