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Review Paper

Application of the infinite matrix theory to the solvability of singular integral equations

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ABSTRACT
We deal with integral equations with a singular kernel of Carlman type. A method to approach to the solution of these equations is given. Infinite matrix theory is used to determine the Fourier coefficients of the solution in the expansion in a series of orthogonal polynomials.
***Key words**: Infinite linear system of linear equations. Hilbert operator. Fourier expansion.*
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I. Introduction

In this paper we introduce a method for solving integral equations with singular kernel of the Carlman type. Although the formulas suitable for expressing the solutions are known [9], the complexity of these formulas leads to choosing the use of numerical techniques for solving the equations in question. Moreover, the approach proposed here can be easily extended to the resolution of more complex singular equations of which there are no known formulas suitable for expressing the solutions. These equations have been extensively treated using the classical numerical methods for their resolution, see for example [7] and the bibliography cited therein. However, the approach proposed here is entirely different. We derive an infinite system to which Schimidt theory [1,4,5] is applicable and whose solution is the sequence of Fourier coefficients of the solution of the integral equation under consideration. More precisely, setting as system of orthogonal functions with respect to which the expansions of the known functions involved in the integral equation are considered, a system of orthogonal polynomials, the method described here allows the computation of the Fourier coefficients of the solution with respect to the same system of orthogonal polynomials. The Chebyshev polynomials are chosen, but with suitable modifications the method can be generalized to any other choice of orthogonal polynomials. The sequence of the Fourier coefficients is determined with the solution of an infinite system. Then the algorithm for the construction of the matrix of this system is determined.

Finally we highlight that the use of infinite systems has already been adopted for the resolution of systems of ordinary differential equations (see for instance [3]).
The remaining part of the paper is organized as follows. In Sections 2 we propose and discuss the method to solve Carlman integral equation and in Section 3 a particular case is considered.

II. A method to solve Carlman integral equation

Denoted by $\alpha(t)$ and $e(t)$ two functions defined in the interval $[-\frac{h}{2},\frac{h}{2}]$ and by λ a given real number, let us consider the equation

$$\alpha(t)u(t)+\lambda\{\mathcal{T}\}_t\{u(\xi)\}=e(t), \quad -\frac{h}{2}<t<\frac{h}{2}, \qquad (2.1)$$

where $u(t)$ is the unknown function and $\mathcal{T}_t$ is the operator which applied to a function $f(t)$ defined for $|t|\le\frac{h}{2}$, carries out the transformation

$$\mathcal{T}_t\{f(\xi)\} = \frac{1}{\pi}\int_{-h/2}^{h/2} \frac{f(\xi)}{\xi - t}d\xi := \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{|\xi-t|>\varepsilon} \frac{f(\xi)}{\xi - t}d\xi\,, \qquad -\frac{h}{2} < t < \frac{h}{2},$$

of the integral taking the Cauchy principal value. The function $\mathcal{T}\{f(\xi)\}$ represents the finite Hilbert transform of $f(t)$. It is not restrictive to identify the interval in which (2.1) is considered with interval $(-1,1)$. Furthermore, if

$$y(t) = u(t)\sqrt{1-t^2}, \qquad a(t) = \frac{\alpha(t)}{\sqrt{1-t^2}},$$

is placed, the equation (2.1) becomes

$$a(t)y(t) + \frac{\lambda}{\pi}\int_{-1}^{1} \frac{y(\xi)}{\xi - t}\frac{d\xi}{\sqrt{1-\xi^2}} = e(t), \quad -1 < t < 1, \qquad (2.2)$$

We will assume in the following that the functions $a(t)$ and $e(t)$ satisfy the Dini-Lipschitz condition, i.e. that the modulus of continuity of these functions goes to 0 faster than $log^{-1}t$ as $t \to 0$. We will find the solution of (2.2) satisfying the same smoothness regularity. By the Dini-Lipschitz theorem (see [2]), we will have

