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Review Paper

The Conditions for Conformality ofRiemannian Manifolds to Spheres

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ABSTRACT

In this chapter, we assume [X,Dρ] r = 0 and obtain conditions for conformality of M to an n-sphere where the bracket [,] is the lie bracket and Dρ is the vector filed associated to the 1-form dρ. we consider a compact orientable Riemannian manifold M and non metric semi-symmetric connection on M and obtain conditions for M to be either conformal or isometric to a sphere without putting restrictions on the Scalar curvature of M. Finally, special cases of our results are also deduced by using the projective change of the Riemannian connection and the conformal change of the Riemannian metric g.

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1. Notations and Formulae:

The raising and lowering of the indices are as usual carried out respectively with g^{ij} and g_{ij} . Thus, tensors are called associated tensors . Let S, and T be the covariant tensors of order s with local components $S_{i_1 i_2 \ldots\ldots i_s}$ and $T_{i_1 i_2 \ldots\ldots i_s}$ respectively. The associated contravariant components of T are $T^{i_1 i_2 \ldots\ldots i_s}$. We define the inner product of S and T by $S_{i_1 i_2 \ldots\ldots i_s} \; T^{i_1 i_2 \ldots\ldots i_s}$ and denote by $<S,T>$. If $S=T$ we write $|S|^2$ for $<S,S>$

(1.1) $\quad £_X r = 2(n-1)\ \Delta\rho - 2r\rho$

where Δ is the Laplace-Beltrami operator on M.

(1.2) $\quad £_X g^{ij} =- 2\rho\, g^{ij}$,

(1.3) $\quad £_x K_{hijk} = 2\rho\, K_{hijk} - g_{hk} \nabla_j \rho_i + g_{hj} \nabla_i \rho_k - g_{ij} \nabla_h \rho_k + g_{ik} \nabla_h \rho_j$,

(1.4) $\quad £_X K_{ij} = \Delta\rho g_{ij} - (n-2) \nabla_i \rho_j$,

(1.5) $\quad \nabla_k \nabla_i y^j - \nabla_i \nabla_k y^j = K^j_{kih} y^h,\ \ g^{kj}(\nabla_k \nabla_i y_j - \nabla_i \nabla_k y_j) = K^h_i y_h$,

where y is any differentiable vector field on M. We write $f_i = \nabla_i f$ and $f^i = g^{ij} f_j$. The tensor Z[3] is given by

(1.6) $\quad Z_{hijk} = K_{hijk} - r\,(g_{hk}\, g_{ij} - g_{hj}\, g_{ik})/\ n(n-1)$,

(1.7) $\quad W_{hijk} = aZ_{hijk} + b_1 g_{hk}\, G_{ij} - b_2\, g_{hj}\, G_{ik}\ \ b_3 g_{ij}\, G_{hk}$
$\qquad\qquad -b_4\, g_{ik}\, G_{hj} + b_5 g_{hi}\, G_{jk} - b_6\, g_{jk}\, G_{hi}$,

where a, b_1,.......b_6 are any constants.

2. Compact-Orientable Riemannian Manifolds of Dimension $n > 2$:

Lemma 2.1:

Let M be a compact, Orientable Riemannian manifold of dimensions $n > 2$. For any vector field y and a differentiable function f, we have

$$\int_M (\nabla_i y^i) dv = 0, \qquad \int_M \Delta f dv = 0$$

the first is well known Green's formula. The second follows as a consequence of the first.

Lemma 2.2:

Let M be a compact, oriented Riemannian manifold of dimensions $n > 2$ admitting an infinitesimal non-isometric transformation satisfying

$$(£_X g)_{ij} = \nabla_i X_j + \nabla_j X_i = 2\rho g_{ij},$$

Then, for any function f on M, we have

$$\int_M \rho f dv = -1/n \int_M £_X f dv$$

Lemma 2.3:

For a manifold M having the same properties as in Lemma 2.2 we have

(a) $$\int_M (\nabla\rho)^2 dv = \int_M \rho^i \nabla_i (\Delta\rho) dv = \int_M \left[K_{ij}\rho^j - g^{Kj}\nabla_k \nabla_j \rho_i\right]\rho^i dv,$$

and if $r =$ constant.

(b) $$\int_M (\nabla\rho)^2 dv = \frac{1}{n-1}\int_M r\rho_i \rho^i dv.$$

Proof:

$\nabla_i(\rho^i \Delta\rho) = \rho^i \nabla_i (\Delta\rho) - (\Delta\rho)^2 = (K_{ij}\ \rho^j - g^{Kj}\ \nabla_k \nabla_j \rho_i)\ \rho^i - (\Delta\rho)^2$ by (1.5).
Integrating and using Lemma 2.1, we get
(a) Setting $£_X r = 0$ in (1.1) and using the result in (a), we get (b).

