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Review Paper

A Study of Pseudo Riemannian Manifold

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ABSTRACT
In this chapter, we shall study minimal space-like submanifolds with constant curvature and derive some theorem for such submanifolds, we consider, the geodesic mappings onto projective birecurrent manifolds and also study the geodesic mappings on S-manifolds.

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1. Local Formulas:

In this section, we have to compute the laplacian of the second fundamental form of a minimal submanifold of a Pseudo-Riemannian manifold.

Let M be an n-dimensional Riemannian manifold immersed in an $(n+p)$ dimensional pseudo Riemannian manifold N. We choose a local field of Pseudo-Riemannian orthonormal frames $e_1,....,e_{n+p}$ in N such that, restricted to M, the vectors $e_1,....e_n$ are space like tangent to M, (and consequently, the remaining vectors e_{n+1}, $,e_{n+p}$ are time-like normal to M). We make use of the following convention on the ranges of indices:

$$1 \le A, B, C, D \le n+p;$$
$$1 \le i, j, k, l \le n;$$
$$n+1 \le \alpha, \beta, \gamma \le n+p$$

and we agree that repeated indices are summed over respective ranges. Let $W_1, ...W_2,....W_{n+p}$ be its dual frame field so that the Pseudo-Riemannian metric of N is given by

$$dS_N^2 = \sum W_1^2 - \sum W_\alpha^2 = \sum \varepsilon_A W_A^2$$

where ε_1 =1 for $1 \le i \le n$ and ε_α =-1 for $(n+1) \le \alpha \le n+p$. Then the structure equations of N are given by

(1.1) $dW_A = \Sigma \varepsilon_B W_{AB} \Lambda W_B,\ W_{AB} + W_{BA} = 0,$

(1.2) $dW_{AB} = \Sigma \varepsilon_c W_{Ac} \Lambda W_{cB} - \frac{1}{2}\Sigma \varepsilon_c \varepsilon_D k_{ABCD} W_c \Lambda W_D.$

we restrict these forms to M. Then

(1.3) $W_\alpha = 0$ for $n+1 \leq \alpha \leq n+p$

and the Riemannian metric of M is written as

$$dS^2_M = \sum_i W_i^2$$

we may put

(1.4) $$W_{i\alpha} = \sum h^{\alpha}_{ij} W_j$$

from these formula, we obtain

(1.5) $dW_i = \Sigma\, W_{ij} \Lambda W_j,$

(1.6) $dW_{ij} = \Sigma\, W_{ik} \Lambda W_{kj} - \frac{1}{2}\, \Sigma R_{ijkl}\, W_k \Lambda W_1,$

(1.7) $R_{ijkl} = K_{ijkl} - \sum \left(h^{\alpha}_{ik} h^{\alpha}_{jl} - h^{\alpha}_{il} h^{\alpha}_{jk}\right),$

$dW_\alpha = -\sum W_{\alpha\beta} \Lambda W_\beta,$

(1.8) $dW_{\alpha\beta} = -\Sigma\, W_{\alpha\gamma} \Lambda W_{\gamma\beta} - \frac{1}{2}\, \Sigma R_{\alpha\beta ij}\, W_i \Lambda W_j,$

(1.9) $R_{\alpha\beta ij} = \sum_k \left(h^{\alpha}_{ki} h^{\beta}_{kj} - h^{\alpha}_{kj} h^{\beta}_{ki}\right).$

The Riemannian connection of M is defined by (W^i_j). The form (W^{α}_{β}) defines as connection in the normal bundle of M. We call $\sum h^{\alpha}_{ij} W_i W_j e_\alpha$ the second fundamental form of the immersed manifold M. Sometimes, we denote the second fundamental form $\sum h^{\alpha} e$ by its components h^{α}_{ij}. We call $\sum_\alpha \left(\frac{1}{n} \sum_i h^{\alpha}_{ii}\right) e_\alpha$ the mean curvature normal or the mean curvature vector . An immersion is said to be minimal if its mean curvature normal vanishes.

identically i.e. if $\sum_i h^{\alpha}_{ii} = 0$ If for all α

Let h^{α}_{ijk} denote the covariant derivative of h^{α}_{ij} so that

(1.10) $\sum h^{\alpha}_{ijk} W_k = dh^{\alpha}_{ij} + \sum h^{\alpha}_{ik} W_{kj} + \sum h^{\alpha}_{kj} W_{ki} - \sum h^{\beta}_{ij} W^{\beta}_{\alpha}$

Then, we have $h^{\alpha}_{ijk} = h^{\alpha}_{ikj}$. Next we take the exterior, derivative of (1.10) and

define the second covariant derivative of h^{α}_{ij} by

(1.11) $\sum h^{\alpha}_{ijkl} W_l = dh^{\alpha}_{ijk} + \sum h^{\alpha}_{ijl} W_{lk} + \sum h^{\alpha}_{ilk} W_{lj} + \sum h^{\alpha}_{ljk} W_{li} - \sum h^{\alpha}_{ijk} W^{\beta}_{\alpha}$

then, we obtain the Ricci formula

(1.12) $h^{\alpha}_{ijkl} - h^{\alpha}_{ijlk} = \sum h^{\alpha}_{im} R_{mjkl} + \sum h^{\alpha}_{mj} R_{mikl} + \sum h^{\beta}_{ij} R_{\alpha\beta kl}$

The Laplacian Δh^{α}_{ij} of the second fundamental form h^{α}_{ij} is defined by

(1.13) $\Delta h^{\alpha}_{ij} = \sum_k h^{\alpha}_{ijkk}$

Using the same method as discussed [3], we have

(1.14) $\Delta h^{\alpha}_{ij} = \sum_k h^{\alpha}_{kkij} + \sum_{mk} h^{\alpha}_{im} R_{mkjk} + \sum_{mk} h^{\alpha}_{mk} R_{mijk} + \sum_{\alpha\beta k} h^{\beta}_{ik} R_{\alpha\beta jk}$

Now, we assume that M is minimal in N so that $\sum h^{\beta}_{kk} = 0$ for all β, then, from (1.14) we obtain

(1.15) $$\sum h^{\alpha}_{ij} \Delta h^{\alpha}_{ij} = \sum (h^{\alpha}_{ij})^2 K_{jkjk} + \sum h^{\alpha}_{ij} h^{\alpha}_{mk} h^{\beta}_{mk} h^{\beta}_{ij} + \sum h^{\alpha}_{ij} h^{\alpha}_{im} h^{\beta}_{mk} h^{\beta}_{kj} + \sum h^{\alpha}_{ij} h^{\beta}_{mj} h^{\beta}_{ik} h^{\beta}_{mk} - 2 \sum h^{\alpha}_{ij} h^{\alpha}_{mk} h^{\beta}_{mj} h^{\beta}_{ik}$$