$$a(t) = \sum_{k=0}^{\infty} a_k T_k(t), \qquad |t| \le 1, \quad (2.3)$$

$$e(t) = \sum_{k=0}^{\infty} e_k T_k(t), \qquad |t| \le 1, \quad (2.4)$$

$$y(t) = \sum_{k=0}^{\infty} c_k T_k(t), \qquad |t| \le 1, \quad (2.5)$$

where $\{T_k(t)\}_{k=0}^{\infty}$ is the system of the first kind Chebyshev polynomials and the sequences $\{a_k\}_{k=0}^{\infty}, \{e_k\}_{k=0}^{\infty}, \{c_k\}_{k=0}^{\infty}$ represent the sequences of the Fourier coefficients of the functions $a(t), e(t), y(t)$, respectively, with respect to the previous system of orthogonal polynomials. Solving equation (2.2) is therefore the same to determining the elements of the sequence $\{c_k\}_{k=0}^{\infty}$.
Assuming that function *y(t)* in the forms (2.5) is a solution of (2.2), we have

$$a(t)\sum_{k=0}^{\infty} c_k T_k(t) + \frac{\lambda}{\pi}\left\{c_0\int_{-1}^{1}\frac{1}{\xi-t}\frac{d\xi}{\sqrt{1-\xi^2}}\right\} + \sum_{k=1}^{\infty} c_k\int_{-1}^{1}\frac{T_k(\xi)}{\xi-t}\frac{d\xi}{\sqrt{1-\xi^2}} = e(t), -1 < t < 1. \quad (2.6)$$

Denoted with $U_n(t)$ the *n*th Chebyshev polynomial of the second kind, it results

$$\mathcal{T}_t\{(1-\xi^2)^{-1/2}T_n(\xi)\} = U_{n-1}(t), \quad -1 < t < 1, n \ge 1, \qquad (2.7)$$

$$\mathcal{T}_t\{(1-\xi^2)^{-1/2}T_0(\xi)\} = 0, \quad -1 < t < 1 \qquad (2.8)$$

(see [6]). By (2.7) and (2.8), the (2.6) becomes

$$a(t)\sum_{k=0}^{\infty} c_k T_k(t) + \lambda\sum_{k=1}^{\infty} c_k U_{k-1}(t) = e(t), \quad -1 < t < 1,$$

and then

$$a(t)\sum_{k=0}^{\infty} c_k T_k(t) + \lambda c_1 T_0(t) + \lambda\sum_{k=1}^{\infty} c_{k+1} U_k(t) = e(t), \quad -1 < t < 1. \qquad (2.9)$$

In order to represent the left side of (2.9) as a series of the first kind Chebyshev polynomials so as to be able to compare the coefficients of this expansion with the some ones of (2.4), we premise the following
Lemma 2.1 *Let $\{T_k(t)\}_{k=0}^{\infty}$ and $\{U_k(t)\}_{k=0}^{\infty}$ be the sequences of the first and second kind Chebyshev polynomials, respectively. Then*

$$U_{2n}(t) = T_0(t) + 2\sum_{k=1}^{n} T_{2k}(t), \qquad n \ge 1, \qquad (2.10)$$

$$U_{2n+1}(t) = 2\sum_{k=0}^{n} T_{2k+1}(t), \qquad n \ge 0. \qquad (2.11)$$

Proof. We will prove (2.10) and (2.11) for $n = 1$ and $n = 0$, respectively. Then supposed true at index n, we will prove true for $n + 1$. Recalling the definition of the first and second kind Chebyshev polynomials, proving (2.10) and (2.11) for $n = 1$ and $n = 0$ respectively, is the same to show

$$\frac{\sin 3\vartheta}{\sin\vartheta}=1+2\cos 2\vartheta,\qquad \frac{\sin 2\vartheta}{\sin\vartheta}=2\cos\vartheta,$$

