Quest Journals
Journal of Research in Applied Mathematics
Volume 9 ~ Issue 7 (2023) pp: 28-33
ISSN (Online): 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Review Paper

Spectral decomposition of a matrix: Some applications on correlation matrices

Şenol Çelik
Biometry and Genetic Unit, Department of Animal Science, Faculty of Agriculture, Bingol University, Bingol 12000, Turkey

Abstract
The spectral decompositions of (2x2), (3x3) and (4x4) matrices were examined in this paper. To determine the spectral decomposition, it is necessary to have a positive definite matrix and to know the eigenvalue and the eigenvector of this matrix.A symmetric matrix and certain non-symmetric matrices are illustrated. The spectral decomposition of correlation matrices was used in an application. A correlation matrix was constructed for the values of several variables from several livestock studies, and the spectral decomposition of these correlation matrices was derived.
Keywords: *Spectral decomposition, matrix, correlation*

*Received 10 July, 2023; Revised 20 July, 2023; Accepted 22 July, 2023 © The author(s) 2023. Published with open access at **www.questjournals.org***

I. Introduction

Matrix spectral decomposition is a common and significant activity in applied mathematics, physics, and engineering. Many application problems necessitate the consideration of three-dimensional matrices with spectral decomposition over real values. Closed form solution techniques must be considered if the functional dependence of the spectral decomposition on the matrix elements is to be retained. Existing closed form expressions are predicated on the use of primary matrices in variations, which have a number of flaws when evaluated in the context of finite precision arithmetic [1].

For mathematicians, physicists, and engineers, spectral decomposition of real valued matrices (eigen decomposition) is critical. In particular, the decomposition of (3x3) matrices is important in three-dimensional space since it is common in many real-world applications on texts. Because of its close relationship to the roots of the cubic equation, this unique spectral decomposition problem has been investigated for decades. With the advent of computers and strong symbolic systems such as Mathematica[2] and SymPy[3], closed form solutions have become increasingly popular. The symbolic technique is a useful tool for computing eigenvalues and eigenvectors while maintaining their functional dependency on matrix members.
The purpose of this research is to investigate the spectral decomposition of any matrix and correlation matrices.

II. Material and Method

Distances and the assumption that the data are multivariate normally distributed are frequently used in the analysis of variation and interrelationships in multivariate data. Quadratic forms are matrix products that can be used to describe squared distances and multivariate normal density. As a result, it is not surprising that quadratic forms are important in multivariate analysis. In this part, we look at quadratic forms that are always nonnegative, as well as the corresponding positive definite matrices. In many circumstances, results involving quadratic forms and symmetric matrices are a direct result of a symmetric matrices expansion known as the spectral decomposition [4].A (kXk) symmetric matrix A's spectral decomposition is given by [4, 5]

$$\underset{(k*k)}{A} = \lambda_1 \underset{(k*1)}{e_1}\underset{(1*k)}{e'_1} + \lambda_2 \underset{(k*1)}{e_2}\underset{(1*k)}{e'_2} + \cdots + \lambda_k \underset{(k*1)}{e_k}\underset{(1*k)}{e'_k}$$

where λ_1, λ_2, …, λ_k are eigenvalues of A and e_1, e_2,…,e_kk are the associated normalized eigenvectors. Thus, $e'_i e_i = 1$ for i=1, 2,…, k, and $e'_i e_j = 0$ for $i \neq j$.

III. Results and Discussion

Calculate the eigenvalue and eigenvector for a square matrix A of (2*2) given below.

$$A = \begin{bmatrix} 2 & -10 \\ -10 & 2 \end{bmatrix}$$

To calculate the eigenvalues for this matrix

$$\det(A - \lambda I) = 0$$

from the formula,

$$det\left(\begin{bmatrix} 2 & -10 \\ -10 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = det\left(\begin{bmatrix} 2 & -10 \\ -10 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$$

$$det\left(\begin{bmatrix} 2-\lambda & -10 \\ -10 & 2-\lambda \end{bmatrix}\right) = 0$$

$$(2-\lambda)(2-\lambda) - 10 * 10 = 0$$

$$\lambda^2 - 4\lambda - 96 = 0$$

The roots of this quadratic equation give the eigenvalues of the matrix.

$$\lambda_1 = 12 \ \ and \ \lambda_2 = -8$$

The eigenvectors obtained from here are respectively,

$$v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \ and \ v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The unit eigenvectors of these eigenvectors are respectively,

$$e_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$e_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

available in the form.

$$A = 12 \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} - 8 \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$A = 12 \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} - 8 \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 6 & -6 \\ -6 & 6 \end{bmatrix} - \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 2 & -10 \\ -10 & 2 \end{bmatrix}$$

is obtained as. Similarly, a (3*3) square matrix A to find the spectral decomposition is as follows.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 8 & 6 & 4 \\ 8 & 4 & 6 \end{bmatrix}$$

The eigenvalues of this matrix are,

$\lambda_1 = 10,\ \lambda_2 = 2$ and $\lambda_3 = 2$ as were founded.