2. Minimal Submanifolds of a Pseudo Riemannian Manifold of Constant Curvature:

Throughout this section, we shall assume that the ambient space N is a space of constant curvature c, then

$$K_{ABCD} = c\, (\delta_{AC}\, \delta_{BD} - \delta_{AD} \delta_{BC})$$

Hence (1.15) reduces to

(2.1) $$\sum h_{ij}^{\alpha}\Delta h_{ij}^{\alpha} = nc\sum \left(h_{ij}^{\alpha}\right)^2 + \sum h_{ij}^{\alpha} h_{mk}^{\alpha} h_{mk}^{\beta} h_{ij}^{\beta} + \sum h_{ij}^{\alpha} h_{im}^{\alpha} h_{mk}^{\beta} h_{kj}^{\beta} + \sum h_{ij}^{\alpha} h_{mj}^{\alpha} h_{ik}^{\beta} h_{mk}^{\beta} - 2\sum h_{ij}^{\alpha} h_{mk}^{\alpha} h_{mj}^{\beta} h_{ik}^{\beta}$$

For each α, let H_α denote the symmetric matrix (h_{ij}^{α}). We denote the square of the length of the second fundamental form by S, i.e.

(2.2) $$S = \sum \left(h_{ij}^{\alpha}\right)^2$$

Now (2.1) may be written as follows

(2.3) $$\sum h_{ij}^{\alpha}\Delta h_{ij}^{\alpha} = ncS + \sum_{\alpha\beta} tr\,(H_\alpha H_\beta - H_\beta H_\alpha)^2 + \sum_{\alpha\beta}(tr H_\alpha H_\beta)^2$$

We derive some theorems with analytical Proofs, as stated below:

Theorem 2.1:

Let H_i $(i \geq 2)$ be symmetric $(n\times n)$ matrices, S. Chern [4],

$$S_i = trH_i^2 \quad and \quad S = \sum_i S_i$$

then

(2.4) $$\sum_{i,j} tr(H_i H_j - H_j H_i)^2 - \sum_{i,j}(trH_i H_j)^2 \geq \frac{-3}{2}S^2$$

and the equality holds if and only if all H_i =0 or there exists two of H_i different from zero. More over, if $H_1 \neq 0$, $H_2 \neq 0$, $H_i = 0$ $(i\neq 1,2)$, Then $S_1 = S_2$ and there exists an orthogonal $(n\times n)$ –matrix.

Theorem 2.2:

If M is a minimal submanifold, then

(2.5) $$\sum_{\alpha,\beta} tr(H_\alpha H_\beta - H_\beta H_\alpha)^2 + \sum_{\alpha\beta}(TrH_\alpha H_\beta)^2 \geq \frac{1}{2}S^2$$

Proof:

Using (2.4), we have

(2.6) $$\sum_{\alpha\beta} tr(H_\alpha H_\beta - H_\beta H_\alpha)^2 + \sum_{\alpha\beta}(trH_\alpha H_\beta)^2 \geq \frac{-3}{2}S^2 + 2\sum(trH_\alpha H_\beta)^2$$

If M is a minimal submanifold, then using the same method as described in [4], we have (2.5).

Now,

(2.7) $$\sum h_{ij}^{\alpha}\Delta h_{ij}^{\alpha} \geq ncS + \frac{1}{2}S^2 \geq \left(\frac{1}{2}S + nc\right)S$$

Theorem 2.3:

If M be an n-dimensional compact oriented Riemannian manifold which is minimally immersed in an $(n+p)$ –dimensional Pseudo-Riemannian space N, then

(2.8) $$\int_M \left(\sum h_{ij}^{\alpha}\Delta h_{ij}^{\alpha}\right)dv = -\int_M \sum \left(h_{ijk}^{\alpha}\right)^2 dv \leq 0$$

Proof:

We have

(2.9) $$\frac{1}{2}\Delta S = \sum \left(h_{ijk}^{\alpha}\right)^2 + \sum h_{ij}^{\alpha}\Delta h_{ij}^{\alpha}$$

Integrating (2.9) over M and appling minimally Green's theorem to the left hand side, we observe that the integral of the left hand side vanishes and hence that of the right hand side also vanishes.

3. Symmetry Conditions for Riemannian or Pseudo Riemannian Manifolds :

Let (M^n,g) be a n-dimensional Riemannian or Pseudo –Riemannian manifold.

Def.1 : M^n is called local symmetric, when $R^P_{ijk,l} = 0$.

Def. 2: M^n is called recurrent [7], when $R^P_{ijk,l} = a_l R^P_{ijk}, R^P_{ijk} \neq 0$.

Def..3 : M^n is called birecurrent [3], when $R^P_{ijk,lm} = a_{lm} R^P_{ijk}, R^P_{ijk} \neq 0$.

Def. 4: M^n is called Projective symmetric [9],Projective recurrent [2], Projective birecurrent, when the projective Weyl-tensor has the correspondent properties.

Theorem 1:

The birecurrence factor of a Riemmanian projective birecurrent manifold is a symmetric tensor.

Proof:

We Introduce the Walker's Lemmas, [10].

Lemma-1:

The curvature tensor of a Riemannian manifold (M^n,g) satisfies the identity.

$$R_{hijk,lm} - R_{hijk,ml} + R_{jklm,hi} - R_{jklm,ih} + R_{lmhi,jk} - R_{lmhi,kj} = 0.$$

Lemma-2:

If a_{ij} , b_k are numbers satisfying $a_{ij} = a_{ji}$, $a_{ij}b_k + a_{jk}\, b_i + a_{ki}\, b_j = 0$ for $i,j, k = 1, \ldots, n$, then all the a_{ij} are zero or all b_k are zero

We now introduce the tensor B, defined by $B_{hijk} = W_{hijk} + g_{hj}\, W_{ik} - g_{hk}\, W_{ij}$ where $W_{hj} = g^{ik} W_{hijk}/n = R_{hj} - Rg_{hj}$, therefore, we have

(a) $B_{hijk} = R_{hijk} - R\,(g_{hj}\, g_{ik} - g_{hk}\, g_{ij})$,

(b) $B_{hijk,lm} = a_{lm}\, B_{hijk}$,

Since $W_{hijk,lm} = a_{lm}\, W_{hijk}$ entails $W_{hj,lm} = a_{lm}\, W_{hj}$. After (a) the tensor B has the same algebraic properties as the curvature tensor, therefore (b) and Lemma 2 imply $a_{lm} = a_{ml}$ or $B_{hijk} = 0$. In the second case, M^n is a manifold of constant curvature an consequently $W_{hijk} = 0$

Theorem 2:

For a Riemannian or Pseudo-Riemannian manifold (M^n,g) the following equations are equivalent.

(1) $$R^P_{ijk,lm} = R^P_{ijk,ml} \quad \text{and} \quad W^P_{ijk,lm} = W^P_{ijk,ml},$$

Proof:

"$\to$" It is a direct consequence of the definition of w.