where$\vartheta=\arccos t$, which are easily satisfied.
From

$$2T_n(t)=U_n(t)-U_{n-2}(t),\qquad n\geq 2,\qquad (2.12)$$

(see [8]), it follows

$$U_{2n+2}(t)=U_{2n}(t)+2T_{2n+2}(t),\qquad n\geq 0\qquad (2.13)$$

and

$$U_{2n+3}(t)=U_{2n+1}(t)+2T_{2n+3}(t),\qquad n\geq 0.\qquad (2.14)$$

Assuming (2.10) and (2.11) true at index *n*, (2.13) and (2.14) become

$$U_{2n+2}(t)=T_0(t)+2\sum_{k=1}^{n}T_{2k}(t)+2T_{2n+2}\,(t),$$

$$U_{2n+3}(t)=2\sum_{k=0}^{n}T_{2k+1}(t)+2T_{2n+3}\,(t),$$

that is

$$U_{2n+2}(t)=T_0(t)+2\sum_{k=1}^{n+1}T_{2k}(t),$$

$$U_{2n+3}(t)=2\sum_{k=0}^{n+1}T_{2k+1}(t).$$

Therefore (2.10) and (2.11) are proved with the index *n+1*. ■
We are now able to determine the infinite system whose solution $\{c_k\}_{k=0}^{\infty}$ represents the sequence of the Fourier coefficients of the solution *y(t)* of (2.2).
Theorem 2.1 *Let* $\{a_k\}_{k=0}^{\infty}$ *and* $\{e_k\}_{k=0}^{\infty}$ *be the sequences of the Fourier coefficients of the known functions a(t) and e(t), respectively, in* (2.3), (2.4). *Then the infinite system*

$$a_0c_0+\frac{1}{2}\sum_{k=1}^{\infty}a_{2k}\,c_{2k}+\sum_{k=0}^{\infty}\left(\frac{1}{2}a_{2k+1}+\lambda\right)c_{2k+1}=e_0,\qquad (2.15)$$

$$\frac{1}{2}\sum_{k=0}^{2n}(a_{2n+1-k}+a_{2n+1+k})c_k+\left(a_0+\frac{1}{2}a_{4n+2}\right)c_{2n+1}+\sum_{k=n}^{\infty}\left[\frac{1}{2}(a_{2k-2n+1}+a_{2k+2n+1})+2\lambda\right]c_{2k+2}$$
$$+\frac{1}{2}\sum_{k=n+1}^{\infty}(a_{2k-2n}+a_{2k+2n+2})c_{2k+1}=e_{2n+1},\quad n=0,1,\ldots,\qquad (2.16)$$

$$\frac{1}{2}\sum_{k=0}^{2n-1}(a_{2n-k}+a_{2n+k})c_k+\left(a_0+\frac{1}{2}a_{4n}\right)c_{2n}+\sum_{k=n}^{\infty}\left[\frac{1}{2}(a_{2k-2n+1}+a_{2k+2n+1})+2\lambda\right]c_{2k+1}$$
$$+\frac{1}{2}\sum_{k=n}^{\infty}(a_{2k-2n+2}+a_{2k+2n+2})c_{2k+2}=e_{2n+2},\quad n=1,2,\ldots,\quad (2.17)$$

ordered taking (2.15) *as the first equation and as following* (2.16) *and* (2.17) *alternately, has as solution the Fourier coefficients* $\{c_k\}_{k=0}^{\infty}$ *of the solution y(t) of* (2.2).

Proof. In view of (2.10), (2.11), from (2.9) we have

$$a(t)\sum_{k=0}^{\infty}c_k\,T_k(t)+\lambda c_1T_0(t)+2\lambda\sum_{k=0}^{\infty}c_{2k+2}\sum_{j=0}^{k}T_{2j+1}(t)+\lambda\sum_{k=1}^{\infty}c_{2k+1}\,T_0(t)+2\lambda\sum_{k=1}^{\infty}c_{2k+1}\sum_{j=1}^{k}T_{2j}(t)$$
$$=e(t),\qquad -1<t<1,$$

i.e.

$$a(t)\sum_{k=0}^{\infty} c_k T_k(t) + \lambda \sum_{k=0}^{\infty} c_{2k+1}\, T_0(t) + 2\lambda \left\{ \sum_{k=0}^{\infty}\sum_{j=k}^{\infty} c_{2j+2}\, T_{2k+1}(t) + \sum_{k=1}^{\infty}\sum_{j=k}^{\infty} c_{2j+1}\, T_{2k}(t) \right\} = e(t), -1 < t < 1$$