Here, the eigenvectors,

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

and corresponding unit eigenvectors are respectively,

$$e_1 = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad e_3 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{bmatrix}$$

is in the form.

$$A = 10 * \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} + 2 * \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} + 2 * \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix}$$

can be written as spectral decomposition.
Let a (3*3) symmetric D matrix be as follows.

$$D = \begin{bmatrix} 1 & 0.60 & 0.38 \\ 0.60 & 1 & -0.79 \\ 0.38 & -0.79 & 1 \end{bmatrix}$$

The eigenvalues and eigenvectors of this matrix were found with the R Studio program as follows. R program codes are given in Table 1 and the program output is given in Table 2.

Table 1. Eigenvalue and eigenvector calculation of matrix in R program

```
D <- as.matrix(data.frame(c(1,0.60,0.38),c(0.60,1,-0.79),c(0.38,-0.79,1)))
D
e<- eigen(D)
e$values
e$vectors
```

Table 2. Program output

```
D <- as.matrix(data.frame(c(1,0.60,0.38),c(0.60,1,-0.79),c(0.38,-0.79,1)))
> D
    c.1..0.6..0.38. c.0.6..1...0.79. c.0.38...0.79..1.
[1,]          1.00          0.60          0.38
[2,]          0.60          1.00         -0.79
[3,]          0.38         -0.79          1.00
> e <- eigen(A)
>e$values
[1] 5 1 1
>e$vectors
        [,1]      [,2]      [,3]
[1,] 0.0000000  0.0000000  0.3779645
[2,] 0.7071068  0.7071068 -0.9124870
[3,] 0.7071068 -0.7071068  0.1565580
> D <- as.matrix(data.frame(c(1,0.60,0.38),c(0.60,1,-0.79),c(0.38,-0.79,1)))
> D
    c.1..0.6..0.38. c.0.6..1...0.79. c.0.38...0.79..1.
[1,]          1.00          0.60          0.38
[2,]          0.60          1.00         -0.79
[3,]          0.38         -0.79          1.00
> e <- eigen(D)
>e$values
[1]  1.8348698  1.3608609 -0.1957307
>e$vectors
         [,1]     [,2]      [,3]
[1,] -0.2752205 0.8181442  0.5048699
[2,] -0.7577098 0.1386273 -0.6376976
[3,]  0.5917174 0.5580524 -0.5817629
```

The eigenvalues of the D matrix given according to the results given in Table 2,

$$\lambda_1 = 1.83487, \quad \lambda_2 = 1.36086, \quad \lambda_3 = -0.19573$$

found as if the unit eigenvectors of this matrix are,

$$e_1 = \begin{bmatrix} -0.27522 \\ -0.75771 \\ 0.59172 \end{bmatrix}$$

$$e_2 = \begin{bmatrix} 0.81814 \\ 0.13863 \\ 0.55805 \end{bmatrix}$$

$$e_3 = \begin{bmatrix} 0.50487 \\ -0.63770 \\ -0.58176 \end{bmatrix}$$

Their spectral decomposition can be written as:

$$D = 1.83487 \begin{bmatrix} -0.27522 \\ -0.75771 \\ 0.59172 \end{bmatrix} [-0.27522 \quad -0.75771 \quad 0.59172] + 1.36086 \begin{bmatrix} 0.81814 \\ 0.13863 \\ 0.55805 \end{bmatrix} [0.81814 \quad 0.13863 \quad 0.55805] - 0.19573 \begin{bmatrix} 0.50487 \\ -0.63770 \\ -0.58176 \end{bmatrix} [0.50487 \quad -0.63770 \quad -0.58176]$$