"$\leftarrow$" contracting with p and j in $W^P_{ijk,lm} = W^P_{ijk,ml}$

We get $R_{ij,lm} = R_{ij,ml}$ The definition of w gives the proof.

Definition 5:

We shall call a Riemannian or Pseudo-Riemannian manifold (M^n,g) satisfying (1) an S-manifold. The following implications hold

Local symmetric $\to$ Recurrent $\longrightarrow$ birecurrent ↘
S-Manifold
Projective symmetric $\to$ Projective Recurrent $\to$ Projective birecurrent ↗

4. Geodesic Mapping onto Riemannian and Pseudo Riemannian Manifolds with Symmetry Conditions

In 1954 Sijukov [8] published the theorem. There does not exist non trivial goodies mapping which takes a (M^n,g) onto a $(\overline{M}^n,\overline{g})$ which is local symmetric but not of constant curvature.

Now, we shall now study the geodesic mapping onto S-manifolds .

Theorem 3:

If it is possible to map geodesically a Riemannian or Pseudo Riemannian manifold (M^n,g) onto an S-manifold $(\overline{M}^n,\overline{g})$, then both manifolds are of constant curvature or $\lambda_{ij} = \Delta g_{ij}$ with Δ= constant.

Proof:

From the Ricci identities and from (1), we infer that

$$\overline{R}^s_{ilm}\ \overline{R}^p_{sjk} + \overline{R}^s_{jlm}\ \overline{R}^p_{isk} - \overline{R}^s_{klm}\ \overline{R}^p_{isj} - \overline{R}^p_{slm}\ \overline{R}^s_{ijk} = 0,$$

Using

$\overline{R}^p_{ijk} = R^p_{ijk} - \delta^p_j\lambda_{ik} + \delta^p_k\lambda_{ij}$, we replace $\overline{R}$ by R;

(1)
$$R^p_{ijk,lm} - R^p_{ijk,ml} = \lambda_{im}R^p_{ljk} + \lambda_{jm}R^p_{ilk} + \lambda_{km}R^p_{ijl}$$
$$-\lambda_{il}R^p_{mjk} - \lambda_{jl}R^p_{imk} - \lambda_{kl}R^p_{ijm} + \delta^p_m\lambda_{sl}R^s_{ijk}$$
$$-\delta^p_l\lambda_{sm}R^s_{ijk} + \delta^p_i\lambda_{sk}R^s_{ibm} + \delta^p_j\lambda_{si}R^s_{klm}$$
$$-\delta^p_k\lambda_{sj}R^s_{ilm} - \delta^p_k\lambda_{si}R^s_{jlm}$$

Let A be the tensor defined by $A_{hijl} = R_{hj}\lambda_{il} - g_{hj}.\lambda_{si}R^s_l$ multiplying (1) by $g_{hp}\,g^{km}$ we obtain the following representation for A:

$$A_{hijl} = C_{hijl} + D_{hijl},\ \text{where}\ C_{hijl} = -C_{ihjl}\ \text{and}\ D_{hijl} = -D_{hilj}$$

from the above result, we infer that the tensor B defined by $B_{hijl} = A_{hijl} + A_{ihlj}$ is skew symmetric in h,i and j, l.

Transvesting $B_{hijl} + B_{hilj}$=0 with g^{il} we get

(2)
$$(n+1)\,\lambda_{hs}R^s_j - \lambda_{sj}R^s_h = n\,\Delta R_{hj} + nR\lambda_{hj} - g_{hj}\lambda^u_s R^s_u,$$

where $\Delta = g^{st}\lambda_{st}/n$ therefore $\lambda_{hs}R^s_j = \lambda_{js}R^s_h$, consequently

$$B_{hijl} = B_{jihl},\ B_{hijl} = B_{hlji}$$

These algebraic properties imply

(3)
$$B_{hijl}=0$$

Contracting (2) with g^{hj} we have $\lambda^u_s R^s_u = nR\Delta$ (2) entails $\lambda_{hs}R^s_j = \Delta\,(R_{hj} - Rg_{hj}) + R\lambda_{hj}$ .Replacing this expression in (3), we get

$$(R_{hj} - Rg_{hj})\,(\lambda_{il} - \Delta g_{il}) + (R_{il} - Rg_{il})\,(\lambda_{hj} - \Delta g_{hj}) = 0$$

The last equations mean that $R_{ij} = Rg_{ij}$, i.e. (M^n,g) is an Einstein manifold or $\lambda_{ij} = \Delta g_{ij}$.

(I) $R_{ij} = Rg_{ij}$ from the Ricci identities and from theorem (2) we have

$$\overline{R}^s_{ilm}W^p_{sjk} + \overline{R}^s_{jlm}W^p_{isk} - \overline{R}^s_{klm}W^p_{isj} - \overline{R}^p_{slm}W^s_{ijk} = 0$$

Replacing $\overline{R}$ by R, lowering the index p we get

$$W_{hijk,lm} - W_{hijk,ml} = \lambda_{im}W_{hljk} + \lambda_{jm}W_{hilk} + \lambda_{km}W_{hijl} - \lambda_{il}W_{hmjk}$$
$$-\lambda_{jl}W_{himk} - \lambda_{kl}W_{hijm} + (g_{mh}\lambda_{st} - g_{lh}\lambda_{sm})W^s_{ijk}$$

Since (M^n,g) is Einstein, we have from $W_{hijk} + W_{ihjk} = 0 \leftrightarrow R_{hj} = Rg_{hj}$ that W is skew symmetric in the first two indices . Therefore, it follows.

(4) $$\lambda_{im} W_{hljk} - \lambda_{il} W_{hmjk} + (g_{mh}\lambda_{sl} - g_{lh}\lambda_{Sm}) W^s_{ijk} + \lambda_{hm} W_{iljk} - \lambda_{hl} W_{imjk} + (g_{mi}\lambda_{sl} - g_{li}\lambda_{sm}) W^s_{hjk} = 0.$$

Counteracting (4) with g^{im} we have $n\Delta\ W_{hljk} + (n+1)\ \lambda_{sl}\ W^S{}_{hjk} + \lambda_{sh}\ W^S{}_{ljk}=0$. The skew –symmetry implies $\lambda_{sl} W^s_{hjk} + \lambda_{sh} W^s_{ljk} = 0.$

Consequently, $\lambda_{Sl} W^S_{hjk} = -\Delta W_{hljk}$ .Replacing this relation in (4), we get

$$(\lambda_{im} - \Delta g_{im})\ W_{hljk} - (\lambda_{hl} - \Delta g_{hl})\ W_{imjk} = (\lambda_{il} - \Delta g_{il})\ W_{hmjk} + (\lambda_{hm} - \Delta g_{hm})\ W_{lijk}$$

Exchange the indices k with l and i with m, we note that

$$(\lambda_{im} - \Delta g_{im})\ W_{hljk} = (\lambda_{hl} - \Delta g_{hl})\ W_{imjk}$$

Using again the skew symmetry we get $(\lambda_{im} - \Delta g_{im})\ W_{hljk} = 0$ consequently $\lambda_{im} = \Delta g_{im}$ or $W_{hljk} = 0$ i.e. (M^n,g) and $(\overline{M}^n, \overline{g})$ are of constant curvature.