Thus, by (2.3) and (2.4), we try

$$\sum_{n=0}^{\infty} a_n\, T_n(t) \sum_{k=0}^{\infty} c_k T_k(t) + \lambda \sum_{k=0}^{\infty} c_{2k+1}\, T_0(t) + 2\lambda \left\{ \sum_{k=0}^{\infty}\sum_{j=k}^{\infty} c_{2j+2}\, T_{2k+1}(t) + \sum_{k=1}^{\infty}\sum_{j=k}^{\infty} c_{2j+1}\, T_{2k}(t) \right\}$$

$$= \sum_{n=0}^{\infty} e_n T_n(t)\,, \qquad -1 < t < 1,$$

and making the Cauchy product

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_k T_k(t) c_{n-k}\, T_{n-k}(t) + \lambda \sum_{k=0}^{\infty} c_{2k+1}\, T_0(t) + 2\lambda \left\{ \sum_{n=0}^{\infty}\sum_{k=n}^{\infty} c_{2k+2}\, T_{2n+1}(t) + \sum_{n=1}^{\infty}\sum_{k=n}^{\infty} c_{2k+1}\, T_{2n}(t) \right\} =$$

$$\sum_{n=0}^{\infty} e_n T_n(t)\,, -1 < t < 1. \qquad (2.18)$$

Taking into account that $2T_m(t)T_n(t) = T_{m+n}(t) + T_{|m-n|}(t)$, (see [8]), from (2.18) we deduce

$$\frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_k c_{n-k}\, T_n(t) + \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k\, c_{n-k} T_{|n-2k|}(t) + \lambda \sum_{k=0}^{\infty} c_{2k+1}\, T_0(t)$$

$$+ 2\lambda \left\{ \sum_{n=0}^{\infty}\sum_{k=n}^{\infty} c_{2k+2}\, T_{2n+1}(t) + \sum_{n=1}^{\infty}\sum_{k=n}^{\infty} c_{2k+1}\, T_{2n}(t) \right\} =$$

$$\sum_{n=0}^{\infty} e_n T_n(t)\,, -1 < t < 1. \qquad (2.19)$$

Being

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k c_{n-k}\, T_{|n-2k|}(t) = \sum_{k=0}^{\infty} a_k\, c_k T_0(t) + \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} (a_k c_{k+n} + c_k a_{k+n}) T_n(t),$$

(2.19) gives

$$\frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k\, c_{n-k} T_n(t) + \frac{1}{2}\sum_{k=0}^{\infty} a_k\, c_k T_0(t) + \frac{1}{2}\sum_{n=1}^{\infty}\sum_{k=0}^{\infty} (a_k c_{n+k} + c_k a_{k+n}) T_n(t) + \lambda \sum_{k=0}^{\infty} c_{2k+1}\, T_0(t)$$

$$+ 2\lambda \left\{ \sum_{n=0}^{\infty}\sum_{k=n}^{\infty} c_{2k+2}\, T_{2n+1}(t) + \sum_{n=1}^{\infty}\sum_{k=n}^{\infty} c_{2k+1}\, T_{2n} \right\} = \sum_{n=0}^{\infty} e_n T_n(t)\,, \qquad -1 < t < 1,$$

i.e.

$$\left\{ a_0 c_0 + \frac{1}{2}\sum_{k=1}^{\infty} a_k c_k + \lambda \sum_{k=0}^{\infty} c_{2k+1} \right\} T_0(t)$$

$$+ \sum_{n=1}^{\infty} \left\{ \frac{1}{2}\sum_{k=0}^{2n} a_k\, c_{2n-k} + \frac{1}{2}\sum_{k=0}^{\infty} (a_k\, c_{k+2n} + c_k\, a_{k+2n}) + 2\lambda \sum_{k=n}^{\infty} c_{2k+1} \right\} T_{2n}$$

$$+ \sum_{n=0}^{\infty} \left\{ \frac{1}{2}\sum_{k=0}^{2n+1} a_k\, c_{2n+1-k} + \frac{1}{2}\sum_{k=0}^{\infty} (a_k\, c_{k+2n+1} + c_k\, a_{k+2n+1}) + 2\lambda \sum_{k=n}^{\infty} c_{2k+2} \right\} T_{2n+1}$$

$$= \sum_{n=0}^{\infty} e_n\, T_n(t)\ , \ -1 < t < 1. \qquad (2.20)$$

Finally, comparing the Fourier coefficients of the functions on left and right sides of (2.20), we have