$$D = 1.83487 \begin{bmatrix} 0.07575 & 0.20854 & -0.16285 \\ 0.20854 & 0.57412 & -0.44835 \\ -0.16285 & -0.44835 & 0.35013 \end{bmatrix} + 1.36086 \begin{bmatrix} 0.66935 & 0.11342 & 0.45656 \\ 0.11342 & 0.01922 & 0.07736 \\ 0.45656 & 0.07736 & 0.31142 \end{bmatrix} - 0.19573 \begin{bmatrix} 0.25489 & -0.32196 & -0.29371 \\ -0.32196 & 0.40666 & 0.37099 \\ -0.29371 & 0.37099 & 0.33845 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0.60 & 0.38 \\ 0.60 & 1 & -0.79 \\ 0.38 & -0.79 & 1 \end{bmatrix}$$

expressed as spectral decomposition.

In a study, the spectral decomposition of a symmetric (4*4) size A square matrix consisting of correlation coefficients between measurements of body biological characteristics [6] in donkeys can be examined.

$$A = \begin{bmatrix} 1 & 0.762 & 0.767 & 0.614 \\ 0.762 & 1 & 0.951 & 0.569 \\ 0.767 & 0.951 & 1 & 0.586 \\ 0.614 & 0.569 & 0.586 & 1 \end{bmatrix}$$

The R program instructions that calculate the eigenvalues and unit eigenvectors of this correlation matrix are presented in Table 3, and the output of the program is presented in Table 4.

Table 3. R program codes of the eigenvalue and unit vector of the given matrix A

```
A <- as.matrix(data.frame(c(1,0.762,0.767, 0.614),c(0.762,1,0.951,0.569),c(0.767,0.951,1,0.586),c(0.614,0.569,0.586,1)))
A
e<- eigen(A)
e$values
e$vectors
```

Table 4. Results of R codes of (4*4) dimensional symmetric square matrix

```
> A <- as.matrix(data.frame(c(1,0.762,0.767,0.614), c(0.762,1,0.951,0.569), c(0.767,0.951,1,0.586), c(0.614,0.569,0.586,1)))
> A
   c.1..0.762..0.767..0.614. c.0.762..1..0.951..0.569. c.0.767..0.951..1..0.586. c.0.614..0.569..0.586..1.
[1,]     1.000     0.762     0.767     0.614
[2,]     0.762     1.000     0.951     0.569
[3,]     0.767     0.951     1.000     0.586
[4,]     0.614     0.569     0.586     1.000
> e <- eigen(A)
>e$values
[1] 3.14220123 0.53080342 0.27823691 0.04875844
>e$vectors
         [,1]        [,2]       [,3]         [,4]
[1,] -0.5019803 -0.01887884  0.8646711 -0.001793548
[2,] -0.5290368 -0.35999118 -0.3164420 -0.700279122
[3,] -0.5322114 -0.32939582 -0.3146842  0.713598792
```

[4,] -0.4299847 0.87266737 -0.2306125 -0.019562679

As seen in Table 4, the eigenvalues of the (4*4) dimension A matrix were found to be 3.142, 0.531, 0.278 and 0.049, respectively. The unit eigenvectors corresponding to these eigenvalues were obtained as follows.

$$e_1 = \begin{bmatrix} -0.501 \\ -0.529 \\ -0.532 \\ -0.430 \end{bmatrix}, e_2 = \begin{bmatrix} -0.019 \\ -0.360 \\ -0.329 \\ 0.873 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0.865 \\ -0.316 \\ -0.315 \\ -0.231 \end{bmatrix}, \ e_4 = \begin{bmatrix} -0.002 \\ -0.700 \\ 0.714 \\ -0.020 \end{bmatrix}$$

$$A = 3.142 \begin{bmatrix} -0.501 \\ -0.529 \\ -0.532 \\ -0.430 \end{bmatrix} [-0.501 \quad -0.529 \quad -0.532 \quad -0.430]$$
$$+ 0.531 \begin{bmatrix} -0.019 \\ -0.360 \\ -0.329 \\ 0.873 \end{bmatrix} [-0.019 \quad -0.360 \quad -0.329 \quad 0.873]$$
$$+ 0.278 \begin{bmatrix} 0.865 \\ -0.316 \\ -0.315 \\ -0.231 \end{bmatrix} [0.865 \quad -0.316 \quad -0.315 \quad -0.231]$$
$$+ 0.049 \begin{bmatrix} -0.002 \\ -0.700 \\ 0.714 \\ -0.020 \end{bmatrix} [-0.002 \quad -0.700 \quad 0.714 \quad -0.020]$$