(II) $\lambda_{ij} = \Delta g_{ij}$, at first we study the general case

Let (M^n,g) and $(\overline{M}^n, \overline{g})$ be geodetically equivalent Riemannian or Pseudo-Riemannian manifolds such that $\lambda_{ij} = \Delta g_{ij}$ by definition, we have

$$\lambda_{i,j} = \lambda_{i,j} - \lambda_i \lambda_j,$$

Putting $\mu := \overline{e}^{\lambda}$ we get $\mu_{i,j} + \mu\Delta g_{ij} = 0$. The Ricci identities imply $\mu_s R^s_{ijk} + g_{ij}\mu_s R^s_k - g_{ik}\mu_s R^s_j = 0$, the last expression is equivalent to $\lambda^t\ W_{itjk}=0$ where $\lambda^t = g^{ts}\lambda_s$,

Theorem 4:

A geodesic mapping form (M^n,g) onto $(\overline{M}^n, \overline{g})$ with $\lambda_{ij} = \Delta g_{ij}$ implies that $\lambda^t W_{itjk}=0$.

As a consequence, we now prove that when $(\overline{M}^n, \overline{g})$ is an S-manifold and $\lambda_{ij} = \Delta g_{ij}$ then Δ is a constant.

From theorem 4, we have $\lambda^t W_{itjk} = 0$. This is equivalent to

(5) $$(\lambda_S g^{ts})\overline{W}_{itjk} = 0$$

$$\lambda_{i|j} = \lambda_{i,j} - 2\lambda_i\lambda_j = \Delta g_{ij} - \lambda_i\lambda_j, g^{ts}_{|l} = 2\lambda_l g^{ts} + \delta^t_l \lambda_u g^{us} + \delta^s_l \lambda_u g^{ut},$$

where "|" denotes the covariant differentiation with respect to the Riemannain connection of $\overline{g}$ in $\overline{M}^n$ .Therefore, the covariant differentiation, with respect to $\overline{g}$, of (5) gives

(6) $$(\lambda_u \lambda_S g^{us} + \Delta)\overline{W}_{iljk} + g^{ts}\lambda_s \overline{W}_{itjk|l} = 0,$$

The covariant differentiation of (6) gives

(7) $$\Delta m \overline{W}_{iljk} = -(\lambda_u \lambda_S g^{us} + \Delta)(\overline{W}_{iljk|m} + \overline{W}_{imjk|l}) - g^{ts}\lambda_s \overline{W}_{itjk|lm}$$

The right side of (7) is symmetric in l and m, therefore $\Delta m \overline{W}_{iljk} = \nabla_l \overline{W}_{imjk}$ and consequently

(8) $$\Delta_m W_{iljk} = \Delta_l W_{imjk}$$

Let now p be a point of M^n. We assume that Δ is not constant. Then we can find, by a linear transformation in M^n_p, a basis $E_1, \ldots, E_n$ in M^n_p, with $E_{1=}$ (f. grad $\Delta)|p. f(p) \neq 0$, consequently (8) implies

(9) $$W_{iljk} = 0 \text{ for all } l \neq 1$$

Now $g^{lk} W_{iljk} = n\,(R_{ij} - Rg_{ij})$

On the other hand (9) implies $g^{lk} W_{iljk} = g^{1k} R_{i1jk} - g_{ij} R^1_1 + \delta^1_i \cdot R_{1j}$ By definition we have $R_{i1jk} = g_{1k} R_{ij} - g_{1j} R_{ik} - W_{1ijk}$ Therefore

(10) $$n(R_{ij} - Rg_{ij}) = R_{ij} - g_{1j} R^1_i - g_{ij} R^1_1 + \delta^1_i R_{1j} - g^{1K} W_{1ijk}$$

Contracting (10) with g^{ij} and using (9) we get $R^1_1 = R$ Then equation (10) becomes $(n-1)(R_{ij} - Rg_{ij}) = -g_{ij} R^1_i + \delta^1_i R_{1j} - g^{1k} W_{1ijk}$. This last expression is equivalent to

$$(n-1)\,(R^p_i - R\delta^p_i) = -\delta^p_1 R^1_i + \delta^1_i R^p_1 - g^{pj} g^{1k} W_{1ijk}$$

for $i \neq 1, p \neq 1$ one has $R^p_i = R\delta^p_i$. for $i \neq 1$, $p = 1$ one has $R^1_i = 0$. for $i = 1$, $p \neq 1$ one has $R^p_1 = 0$

Hence (M^n, g) is an Einstein manifold. In this case, the algebraic properties of W, the expression (9) and theorem of Bellrami imply that both manifolds are of constant curvature. Consequently, from $\overline{R}_{ij} = R_{ij} - \lambda_{ij}$, we have $\overline{R}\overline{g}_{ij}(R - \Delta)g_{ij}$. This means that $\overline{R} = 0$ and $R = \Delta$ = constant or that $(R - \Delta)/\overline{R}$ =constant. i.e. Δ = constant, since a geodesic and conformal mapping is a homothety. This contradicts our assumption $\Delta \neq$ constant.

We now consider two special kinds of S-manifolds, the recurrent and the projective recurrent manifolds.

Theorem-5:

There does not exist a non-trivial geodesic mapping which takes a Riemannian or Pseudo-Riemannian manifold (M^n, g) onto an $(\overline{M}_n, \overline{g})$ which is recurrent but of constant curvature.

Proof:

By theorem –3 it is sufficient to study the case $\lambda_{ij} = \Delta g_{ij}$. We introduce the tensor B, defined by $B^p_{ijk} = R^p_{ijk} - \Delta(\delta^p_j g_{ik} - \delta^p_k g_{ij})$ We have $B^p_{ijk} = R^p_{ijk}, g_{hp} B^p_{ijk} = B_{hijk} = -B_{ihjk}$,

(11) $$B^p_{ijk\,|l} = a_l B^p_{ijk},$$

Where a_l is the recurrence vector of $(\overline{M}^n, \overline{g})$. From $\overline{\Gamma}^k_{ij} = \Gamma^k_{ij} + \delta^k_i \lambda_j + \delta^k_j \lambda_i$, and (11), we have.

$$a_l B_{hijk} = B_{hijk,l} + g_{lh} \lambda_s B^S_{ijk} - 2\lambda_l B_{hijk} - \lambda_i B_{hljk} - \lambda_j B_{hilk} - \lambda_k B_{hijl}$$

from the skew symmetry we get $\lambda_s B^S_{ijk} = 0$. consequently $\lambda_i B_{hljk} + \lambda_h B_{iljk} = 0$ therefore $B = 0$ i.e. $(\overline{M}^n, \overline{g})$ is local Euclidean and (M^n, g) is a manifold of constant curvature, or the mapping is trivial.