Lemma 2.4:

Let M be a manifold having the same Properties as in Lemma 2.2 and satisfying the condition $[X, D\rho]\ r = 0$, then

(a) $$\int_M r\rho_i \rho^i dv = (n-1)\int_M (\Delta\rho)^2 dv - \frac{n-2}{2n}\int_M £_X £_{D\rho} r dv,$$

and if $£_X r = 0$

(b) $$\int_M r\rho_i \rho^i dv = \frac{1}{n-1}\int_M r^2 \rho^2 dv.$$

Proof:

Form (1.6) we have

$$0 = [X_1, D\rho\ ] r = £_X £_{D\rho}\ r - £_{D\rho}\ £_X r.$$

Therefore, by using (1.1),

(2.1) $$£_X £_{D\rho}\ r = £_{D\rho}\ £_X r = £_{D\rho}\ [2(n\text{-}1)\Delta\rho - 2r\rho]$$
$$= 2(n\text{-}1)\ \rho^i\ \nabla_i (\Delta\rho) - 2r\rho_i \rho^i - 2\rho\rho^i\ \nabla_i r.$$

Integrating, using Lemmas 2.3 and 2.2 we get (a). When $£_X r = 0$, $\Delta\rho = r\rho/(n\text{-}1)$ and $£_X £_{D\rho}\ r = £_{D\rho}\ £_X r = 0$ and substituting these values in (a) we get (b).

Theorem 2.1:

If a compact, orientable, smooth Riemannian manifold M of dimension n>2 admitting an infinitesimal conformal transformation X : $£_X g = 2\rho g$, $\rho \neq$ constant, with $[X, D\rho] r = 0$ satisfies

$$\int_M \left[\Lambda_{ij}\rho^i \rho^j - \frac{\alpha(n-2)}{2n^2} £_X £_{D\rho} r\right] dv \geq 0,$$

where $\Lambda_{ij} = K_{ij} - r\alpha g_{ij}/n$ and $\alpha = 1$ then M is conformal to an n-sphere.

Proof:

By writing $\nabla^j = g^{ji}\ \nabla_i$, we find, for an arbitrary vector field y that

$\nabla^j\ [\ (\nabla_j\ y_i + \nabla_i y_j - (2\alpha/n)\ (\nabla_t y^t) g_{ij}\ ) y^i]$
$\quad = (\nabla^j \nabla_j\ y_i + \nabla_j \nabla_i\ y^j - (2\alpha/n)\ \nabla_i\ (\nabla_t y^t)) y^i + (\nabla_j y_i + \nabla_i y_j - (2\alpha/n)\ (\nabla_t y^t) g_{ij}) \nabla^j y^i$

$= (g^{jk}\nabla_k\nabla_j y_i + \nabla_i\nabla_j y^j + K_{jih}{}^j y^h - (2\alpha/n)\nabla_i\nabla_t y^t)y^i + (2\alpha/n)(1-\alpha)(\nabla_t y^t)^2$
$+ \frac{1}{2}(\nabla_j y_i + \nabla_i y_j - (2\alpha/n)(\nabla_t y^t)g_{ij})(\nabla^j y^i + \nabla^i y^j - (2\alpha/n)(\nabla_t y^t)g^{ij})$

Here we use (1.5), using Lemmas 2.1 and 2.3 and substituting $y^i = \rho^i$ then integrating we get

$$\int_M K_{ij}\rho^i\rho^j dv + \frac{-n+2\alpha-\alpha^2}{n}\int_M (\Delta\rho)^2 dv + \int_M \left|\nabla\nabla\rho + \frac{\alpha}{n}\Delta\rho . g\right|^2 dv = 0$$

Setting $K_{ij} = \Lambda_{ij} + r\alpha g_{ij}/n$, we obtain

$$\int_M \Lambda_{ij}\rho^i\rho^j dv + \frac{-n+2\alpha-\alpha^2}{n}\int_M (\Delta\rho)^2 dv + \frac{\alpha}{n}\int_M r\rho_i\rho^i dv + \int_M \left|\nabla\nabla\rho + \frac{\alpha}{n}\Delta\rho . g\right|^2 dv = 0$$

Substituting from Lemma 2.4(a) and simplifying, we obtain finally

(1)
$$\int_M \left[\Lambda_{ij}\rho^i\rho^j - \frac{\alpha(n-2)}{2n^2}\pounds_X\pounds_{D\rho} r \right] dv + \int_M \left|\nabla\nabla_\rho + \frac{1+\sqrt{(\alpha-1)(n-1)}}{n}\nabla\rho . g\right|^2 dv = 0$$

Theorem 2.2:

Let M be an orientable, smooth Riemannian manifold of dimension $n > 2$ admitting an infinitesimal conformal transformation X satisfying $(\pounds_X g)_{ij} = \nabla_i X_j + \nabla_j X_i = 2\rho g_{ij}$, such that $\rho \neq$ constant and $[X, D\rho]r = 0$, then M is conformal to an n-sphere if $\pounds_X|G|^2 = 0$ where $G_{ij} = K_{ij} - (r/n)\, g_{ij}$.