$$A = 3.142 \begin{bmatrix} 0.252 & 0.266 & 0.267 & 0.216 \\ 0.266 & 0.280 & 0.282 & 0.227 \\ 0.267 & 0.282 & 0.283 & 0.229 \\ 0.216 & 0.227 & 0.229 & 0.185 \end{bmatrix} + 0.531 \begin{bmatrix} 0.00036 & 0.0068 & 0.0062 & -0.0165 \\ 0.0068 & 0.1296 & 0.1186 & -0.31415 \\ 0.0062 & 0.1186 & 0.1085 & -0.28745 \\ -0.0165 & -0.31415 & -0.28745 & 0.7615 \end{bmatrix}$$
$$+ 0.278 \begin{bmatrix} 0.748 & -0.274 & -0.272 & -0.199 \\ -0.274 & 0.100 & 0.099 & 0.0730 \\ -0.272 & 0.099 & 0.099 & 0.0726 \\ -0.199 & 0.0730 & 0.0726 & 0.053 \end{bmatrix}$$
$$+0.049 \begin{bmatrix} 0.000003 & 0.00126 & -0.00128 & -0.000035 \\ 0.00126 & 0.4904 & -0.4997 & 0.0137 \\ -0.00128 & -0.4997 & 0.509 & -0.014 \\ -0.000035 & 0.0137 & -0.014 & 0.00038 \end{bmatrix}$$

and

$$A = \begin{bmatrix} 1 & 0.762 & 0.767 & 0.614 \\ 0.762 & 1 & 0.951 & 0.569 \\ 0.767 & 0.951 & 1 & 0.586 \\ 0.614 & 0.569 & 0.586 & 1 \end{bmatrix}$$

Other applications have been made in different fields regarding spectral decomposition [7, 8].

IV. Conclusion

The solution technique to spectral decomposition of real-valued matrices with real eigenvalues is investigated in this study. Applications were made on (2*2), (3*3) and (4*4) type matrices. The eigenvalues and eigenvectors of the matrices were used to obtain the spectral decomposition. The spectral decomposition of the correlation matrix was found and the relationship between them was examined. The application was realized by giving an example with animal husbandry data. It is concluded that very different and wide application of spectral decomposition is possible.

References

[1]. Habera, M.,Zilian, A. 2021. Symbolic spectral decomposition of 3x3 matrices. Numerical Analysis (math.NA); Mathematical Software (cs.MS). arXiv:2111.02117 [math.NA]. https://doi.org/10.48550/arXiv.2111.02117.

[2]. Wolfram, S. 1991. Mathematica: a system for doing mathematics by computer. Addison Wesley Longman Publishing Co.,Inc.

[3]. Meurer, A., Smith,C.P., Paprocki, M., ˇCertík, O., Kirpichev,S.B., Rocklin, M., Kumar, A., Ivanov, S., Moore,J.K., Singh, S., Rathnayake, T., Vig, S., Granger,B.E., Muller,R.P., Bonazzi, F., Gupta, H., Vats, S., Johansson, F., Pedregosa, F., Curry,M.J., Terrel,A.R., Rouˇcka, V., Saboo, A., Fernando, I., Kulal, S., Cimrman, R., Scopatz, A. 2017. Sympy:symboliccomputinginpython.PeerJComputerScience,3:e103,ISSN2376-5992. doi:10.7717/peerj-cs.103.URLhttps://doi.org/10.7717/peerj-cs.103.

[4]. Johnson, R.A., Wichern, D.W. 2007. Applied Multivariate Statistical Analysis. Pearson Education, Inc., London.

[5]. Poole, D. 2011. Linear Algebra A modern Introduction. Brooks/Cole, Cengage Learning, Canada. ISBN-13: 978-0-538-73545-2.

[6]. Çelik, Ş. Yılmaz, O. 2023. Investigation of the Relationships between Coat Colour, Sex, and Morphological Characteristics in Donkeys Using Data Mining Algorithms. Animals, 13, 2366. https://doi.org/10.3390/ani13142366.

[7]. Gronevelt, W., Ismail, M.E.H., Koelink, E. 2013. Spectral decomposition and matrix-valued orthogonal polynomials. Advances in Mathematics, 244: 91–105.

[8]. Watson, G.S. 1996. Spectral Decomposition of the Covariance Matrix of a Multinomial. J. B. Statist. Soc. B., 58(1):289-291.