Theorem-6:
There does not exist a non-trivial geodesic mapping which takes a Riemannian or Pseudo Riemannian on manifold (M^n, g) onto an $(\overline{M}^n, \overline{g})$ which is projective recurrent but not of constant curvature.

Proof:

By the theorem of Matusmoto and the theorems 3 and 4, we have only to study the case, where $\lambda_{ij} = \Delta g_{ij}$ and $(\overline{M}^n, \overline{g})$ is an Einstein manifold

The inerrability condition of $\lambda_{ij} = \Delta g_{ij}$, Δ = constant, may be written as $\lambda_s \overline{R}^s_{ijk} = 0$. Contracting with $\overline{g}^{ik}$ we get $\lambda_j \overline{R} = 0$. consequently the mapping is trivial or $\overline{R} = 0$.

In the second case $(\overline{M}^n, \overline{g})$ is a special Einstein manifold, i.e $\overline{R}_{ij} = 0$. for such a manifold we have $\overline{W}^p_{ijk} = \overline{R}^p_{ijk}$ and therefore $\overline{R}^p_{ijk|l} = a_l \overline{R}^p_{ijk}$, i.e $(\overline{M}^n, \overline{g})$ is a recurrent manifold.

Theorem 5 completes the proof

5. The Jacobi Identity:

Let X, y, and Z be three arbitrary elements of X (M). We prove that in the case of a non-symmetric metric connection the Jacobi identity is given by

(5.1) $\quad S\{[X,[y,Z]]\} = S\{R(X,y)Z - \nabla_{T(X,y)}Z\}$

According to the definition of the bracket operation, with respect to ∇, on X (M) we obtain

(5.2) $\quad S\{[X,[y,Z]]\} = S\{\nabla_X\nabla_y Z - \nabla_y\nabla_X Z - \nabla_{[X,y]}Z\}$

Taking account of the equation

$$R(X,y)Z = \nabla_X\nabla_y Z - \nabla_y\nabla_X Z - \nabla_{[X,y]}Z + \nabla_{T(X,y)}Z.$$

we get

(5.3) $\quad S\{\nabla_X\nabla_y Z - \nabla_y\nabla_X Z - \nabla_{[X,y]}Z\} = S\{R(X,y)Z - \nabla_{T(X,y)}Z\}$

Hence, from (5.2) and (5.3) the result follows. The formula (5.1) of the Jacobi identity, with the help of the Bianchi identity takes the form.

(5.4) $\quad S\{[X,[y,Z]\} = S\{T(T(X,y),Z) + (\nabla_X T)(y,Z) - \nabla_{T(X,y)}Z\}$

If the connection ∇ is a π-symmetric metric connection ∇^*.

(5.5) $\quad S\{|X,[y,Z]^*|^*\} = S\{X(\nabla^*\pi(Z,y) - \nabla^*\pi(y,Z) - \nabla^*_{\pi(y)X - \pi(X)y} Z\}$

At last, in the case of the special π-semi symmetric connection, we get

(5.6) $\quad S\{[X,[y,Z]^{**}]^{**}\} = S\{\nabla^{**}_{p(X)y - p(y)X} Z\}$

6. Pseudo Lie Algebras:

Let □,be a vector space over a filed k. The set □ will be called a pseudo Lie algebra over K. If there is given an internal product in □, which is called the bracket operation, which is k-bilinear, skew symmetric and the internal product in y satisfies the generalized Jacobi identify (5.1).

It is known that the set X (M) of all C^∞ vector fields, which are defined on the C^∞ Riemannian manifold M is a vector space over R. The vector space X (M), in addition to its vector space structure possesses a bracket operation, i.e., a map X (M) $\times$ X (M) $\rightarrow$ X (M) taking the pair (X,y) to the element $[X,y]$ of X (M) ,which has the following properties;

(i) R-bilinearity $\quad [aX+by,Z] = a[X,z] + b[y,Z]$

$[Z, aX+by] = a[Z,X]+b[Z,y],$

(ii) Skew symmetry $[X,y] = -[y,X],$

(iii) Jacobi identity $S \{[X,[y,Z]]\} = S \{R(X,y)Z - \nabla_{T(X,y)}Z\}$

for all $X,y,Z, \in X(M)$ and $a,b \in R$, (R the field of real numbers) Hence $X(M)$ is a Pseudo-Lie algebra over R, with respect to the bracket operation $[X,y]$.

7. Geodesic Mappings onto Projective Bircurrent Manifolds:

Definition:

A Riemannian or Pseudo-Riemannian manifolds (M^n,g) $n \geq 3$, is called M Projective bircurrent if $W^P_{ijk,lm} = a_{lm} W^P_{ijk}$. If a_{lm} is not the zero tensors on (M^n,g) we call(M^n, g) a strictly projective bircurrent manifold. We already know, that a_{lm} are the components of a symmetric tensor.

Theorem-1:

An Einstein projective birecurrent manifold (M^n,g) reduces to a manifold of constant curvature or a_{lm} satisfies $a^{lm} a_{lm} = 0$ where $a^{lm} = g^{tl} g^{Sm} a_{ts}$.

Proof:

It is easy to prove that the projective Weyl tensor satisfies the 2, Bianchi's identity If and only if $R_{ij,k} = R_{ik,j}$. This is the case for an Einstein manifold consequently, we have

(1) $$a_{lm} W_{hijk} + a_{jm} W_{hikl} + a_{km} W_{hilj} = 0$$

contracting this equation with g^{hl}, we get

(2) $$a_{Sm} W^S_{ijk} = 0.$$

Since (M^n,g) is Einstein, W satisfies the same algebraic Properties of the curvature tensor, consequently (2) implies .

(3) $$a^{Sm} W_{sijk} = - a^{Sm} W_{jkis} = 0$$

Contracting (1) by a^{lm} we get by (3) $(a^{lm} a_{lm}) w_{hijk} = 0$ the theorem of Beltrami completes the proof.

Theorem-2:

In The Riemannian case there is no strictly projective birecurrent Einstein manifold

Proof:

In the Riemannian case $a^{lm} a_{lm} = 0$ entails $a_{lm} = 0$.