Proof:

From (1.2) and (1.6) we get

$$\pounds_X|G|^2 = 2 < \pounds_X G, G > - 4\rho\, |G|^2 = -2(n-2) < G, \nabla\nabla\rho > -4\rho\, |G|^2$$

i.e.

(2) $< G, \nabla\nabla\rho > = -\{2\rho|G|^2 + \frac{1}{2}\pounds_X|G|^2\}/(n-2)$

On the other hand

(3) $\nabla^i(G_{ij}\rho\rho^j) = G_{ij}\,\rho^i\rho^j + \rho < G, \nabla\nabla\rho > + (n-2)\,\rho\,(\rho^i\nabla_i r)/2n$

Multiplying (2) by ρ and integrating, then Integrating (3) and eliminating $\int_M \rho < G, \nabla\nabla\rho > dv$ so that

We get the integral formula.

(4)
$$\int_M \left[G_{ij}\rho^i\rho^j - \frac{(n-2)}{2n^2}\pounds_X\pounds_{D\rho} r \right] dv = \frac{2}{n-2}\int_M \left[\rho^2|G|^2 + \frac{1}{4}\rho\pounds_X|G|^2 \right] dv$$

Theorem (2.2) follows from theorem (2.1) and the integral formula (4).

Theorem 2.3:

Let M be an orientable smooth Riemannian manifold of dimension $n > 2$ admitting an infinitesimal conformal transformation X satisfying $(\pounds_X g)_{ij} = \nabla_i X_j + \nabla_j X_i = 2\rho g_{ij}$ such that $\rho \neq$ constant and $[X, D\rho]\, r = 0$. Then M is conformal to an n-sphere if $\pounds_X|W|^2 = 0$ where W is a tensor.

Proof:

From (1.7), (1.6), (1.3), (1.4) and (1.2), we get

(5) $< \pounds_X W, W > = 2\rho\, |W|^2 - c < G, \nabla\nabla\,\rho >$

where c is a constant given by

$$\frac{c-4a^2}{n-2} = 2a\sum_{i=1}^{4} b_i + \left[\sum_{i=1}^{6}(-1)^{i-1}b_i\right]^2 + (n-1)\sum_{i=1}^{6} b_i^2 - 2(b_1b_3 + b_2b_4 - b_5b_6).$$

Here $c \geq 0$, By using (1.2), we get

(6) $$\pounds_X|W|^2 = 2 < \pounds_X W, W > -8\rho|W|^2$$

Hence from (5) and (6) we get $\pounds_X|W|^2 = -4\rho\,|W|^2 - 2c<G, \nabla\nabla\,\rho>$,or

$$\int_M 2c\rho < G, \nabla\nabla\rho > dv = -4\int_M \rho^2|W|^2\,dv - \int_M \rho\pounds_X|W|^2\,dv$$

Also from (3) and lemma (2.2), we get

(7) $$c\int_M\left[G_{ij}\rho^i\rho^j - \frac{n-2}{2n^2}\pounds_X\pounds_{D\rho}r\right]dv = 2\int_M \rho^2|W|^2\,dv + \frac{1}{2}\int_M \rho\pounds_X|W|^2\,dv$$

theorem (2.3) follows form theorem (2.1) and the integral formula (7)

3. Riemannian Manifold is Isometric to Sphere:

Let $\alpha = 1$ and r = constant, then $\Lambda_{ij} = G_{ij}$ and $\pounds_X\pounds_{D\rho}\,r = 0$. The condition for M to be conformal to a sphere in theorem (2.1) reduces to $\int_M G_{ij}\rho^i\rho^j\,dv \geq 0,$ which is a known condition, [3] for M to be isometric to a sphere with this, theorem (2.3) is the one due to Hsiung [1].

4. Non-Metric, Semi-Symmetric Connection $\overset{*}{\nabla}$:

Let ∇ be a Riemannian connection on M with components $\left\{\begin{matrix} h \\ j\,i \end{matrix}\right\}$, called the Christofel symbols. If g^{ji} are the contra variant components of the metric g, then the raising and lowering of indices of a tensor are carried out using g^{ji} and g_{ji}. In this chapter, Einstein Summation conventions are used, i.e. one index below and the same index above to represent the summation. For a smooth function f on M, the Laplacian of f, i.e. Δf, is defined by

$$\Delta f = g^{ji}\,\nabla_j\nabla_i f,$$

Where ∇_i denote the covariant differentiation with respect to the index i and the Laplace -Beltrami operator is given by

$$\Delta = g^{ij}\,\nabla_j\nabla_i$$

The gradient of a smooth function f on M is given by a unique vector filed y.

$$df(y) = \text{grad}\, f(y) = yf$$

for all vector fields y or in terms of local co-ordinates $f^i = g^{ji}\,\nabla_j f$. The divergence of a smooth vector field X on M is given by

$$\text{div}\, X = g^{ji}\,\nabla_j X_i = \nabla^i X_i,$$

where $X_i = g_{ji}X^j$ and X^j are the components of the vector field X on M. For a smooth function ρ on M, If $D\rho$ is the vector field associated with the closed 1-form $d\rho$, then we denote the components of $D\rho$ by ρ^i such that $\rho^i = g^{ji}\rho_j$ and $\rho_j = \nabla_j\,\rho$ are the components of $d\rho$.

If S and T are any two tensor fields on M of same order 2 or 4, then accordingly, we write

$$g(S,T) = S^{ji}T_{ji} \;\; or \;\; g(S,T) = S^{kjih}T_{kjih}$$

In particular, the absolute value of a tensor S of order 2 or 4 is given by

(4.1a) $$|S|^2 = g(S,S) = S^{ji}S_{ji} \;\; or \;\; |S|^2 = g(S,S) = S^{Kjih}S_{Kjih}$$

The non-metric semi-symmetric connection $\overset{*}{\nabla}$ on M with components Γ_{ji}^{h} is defined by

(4.1) $$\Gamma_{ji}^{h} = \left\{ \begin{matrix} h \\ ji \end{matrix} \right\} + \delta_i^h \rho_j$$

The curvature tensors with respect to $\overset{*}{\nabla}$ and ∇ that is $\overset{*}{K}_{kjih}$ and K_{kjih} are related by (Nirmala and Chafle 1992, Biswas and De 1997)

(4.2) $$\overset{*}{K}{}_{kji}^{h} = K_{kji}^{h} - C_{ji}\delta_k^h + C_{ki}\delta_j^h,$$

where

(4.3) $$C_{ji} = \nabla_j \rho_i - \rho_j \rho_i$$

contracting (4.2) with respect to the indices h and k, we have

(4.4) $$\overset{*}{K}_{ji} = K_{ji} - (n-1)C_{ji},$$

where C_{ji} is given by (4.3) $\overset{*}{K}_{ji} = \overset{*}{K}{}_{tji}^{t}$ and $K_{ji} = K_{tji}^{t}$, further, transvecting (4.4) by g^{ji} on both sides, we have

(4.5) $$\overset{*}{r} = r - (n-1)\,\Phi$$

where $\overset{*}{r} = g^{ji}\overset{*}{K}_{ji}$, $r = g^{ji} K_{ji}$ and

(4.6) $$\Phi = g^{ji} C_{ji} = \Delta\rho - |d\rho|^2$$

The tensor G which measures the deviation of M from an Einstein manifold is defined by Yano [4].

(4.7) $$G_{ji} = K_{ji} - \frac{r}{n} g_{ji}$$

and the concircular tensor Z which measures the deviation of M from the manifold of constant curvature by Yano [4].

(4.8) $$Z_{kjih} = K_{kjih} - \frac{r}{n(n-1)}\left(g_{Kh}g_{ji} - g_{jh}g_{ki}\right)$$

further, the tensor W is also given [2].

(4.9) $W_{kjih} = a\, Z_{kjih} + b_1 g_{kh} G_{ji} - b_2 g_{Ki}\, G_{jh} + b_3 g_{ji}\, G_{kh} - b_4 g_{jh}\, G_{Ki} + b_5\, g_{Kj}\, G_{ih} - b_6 g_{ih}\, G_{Kj}$

where a, b_1, b_2, b_3, b_4, b_5 and b_6 are some constants, if we take

$$a = 1,\ b_1 = b_2 = b_3 = b_4 = b_5 = b_6 = 0$$

in (4.9), then $W_{Kjih} = Z_{Kjih}$ from (4.8), it is easy to see that

$$Z_{kjih}\, g^{kh} = G_{ji},\ Z_{kjih}\, g^{ji} = G_{kh},\ Z_{kjih}\, g^{ki} = -G_{jh}$$

(4.9a) $$Z_{kjih}\, g^{jh} = -G_{ki},\ Z_{kjih}\, g^{ih} = 0, \text{ and } Z_{kjih} g^{kj} = 0$$

we define a positive smooth function u induced by ρ by

(4.10) $$u(X) = e^{-\rho(X)}$$

for all X in M. If Du is the vector field associated with the closed 1-form du, then the components of Du are denoted by $u^i = g^{ji}\, u_j$ and $u_j = \nabla_j u$ using (4.10), the following equations can be deduced.