Geodesic mapping on projective bircurrent manifolds. Let $(\overline{M}^n, \overline{g})$ be a projective birecurrent Riemannian or Pseudo Riemannian manifold and (M^n,g) a Riemannian or Pseudo Riemannian manifold geodescially equivalent to $(\overline{M}^n, \overline{g})$, $n \geq 3$. We have that both manifolds are of constant curvature or that $\lambda_{ij} = \Delta g_{ij}$ with Δ constant. In the non trivial case $\lambda_{ij} = \Delta g_{ij}$, we get

(4) $$\lambda^t \overline{W}_{itjk} = 0, \qquad \text{where } \lambda^t = g^{ts} \lambda_s$$

The covariant differentiation of (4) with respect $\overline{g}$, to using the fact that Δ = constant, implies.

$$(\lambda_u \lambda_s g^{us} + \Delta).(\overline{W}_{iljk|m} + \overline{W}_{imjk|l}) - g^{ts} \lambda_s \overline{W}_{itjk|lm} = 0,$$

By assumption, using (4), we get

(5) $$(\lambda_u \lambda_s g^{us} + \Delta).(\overline{W}_{iljk|m} + \overline{W}_{imjk|l}) = 0.$$

Consequently we have the two cases:

Case-I $\lambda_u \lambda_s g^{us} + \Delta = 0,$

Case-II $\overline{W}_{iljk|m} + \overline{W}_{imjk|l} = 0.$

Case-I : $\lambda_u\lambda_s g^{us}+\Delta=0$, since by definition $\lambda_{i,j} = \lambda_{ij} + \lambda_i\lambda_j = \Delta g_{ij}+\lambda_i\lambda_j$ we get $\lambda^i_{,i} = n\Delta + \lambda^i\lambda_i$, Therefore in this case we have

(5) $\lambda^i_{,i} = (n-1)\Delta$

Note: In the Riemannian case $\lambda_u\lambda_s g^{us} + \Delta=0$ implies that the mapping is affine or that $\Delta < 0$.

Case II: $\overline{W}_{iljk|m} + \overline{W}_{imjk|l} = 0$, since $\overline{W}_{imjk} + \overline{W}_{ijkm} + \overline{W}_{ikmj} = 0$

We infer $\overline{W}_{iljk|m} + \overline{W}_{ilkm|j} + \overline{W}_{ilmj|k} = 0$ i.e. $\overline{W}$ satisfies the Bianchi identity contracting this identify with $\overline{g}^{im}$ we have $\overline{W}^s_{ljk|s} = 0$, so that, by the definition of $\overline{W}$,we get

(6) $\overline{R}_{lj|k} = \overline{R}_{lk|j}$

This equation entails $\overline{R} =$ constant, where $\overline{R}$ denotes the scalar curvature of $(\overline{M}^n, \overline{g})$ defined by $\overline{R} = \overline{g}^{st}\overline{R}_{st} / n$. denoting with $\overline{W}_{ij}$ the components of the tensor defined by $\overline{W}_{ij} = (1/n)\overline{g}^{st}\overline{W}_{istj} = \overline{R}_{ij} - \overline{R}\overline{g}_{ij}$, we have by the assumption that $(\overline{M}^n, \overline{g})$ is projective birecurrent : $\overline{W}_{ij|lm} = a_{lm}\overline{W}_{ij}$. Applying the fact that $\overline{R} =$ constant, we get

(7) $\overline{R}_{ij|lm} = a_{lm}\overline{R}_{ij} - a_{lm}\overline{R}\overline{g}_{ij}$

contracting (7) with $\overline{g}^{il}$ we have $0 = \frac{1}{2}n\overline{R}_{|jm} = a_{Sm}\overline{R}^S_i - a_{im}\overline{R}$, which is equivalent to

(8) $a^{sm}\overline{R}_{sj} - a^m_j\overline{R} = 0$, where $a^m_j := \overline{g}^{mt}a_{atj}, a^{ms} := \overline{g}^{mt}a^s_t$

On the other hand equations (8) and (7) imply $a_{lm}\overline{R}_{ij} - a_{lm}\overline{R}\overline{g}_{ij} = a_{jm}\overline{R}_{il} - a_{jm}\overline{R}\overline{g}_{il}$. Contracting this equation with a^{lm} , we have

(9) $(a^{lm}a_{lm})(\overline{R}_{ij} - \overline{R}\overline{g}_{ij}) = 0$, Therefore $a^{lm}a_{lm} = 0$ or

$\overline{R}_{ij} = \overline{R}\overline{g}_{ij}$,i.e. $(\overline{M}^n, \overline{g})$ is an Einstein manifold. Applying theorem-1,we infer that the condition $\overline{W}_{iljk|m} + \overline{W}_{imjk|l} = 0$ implies $a^{lm}a_{lm} = 0$.

Theorem-3:

Let (M^n,g) and $(\overline{M}^n, \overline{g})$ be Riemannian or Pseudo Riemannian geodesically equivalent manifolds, with $(\overline{M}^n, \overline{g})$ Projective birecurrent with birecurrence factor a_{lm}. Then from equations (1), we observe that both manifolds are of constant curvature.

(2) $\lambda_{ij} = \Delta g_{ij}$, Δ =constant and $\lambda^S\lambda_S + \Delta = 0$ or

(3) $\lambda_{ij} = \Delta g_{ij}$, Δ= constant and $a^{lm}a_{lm} = 0$

Remarks:

(1) If $(\overline{M}^n, \overline{g})$ be a strictly projective birecurrent Riemmanian manifold then (M^n,g) and $(\overline{M}^n, \overline{g})$ are of constant curvature or $\lambda_{ij} = \Delta g_{ij}$, Δ=constant and $\lambda^S\lambda_S + \Delta = 0$,

(2) If (M^n,g) and $(\overline{M}^n,\overline{g})$ be the compact Riemannian manifolds, then we have

(a) $\overline{W}^p_{ijk|lm} = 0$, using the lemma of Bochner [1] implies, $\overline{W}^p_{ijk|l} = 0$, both manifolds are of constant curvature or the mapping is affine

(b) $\lambda^S\lambda_S+\Delta = 0$ and $\lambda_{ij} = \Delta g_{ij}$ imply $\lambda^i{}_{,i} = (n-1)\,\Delta$, Δ=Constant, this condition, this condition by the lemma of Bochner [1], implies λ = constant i.e. the mapping is affine.

8. Geodesic Mappings of Riemannian or Pseudo-Riemannian Manifolds Satisfying $R^S_{i,Sl} = R^S_{i,ls}$ Onto S-Manifold:

In this section, we consider the following situation (M^n,g) = n-dim. Riemannian or Pseudo-Riemannian manifold which satisfies the below-mentioned equation:

(1)
$$R^S_{i,Sl} = R^S_{i,ls}$$

$(\overline{M}^n,\overline{g})$ = n-dim S-manifold $n\geq 3.(M^n,g)$ and $(\overline{M}^n,\overline{g})$ are geodesically equivalent.