(4.11) (i) $u_i = -u\rho_i$ (ii) $\nabla_j u_i = u\,(\rho_j \rho_i - \nabla_j \rho_i)$ (iii) $\Delta\, u = u\,(\rho^k \rho_k - \Delta\rho)$

corresponding to G,Z and W, from (4.7), (4.8) and (4.9), we define $\overset{*}{G}$, $\overset{*}{Z}$, $\overset{*}{W}$ on M with respect to non-metric semi-symmetric connection $\overset{*}{\nabla}$ given by (4.1)

(4.12) $$\overset{*}{G}_{ji} = \overset{*}{K}_{ji} - \frac{\overset{*}{r}}{n} g_{ji}$$

(4.13) $$\overset{*}{Z}_{kjih} = \overset{*}{K}_{kjih} - \frac{\overset{*}{r}}{n(n-1)}\left(g_{kh}g_{ji} - g_{jh}g_{ki}\right)$$

(4.14) $$\overset{*}{W}_{kjih} = a\overset{*}{Z}_{kjih} + b_1 g_{kh}\overset{*}{G}_{ji} - b_2 g_{ki}\overset{*}{G}_{jh} + b_3 g_{ji}\overset{*}{G}_{kh} - b_4 g_{jh}\overset{*}{G}_{ki} + b_5 g_{kj}\overset{*}{G}_{ih} - b_6 g_{ih}\overset{*}{G}_{kj}$$

where a, b_1, b_2, b_3, b_4 b_5 and b_6 are some constants which occur in (4.9). It is easy to see from (4.12), (4.8) and (4.7) that

(4.14a) $$\overset{*}{G}_{ji}\, g^{ji} = 0 \quad \text{and} \quad G_{ji} g^{ji} = 0$$

respectively

Substituting for $\overset{*}{K}_{ji}$ and $\overset{*}{r}$, from (4.4) and (4.5) respectively in (4.12), we have

(4.15) $$\overset{*}{G}_{ji} = G_{ji} + (n-1)A_{ji},$$

where A_{ji} is given by

(4.16) $$A_{ji} = -\left(C_{ji} - \frac{\Phi}{n} g_{ji}\right) = -\left(\nabla_j\rho_i - \rho_j\rho_i + u^{-1}\frac{\Delta u}{n}\right) = u^{-1}\left(\nabla_j u_i - \frac{\Delta u}{n} g_{ji}\right)$$

where in we have used (4.11) and (4.6). It is easy to see that

(4.16a) $g^{ji} A_{ji}=0$

Again substituting for $\overset{*}{K}_{jih}$ and $\overset{*}{r}$ from (4.2) and (4.5) respectively in (4.13), we have

(4.17) $$\overset{*}{Z}_{kjih} = Z_{kjih} + A_{ji}g_{kh} - A_{ki}g_{jh}$$

Further substitution for $\overset{*}{Z}_{kjih}$ and $\overset{*}{G}_{ji}$ from (4.15) and (4.17) respectively in (4.14), we have

(4.18) $$\overset{*}{W}_{kjih} = W_{kjih} + T_{kjih},$$

where T_{kjih} is given by

$$\frac{T_{kjih}}{n-1} = \left(\frac{a}{n-1} + b_1\right) A_{ji}g_{hk} - \left(\frac{a}{n-1} + b_4\right) A_{ki}g_{hj} - b_2 g_{ki}A_{jh} + b_3 g_{ji}A_{kh} + b_5 g_{Kj}A_{ih} - b_6 g_{ih}A_{kj},$$

where A_{ji} are given by (4.16)

It is easy to see that

(4.19) $$\overset{*}{G}_{ji}\overset{*}{G}^{ji} = G_{ji}G^{ji} + 2(n-1)A^{ji}G_{ji} + (n-1)^2 A^{ji}A_{ji}$$

form (4.19) and (4.1a), we write in the index, free notations as

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(4.20) $$\left|\overset{*}{G}\right|^2 = |G|^2 + 2(n-1)g(A,G) + (n-1)^2 |A|^2$$

using (4.18) and (4.9a), and performing same as in (4.19), we find that

$$\overset{*}{W}_{kjih}\overset{*}{W}^{kjih} = W_{kjih}W^{kjih} + 2W_{kjih}T^{kjih} + T_{kjih}T^{kjih}$$

form which, we have

$$\overset{*}{W}_{kjih}\overset{*}{W}^{kjih} = W_{kjih}W^{kjih} + 2e(n-1)A^{ji}G_{ji} + b(n-1)^2 A_{ji}A^{ji}$$

or in the index free notations.

(4.21) $$\left|\overset{*}{W}\right|^2 = |W|^2 + 2e(n-1)g(A,G) + b(n-1)^2 |A|^2$$

where b and e are obtained after careful simplifications using (4.14a) and (4.16a) respectively so that

(4.22) $$b = \frac{2a^2}{n-1} + \frac{2a}{n-1}\{(n-1)(b_1+b_4) - b_2 - b_3\} + (n-1)\sum_{i=1}^{6} b_1^2 + \left(\sum_{i=1}^{6}(-1)^{i-1}b_1^2\right)^2 - 2(b_1b_3 + b_2b_4 - b_5b_6)$$

(4.22a) $$e = \frac{2a^2}{n-1} + \frac{a}{n-1}\{2(n-1)(b_1+b_4) + (n-2)(b_2+b_3)\} + (n-1)\sum_{i=1}^{6} b_i^2 + \left(\sum_{i=1}^{6}(-1)^{i-1}b_i^2\right)^2 - 2(b_1b_3 + b_2b_4 - b_5b_6)$$

form (4.22) and (4.22a), we have

$$e - b = \frac{na}{n-1}(b_2 + b_3)$$

Note that $\overset{*}{G}$, $\overset{*}{Z}$ and $\overset{*}{W}$ are formed with the help of non metric semi symmetric connection induced by ρ

Theorem 4.1:

If a compact Orientable Riemannian manifold *M* of dimension n ≥2 admits a non constant function ρ on *M* such that

(4.23) $$\nabla_j \nabla_i \rho = \frac{1}{n}\Delta\rho g_{ji}$$

then *M* is conformal to a sphere

Theorem 4.2:

For a compact orientable Riemannian manifold *M*, we have

(4.24) $$\int_M \Delta f dV = 0$$

where *f* is a smooth function on *M* and *dV* is the volume element of *M*.

Lemma 4.1:

Suppose *M* of dimension $n \geq 2$ is compact and admits a non-constant function ρ on *M*. If the tensor field with components A_{ji} is identically zero on *M*.*M* is conformal to a sphere

Lemma 4.2:

Suppose M of dimension $n \geq 2$ is complete, if

$$L_{Du}r = 0 \quad \text{and} \quad \nabla_j \nabla_i u = \frac{\Delta u}{n} g_{ji}$$

holds for a non-constant function u on M, then M is isometric to a sphere

Lemma 4.3:

Suppose M is compact and orientable Riemannian manifold, then

$$(4.25) \qquad \int_M G^{ji} \nabla_j u_i dV = -\frac{n-2}{2n} \int_M L_{Du} r dV$$

Where Du is the vector field on M associated with the closed 1-form du

5. Intergral Formulae and Theorems:

In this section, we prove our integral formulas, Lemmas and theorems. Now from (4.5) and (4.11), we have

$$u\,(\overset{*}{r} - r) = (n-1)\,\Delta\, u$$

multiplying by u on both sides, we get

$$u^2\,(\overset{*}{r} - r) = (n-1)\, u\,\Delta\, u$$

further, integrating over M

$$(5.1) \qquad \int_M u^2 (r^* - r) dV = (n-1) \int_M u \Delta u dV$$

Since $\Delta u^2 = 2u\, \Delta u + 2(\text{grad } u)^2$, integrating over M and using (4.4), we have

$$(5.2) \qquad \int_M u \Delta u dV = -\int_M (grad\ u)^2 dV$$

Thus in view of (5.2), Integral equation (5.1) becomes

$$(5.3) \qquad \int_M \left[u^2 (r^* - r) + (n-1)(grad\ u)^2 \right] dV = 0$$

Thus we have the following Lemma

Lemma 5.1:

For a compact Orientable smooth Riemannian manifold M, a smooth function ρ is constant in M if and only if $\overset{*}{r}$ is identically equal to the scalar curvature r of M.

Proof :

The proof follows from equation (5.3).

Lemma 5.2:

Suppose M is orientable and compact. Then the following integral formulas hold for M.

$$(5.4) \qquad \int_M [nu \left(\left| \overset{*}{G} \right|^2 - \left| G \right|^2 \right) + (n-1)(n-2) L_{Du} r - n(n-1)^2 u |A|^2] dV = 0$$

$$(5.5) \qquad \int_M [nu \left(\left| \overset{*}{W} \right|^2 - \left| w \right|^2 \right) + e(n-1)(n-2) L_{Du} r - bn(n-1)^2 u |A|^2] dV = 0$$

where b and e are given by (4.22) and (4.22a) respectively

Proof:

Since $g^{ji}\, G_{ji} = 0$ by virtue of (4.16) the equations (4.19) or (4.20) and (4.21) may be simplified to the following form

(5.6) $$nu\left(|\overset{*}{G}|^2-|G|^2\right)-(n-1)\,G_{ji}\nabla^j u^i-n(n-1)^2 u|A|^2=0$$

(5.7) $$nu\left(|\overset{*}{W}|^2-|W|^2\right)-e(n-1)\,G_{ji}\nabla^j u^i-bn(n-1)^2 u|A|^2=0$$

where b and c are given by (4.22) and (4.22a) respectively. Integrating (5.6) and (5.7) over the volume of M, and then using (4.25) of Lemma 4.3. We obtain the integral formulas (5.4) and (5.5). This completes the proof of Lemma 5.2.