We may restrict, our study on the case $\lambda_{ij} = \Delta g_{ij}$, Δ =constant from

$$(n-1)\,\Delta\,(g_{il}R_{hj} - g_{hj}R_{il}) = g^{km}\,(R_{hijk,ml} - R_{hijk,lm}) + \Delta(n-1)\,R_{hijl}$$

Contracting this equation with g^{hj} we get

$$(n-1)\,\Delta\,(nRg_{il} - nR_{il}) = (n-1)g^{km}\,(R_{ik,ml} - R_{ik,lm}) + (n-1)^2\,\Delta\,R_{il}$$

By assumption we get $\Delta\,(R_{il} - Rg_{il}) = 0$. Therefore $\Delta = 0$ or $R_{il} = Rg_{il}$ i.e. (M^n,g) is an Einstein manifold .In the second case, we have

(2)
$$\overline{R}_{ij} = (R-\Delta)g_{ij}$$

The conditions of integrability of $\lambda_{ij} = \Delta g_{ij}$, Δ = constant, can be written as $\lambda_S \overline{R}^S_{ijk} = \lambda_S R^S_{ijk} - \Delta(\lambda_j g_{ik} - \lambda_k g_{ij}) = 0$. .Contracting this equation with g^{ki} .

We have $\lambda_s R^S_j = \Delta\lambda_j$ .Being (M^n,g) Einstein, we get $\lambda_j R = \lambda_j\Delta$ consequently $\lambda_j = 0$ for all j i.e. the mapping is affine or $R = \Delta$ and $\overline{R}_{ij} = 0$, i.e. $(\overline{M}^n,\overline{g})$ is a special Einstein manifold.

9. Geodesic Mapping with $\lambda_{ij} = \Delta g_{ij}$:

Let (M^n,g) and $(\overline{M}^n,\overline{g})$ be geodesically equivalent Riemnnain or Pseudo-Riemannian manifolds.

Theorem :

$$g_{hp}\overline{R}^p_{ijk} + g_{ip}\overline{R}^p_{hjk} = 0 \Leftrightarrow \lambda_{ik} = \Delta g_{ik}$$

Proof:

"$\Rightarrow$" we have $\overline{R}^p_{ijk} = R^p_{ijk} - \delta^p_j\lambda_{ik} + \delta^p_k\lambda_{ij}$ from $g_{hp}\overline{R}^p_{ijk} + g_{ip}\overline{R}^p_{hjk} = 0$. It follows that $g_{hj}\,\lambda_{ik} - g_{hk}\,\lambda_{ij} + g_{ij}\,\lambda_{hk} - g_{ik}\,\lambda_{hj} = 0$, contracting this equation with g^{hj}, we get $\lambda_{ik} = \Delta g_{ik}$.

"$\Leftarrow$" $\overline{R}^p_{ijk} = R^p_{ijk} - \Delta(\delta^p_j g_{ik} - \delta^p_k g_{ij})$ had the desired property

We now consider the linear mapping ϕ defined by $\overline{g}(X,y) = g(\phi(X),y)$ where g and $\overline{g}$ are geodesically equivalent metrics, and the related eigen value problem, Assuming that all eigen value of ϕ are distinct, It is not difficult to prove the existence of local coordinates u^1, u^2, u^n such that

(1)
$$dS^2 = \sum_{i=1}^{n} g_{ii}(du^i)^2,$$
$$d\bar{S}^2 = \sum_{i=1}^{n} \rho_i g_{ii}(du^i)^2,$$

where ρ_i = i-th eigen value of ϕ

Now, assuming that $\lambda_{ij} = \Delta g_{ij}$, we infer $g_{hh}\overline{R}^h_{ijk} + g_{ii}\overline{R}^i_{hjk} = 0$ since $\bar{g}_{ii} = \rho_i g_{ii}$, we get ,

$$\left(1/\rho_h\right)\bar{g}_{hh}\overline{R}^h_{ijk} + \left(1/\rho_i\right)\bar{g}_{ii}\overline{R}^i_{hjk} = 0$$

consequently $\left(1/\rho_h - 1/\rho_i\right)\overline{R}_{hijk} = 0$, for $h \neq i$ we have $\overline{R}_{hijk} = 0$ trivially $\overline{R}_{hhjk} = 0$

consequently, by the theorem of Beltrami, (M^n, g) is of constant curvature.

10. Conformal and Concircular Mappings :

Let (M^n, g) and $(\overline{M}^n, \bar{g})$, $n \geq 3$ be two n-dimensional Riemannian or Pseudo Riemannian manifolds and let $\psi : M^n \rightarrow \overline{M}^n$ be a conformal mapping. We assume that $\bar{g} = \rho^2 g$, where ρ is a positive valued function on M^n. It is easy to verify that in the local coordinates $u^1, \ldots, u^n$ the christoffel symbols, the Riemannian curvature tensors and the Ricci tensors of $(\overline{M}^n, \bar{g})$ and (M^n, g) are related as follows:

(1) $\overline{\Gamma}^k_{ij} = \Gamma^k_{ij} + \delta^k_i \lambda_j + \delta^k_j \lambda_i - g_{ij}\lambda^k$ 2)

where $\lambda_i := \partial(\log \rho)/\partial u^i$, $\lambda^k = g^{ki}\lambda_i$.

Remarks :

If $\lambda_i = 0$ for all i, i.e. if ρ=constant, then the mapping is called a homothety or trivial mapping.

(2) $\overline{R}^p_{ijk} = R^p_{ijk} - \delta^p_j \lambda_{ik} + \delta^p_k \lambda_{ij} - g_{ik}\lambda^p_j + g_{ij}\lambda^p_k$

where

(3) $\lambda_{ij} = \lambda_{i,j} - \lambda_i \lambda_j + \frac{1}{2} g_{ij}\lambda^t \lambda_t$ 3)

(4) $\overline{R}_{ik} = R_{ik} - (n-2)\lambda_{ik}/(n-1) - n(Rg_{ik} - \overline{R}\bar{g}_{ik})/2(n-1),$

where $R_{ik} = R^S_{isk}/(n-1), R = g^{st}R_{st}/n$

Introduced the following tensor,

$$C^p_{ijk} := R^p_{ijk} + (n-1)\left(\delta^p_k R_{ij} - \delta^p_j R_{ik} + g_{ij}R^p_k - g_{ik}R^p_j\right)/(n-2)$$
$$+ nR\left(\delta^p_j g_{ik} - \delta^p_k g_{ij}\right)/(n-2)$$

and proved that

(5)
$$C^p_{ijk} = \overline{C}^p_{ijk}$$

The tensor C is called the conformal Weyl-tensor; it is invariant under conformal mappings.