Remarks 5.1:

If we substitute for a =1, $b_1 = b_2 = b_3 = b_4 = b_5 = b_6 = 0$ in (4.14) and (4.22), then

$$\overset{*}{W}_{kjih}=\overset{*}{Z}_{kjih} \text{ and } b=e=\frac{2}{n-1}$$

Thus, the integral formula (5.5) becomes

(5.8) $$\int_M\left[nu(|\overset{*}{Z}|^2-|Z|^2)+2(n-2)L_{Du}r-2n(n-1)u|A|^2\right]dV=0$$

Theorem 5.1:

Suppose M be compact and Orientable smooth Riemannian manifold of dimension $n \geq 2$. If ρ be a smooth function on M and $u = e^{-\rho}$, then

(5.9) $$\int_M\left[nu(|\overset{*}{G}|^2-|G|^2)+(n-1)(n-2)L_{Du}r\right]dV\geq 0$$

(5.10) $$\int_M\left[nu(|\overset{*}{W}|^2-|W|^2)+e(n-1)(n-2)L_{Du}r\right]dV\geq 0,(n>0)$$

where $\overset{*}{G}$ and $\overset{*}{W}$ formed with the help of non-metric semi-symmetric connection induced by ρ. If ρ is such that the equality in (5.9) or in (5.10) holds If and only if M is conformal to a sphere.

Proof:

If follows from Lemma 5.2, theorem 4.1 and Lemma 4.2.

6. Special Case:

(I) If a smooth non constant function ρ on M induces the projective change of the connection i.e. $\overset{\circ}{\nabla}$, then $\overset{\circ}{\nabla}$ is projectively related to ∇ by Yano [4] .

$$\overset{\circ}{\Gamma}{}^h_{ji}=\left\{\begin{matrix}h\\ ji\end{matrix}\right\}+\delta^h_i\rho_j+\delta^h_j\rho_i,$$

where $\rho^i = g^{ji}\rho_j$ and $\rho_j = \nabla_j\rho$ are the components of $d\rho$ If $\overset{\circ}{K}$ and K are the curvature tensors of $\overset{\circ}{\nabla}$ and ∇ respectively , then we have Yano [4].

$$\overset{\circ}{K}{}^h_{kji}=K^h_{kji}-\rho_{ji}\delta^h_k+\rho_{ki}\delta^h_j+(\rho_{kj}-\rho_{jk})\delta^h_i$$

where $\rho_{ji} = \nabla_j\rho_i - \rho_j\rho_i$. Since ρ_{jk} are symmetric with respect to the indices j and k, above equations reduces to

(6.1) $$\overset{\circ}{K}{}^h_{kji}=K^h_{kji}-\rho_{ji}\delta^h_k+\rho_{Ki}\delta^h_j$$

Suppose the smooth function on M which induces the projective change of the connection that is $\overset{\circ}{\nabla}$, also induces the non-metric semi-symmetric connection. $\overset{*}{\nabla}$,

then comparing the expressions of $\overset{*}{K}{}^h_{kji}$ and $\overset{\circ}{K}{}^h_{kji}$ of from (4.2) and (6.1), we have

$$\overset{\circ}{K}{}^h_{kji} = \overset{*}{K}{}^h_{kji} = \overset{\circ}{K}_{ji} = \overset{*}{K}_{ji} \text{ and } \overset{\circ}{r} = \overset{*}{r}$$

Thus we observe that $\rho_{ji} = C_{ji}$

$$\overset{\circ}{G} = \overset{*}{G},\ |\overset{\circ}{G}| = |\overset{*}{G}|,\ \overset{\circ}{W} = \overset{*}{W} \text{ and } |\overset{\circ}{W}| = |\overset{*}{W}|$$

where the definitions of $\overset{\circ}{G}$ and $\overset{\circ}{W}$ are similar to that of G and W respectively.

Finally, we have the following integral formulas

(6.2) $$\int_M \left[nu(|\overset{\circ}{G}|^2 - |G|^2) + (n-1)(n-2)L_{Du}r - n(n-1)^2 u|A|^2 \right] dV = 0$$

(6.3) $$\int_M \left[nu(|\overset{\circ}{W}|^2 - |W|^2) + e(n-1)(n-2)L_{Du}r - bn(n-1)^2 u|A|^2 \right] dV = 0$$

Hence from (6.2) and (6.3) respectively, we have

(6.4) $$\int_M \left[nu(|\overset{\circ}{G}|^2 - |G|^2) + (n-1)(n-2)L_{Du}r \right] dV \geq 0$$

(6.5) $$\int_M \left[nu(|\overset{\circ}{W}|^2 - |W|^2) + e(n-1)(n-2)L_{Du}r \right] dV \geq 0,\ (b > 0)$$

Equality in (6.4) or in (6.5) holds if and only if M is conformal to a sphere.

(II) Suppose ρ on M is such that

(6.6) $$K^h_{kji} = e^{-\rho}\left(\rho_{ji}\delta^h_k - \rho_{ki}\delta^h_j\right)$$

Then, from (4.2), (4.4) and (4.5), we find

$$\overset{*}{K}{}^h_{kji} = (1-u^{-1})K^h_{kji},\quad \overset{*}{K}_{ji} = (1-u^{-1})K_{ji} \text{ and } \overset{*}{r} = (1-u^{-1})r$$

therefore, we have

(6.7) $$\overset{*}{G}_{ji} = (1-u^{-1})G_{ji}$$

(6.8) $$\overset{*}{W}_{kjih} = (1-u^{-1})W_{kjih}$$

Using (6.7) and (6.8) in the integral formulas (6.2) and (6.3), we have

(6.9) $$\int_M \left[n(u^{-1}-2)|G|^2) + (n-1)(n-2)L_{Du}r \right] dV \geq 0$$

(6.10) $$\int_M \left[n(u^{-1}-2)|W|^2 + e(n-1)(n-2)L_{Du}r \right] dV \geq 0, \qquad (b > 0)$$

It follows from (6.9), (6.10), Lemma (5.2) and Lemma (4.1) that if ρ on M is such that the equality in (6.9) or in (6.10) holds If and only if M is conformal to a sphere.

If M is an Einstein manifold, then from (6.7) and (6.8). It is easy to see that

$$\overset{*}{G} = 0 \ \ or\ \ \overset{*}{W} = 0$$

in M, therefore M is isometric to a sphere.

(III) Let ρ be a smooth function on M arising from the conformal change of the metric given by

$$\overline{g} = e^{2\rho} g \ \ or \ \ \overline{g}_{ij} = e^{2\rho} g_{ji}$$

Then, the conformal change of the Riemannian connection ∇ that is $\overline{\nabla}$ induced by ρ with the christoffel symbols $\overline{\Gamma}_{ji}^{h}$ is given by Yano [4].

$$\overline{\Gamma}_{ji}^{h} = \begin{Bmatrix} h \\ ji \end{Bmatrix} + \delta_i^h \rho_j + \delta_j^h \rho_i - g_{ji} \rho^h$$

where $\rho^i = g^{ji} \rho_j$ and $\rho_j = \nabla_j \rho$. The curvature tensors $\overline{K}$ and K of $\overline{\nabla}$ and ∇ respectively are related by

$$\overline{K}_{kji}^{h} = K_{kji}^{h} - p_{ji}\delta_k^h + p_{ki}\delta_j^h - p_k^h g_{ji} + g_{ki} p_j^h \tag{6.11}$$

$$p_{ji} = \nabla_j \rho_i - \rho_j \rho_i + \frac{1}{2}(\rho^t \rho_t) g_{ji} \tag{6.12}$$

If the non constant function ρ on M also induces the non-metric semi-symmetric connection $\overset{*}{\nabla}$ given by (4.1), then(6.12) becomes

$$p_{ji} = C_{ji} + \frac{1}{2}(\rho^t \rho_t) g_{ji} \tag{6.13}$$

where C_{ji} is given by (4.3) and $p_k^h = p_{ki} g^{th}$. Thus substituting form p_{ji} from (6.13) into (6.11), we have

$$\overline{K}_{kji}^{h} = \overset{*}{K}_{kji}^{h} - \left\{ g_{ji} C_k^h - g_{ki} C_j^h - (\rho^t \rho_t)(\delta_k^h g_{ji} - \rho_j^h g_{ki}) \right\} \tag{6.14}$$

If on M is such that

$$\overset{*}{K}_{kji}^{h} = e^{-\rho} \left\{ g_{ji} C_k^h - g_{ki} C_j^h - (\rho^t \rho_t)(\delta_k^h g_{ji} - \rho_j^h g_{ki}) \right\} \tag{6.15}$$

then from (6.14)

$$\overset{*}{K}_{kji}^{h} = (1-u^{-1})^{-1} \, \overline{K}_{kji}^{h}, \ \overset{*}{K}_{ji} = (1-u^{-1})^{-1} \overline{K}_{ji} \ and \ \overset{*}{r} = (1-u^{-1})^{-1} u^{-2} \overline{r} \tag{6.16}$$

form the equations, it is easy to see that

$$\overset{*}{G}_{ji} = (1-u^{-1})^{-1} \left\{ \overline{G} - (1-u^{-1}) \frac{\overset{*}{r}}{n} g_{ji} \right\}$$

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