K.Yano [12] proved that a conformal mapping $\bar{g}_{ij} = \rho^2 g_{ij}$ of (M^n, g) to $(\overline{M}^n, \bar{g})$ concircular if and only if the equation $\lambda_{ij} = \phi g_{ij}$ holds for a certain function

ϕ. In this case $\phi = \frac{1}{2}(R-\rho^2 \overline{R})$. From (3) we have $\lambda_{i,j} = \psi g_{ij} + \lambda_i\lambda_j$. By substiting $\psi := \phi - \frac{1}{2}\lambda^t \lambda_t$ (1) we get $\lambda_{i|j} = \lambda_{i,j} - 2\lambda_i\lambda_j + g_{ij}\, \lambda^t\lambda_t$ consequently

(6) $$\lambda_{i|j} = \Omega \overline{g}_{ij} - \lambda_i\lambda_j,$$

where we have put $\Omega := (\phi + \frac{1}{2}\lambda^t\lambda_t)/\rho^2$. Since $\rho = e^{\lambda}$, we have

(7) $$\rho_{i|j} = \rho\Omega \overline{g}_{ij}$$

therefore

(8) $$\rho_{i|jk} = (\rho\Omega)_k\, \overline{g}_{ij}$$

Applying the Ricci-identities to (8) we have

(9) $$\rho_s \overline{R}^S_{ijk} = (\rho\Omega)_k\, \overline{g}_{ij} - (\rho\Omega)_j\, \overline{g}_{ik}$$

Trasvecting (9) with $\overline{g}^{ik}$ we get $\rho_s \overline{R}^S_j = -(\rho\Omega)_j$ consequently (9) becomes $\rho_s\left(\overline{R}^s_{ijk} + \overline{g}_{ij}\overline{R}^S_k - \overline{g}_{ik}\overline{R}^S_j\right) = 0$. or equivalently

(10) $$\overline{\rho}^s\left(\overline{R}_{sijk} + \overline{g}_{ij}\overline{R}_{sk} - \overline{g}_{ik}\overline{R}_{sj}\right) = 0.$$

where $\overline{\rho}^S := \overline{g}^{st}\rho_t$

Using the definition of the projective Weyl-tensor, we may write (10) as $\overline{\rho}^S \overline{W}_{isjk} = 0.$

11. Concricular Mappings on S-Manifold:

Definition:

We define an S-manifold to be a Riemannian or a Pseudo Riemannian manifold (M^n, g) satisfying $R^p_{ijk,lm} = R^p_{ijk,ml}$

Examples of S-manifold are the locally symmetric

Theorem-1:

A Riemannian or Pseudo-Riemannian manifold is an S-manifold if and only if $W^p_{ijk,lm} = W^p_{ijk,ml}$,where W denotes the projective Weyl -Tensor.

Proof:

"$\Rightarrow$" It is a direct consequence of the definition of W. "$\Leftarrow$" Transvecting $W^p_{ijk,lm} = W^p_{ijk,ml}$, with g^{ik} we get $R^P_{j,lm} = R^P_{j,ml}$ .This expression and the definition of w complete the proof.

We now assume that (M^n, g) and $(\overline{M}^n, \overline{g})$ are concircular related Riemannian or Pseudo-Riemannian manifolds and that $(\overline{M}^n, \overline{g})$ is an S-manifold.

We know that $(\overline{g}^{st}\rho_t)\overline{W}_{isjk} = 0$, by covariant differentiation with respect to $\overline{g}$, we get $\overline{g}^{st}\rho_{t|l}\overline{W}_{isjk} + \overline{g}^{st}\rho_t\overline{W}_{isjk|l} = 0$ and $\rho\Omega\overline{W}_{iljk} + \overline{g}^{st}\rho_t\overline{W}_{isjk|l} = 0$ therefore.

$$(\rho\Omega)_m \overline{W}_{iljk} + \rho\Omega\overline{W}_{iljk|m} + \overline{g}^{st}\rho_{t|m}\overline{W}_{isjk|l} + \overline{g}^{st}\rho_t\overline{W}_{isjk|lm} = 0$$

(11.1) $$(\rho\Omega)_m \overline{W}_{iljk} + \rho\Omega(\overline{W}_{iljk|m} + \overline{W}_{imjk|l}) + \overline{g}^{st}\rho_t\overline{W}_{isjk|lm} = 0$$

from (11.1), we have

(11.2) $$(\rho W)_m\overline{W}_{iljk} = (\rho W)_l\overline{W}_{imjk}$$

Let P be a point of $\overline{M}^n$ .We assume that $\rho\Omega$ is not constant then we may find by a linear transformation in $T_P(M^n)$ a basis $E_1,\ldots,E_n$ in $T_P(M^n)$ with $E_1 = (f.\text{grad}\,(\rho\Omega))/P$, $f(\text{p}) \neq 0$.

from (11.2), we have

(11.3) $\overline{W}_{iljk} = 0$ for all $l \neq 1$.

the definition of $\overline{W}$ entails $g^{lk}\overline{W}_{iljk} = n\left(\overline{R}_{ij} - \overline{R}g_{ij}\right)$

on the other hand (11.3) implies

$$\overline{g}^{lk}\overline{W}_{iljk} = g^{1k}\overline{R}_{i1jk} - \overline{g}_{ij}\overline{R}^{1}_{1} + \delta^{1}_{i}\overline{R}_{ij}$$

By the definition of $\overline{W}$,we have

$$\overline{R}_{i1jk} = g_{1k}\overline{R}_{ij} - g_{1j}\overline{R}_{ik} - \overline{W}_{1ijk}$$

Therefore,

(11.4) $$n\left(\overline{R}_{ij} - \overline{R}\overline{g}_{ij}\right) = \overline{R}_{ij} - \overline{g}_{1ij}\overline{R}^{1}_{i} - \overline{g}_{ij}\overline{R}^{1}_{1} + \delta^{1}_{i}\overline{R}_{1j} - \overline{g}^{1k}\overline{W}_{1ijk}$$

Contracting (11.4) with $\overline{g}^{ij}$ and using (11.3) we get $\overline{R}^{1}_{1} = \overline{R}$ therefore (11.4) becomes

$$(n-1)\left(\overline{R}_{ij} - \overline{R}\overline{g}_{ij}\right) = -\overline{g}_{1j}\overline{R}^{1}_{i} + \delta^{1}_{i}\overline{R}_{1j} + \overline{g}^{1k}\overline{W}_{1ijk}$$

This last expression is equivalent to

$$(n-1)\left(\overline{R}^{P}_{i} - \overline{R}\delta^{P}_{i}\right) = -\delta^{P}_{1}\overline{R}^{1}_{i} + \delta^{1}_{i}\overline{R}^{P}_{i} + \overline{g}^{Pj}\overline{g}^{1k}\overline{W}_{1ijk}$$

for $i \neq 1$, $P \neq 1$ one has $\overline{R}^{P}_{i} = \overline{R}\,\delta^{P}_{i}$.

for $i \neq 1$, $P=1$, one has $\overline{R}^{1}_{i} = 0$.

for $i=1$, $P \neq 1$, one has $\overline{R}^{P}_{1} = 0$.

Hence $\left(\overline{M}^{n}, \overline{g}\right)$ is an Einstein manifold. In this case the algebraic properties of $\overline{W}$ and the expression (11.3) imply that $\left(\overline{M}^{n}, \overline{g}\right)$ is a manifold of constant curvature.

Since $\left(\overline{M}^{n}, \overline{g}\right)$ is an Einstein manifold and the map is concircular we infer that (M^n, g) is also an Einstein manifold . Moreover (M^n, g) is conformal to a manifold of constant curvature and consequently (M^n, g) is a manifold of constant curvature.

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