$$a_0 c_0 + \frac{1}{2}\sum_{k=1}^{\infty} a_k c_k + \lambda \sum_{k=0}^{\infty} c_{2k+1} = e_0$$

$$\frac{1}{2}\sum_{k=0}^{2n+1} a_k c_{2n+1-k} + \frac{1}{2}\sum_{k=0}^{\infty} (a_k c_{k+2n+1} + c_k a_{k+2n+1}) + 2\lambda \sum_{k=n}^{\infty} c_{2k+2} = e_{2n+1}, \qquad n = 0,1,....$$

$$\frac{1}{2}\sum_{k=0}^{2n} a_k c_{2n-k} + \frac{1}{2}\sum_{k=0}^{\infty}(a_k c_{k+2n} + c_k a_{k+2n}) + 2\lambda \sum_{k=n}^{\infty} c_{2k+1} = e_{2n}, n = 1,2, \ldots.$$

and then the system (2.15)-(2.17). ■

The next result ensures the applicability of the Schimidt theory on infinite systems to (2.15)-(2.17).

Theorem 2.2 *Assume that a(t) satisfies the Dini-Lipschitz condition. Let* $\{b_{n,k}\}_{n,k=0}^{\infty}$ *be the matrix of the system* (2.15)-(2.17).*Then the series*

$$\sum_{n=0}^{\infty} b_{n,k}^2, \qquad k = 0,1, \ldots, \qquad (2.21)$$

are convergent.

Proof. We begin by remarking that the series (2.21) with *k=0* is the series of the squares of the Fourier coefficients of *a(t)* with respect to the first kind Chebyshev polynomials. The assumption on the function *a(t)* ensures that the Parseval identity

$$\sum_{n=0}^{\infty} a_n^2 = \left(\int_{-1}^{1} a^2\,(t)dt\right)^{\frac{1}{2}},$$

is true. Therefore the convergence of (2.21) is proved with *k=0*.

In order to prove the convergence of (2.21) for *k>0*, we consider the other series

$$\frac{1}{4}\sum_{n=k+1}^{\infty}(a_{n-k} + a_{n+k})^2, \qquad k = 1,2, \ldots, \qquad (2.22)$$

obtained from (2.21) deleting the first *k* terms. Therefore the convergence of (2.21) will be ensured if we try the convergence of (2.22). This again follows from the Parseval identity true for the function *a(t).*■

The solution of the infinite system (2.15)-(2.17) is given by

$$c_k = \lim_{r\to\infty} c_k^{(r)}, \qquad k = 0,1, \ldots,$$

With

$$c_k^{(r)} = \frac{\triangle_k^{(r)}}{\triangle^{(r)}},$$

where

$$\triangle^{(r)} = \begin{vmatrix} \alpha_{00}, & \ldots, & \alpha_{r0} \\ \ldots, & \ldots, & \ldots \\ \alpha_{0r} & \ldots, & \alpha_{rr} \end{vmatrix},$$

$$\triangle_k^{(r)} = \begin{vmatrix} \alpha_{00} & \alpha_{10} & \cdots & \alpha_{r0} & e_0 \\ \alpha_{01} & \alpha_{11} & \cdots & \alpha_{r1} & e_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{0r} & \alpha_{1r} & \cdots & \alpha_{rr} & e_r \\ b_{0k} & b_{1k} & \ldots & b_{rk} & 0 \end{vmatrix}$$

$$\alpha_{nk} = \sum_{j=0}^{\infty} b_{nj}\, b_{kj}, \qquad n,k = 0,1, \ldots.$$

In this way the extremal solution $\{c_k\}_{k=0}^{\infty}$ of the system (2.15)-(2.17) is determined, i.e. the solution which minimizes the sum of the series $\sum_{k=0}^{\infty} c_k^2$. We observe that the square integrable solution *y(t)* of (2.2) corresponds to the extremal solution, which minimizes the integral $\int_{-1}^{1} y^2(t)dt$.

III. A particular case

If the function *a(t)* is constant, in which case can assume *a(t)=1*, the infinite system (2.15)-(2.17) simplifies considerable. In fact, being

$$a_k = \begin{cases} 0, & k \neq 0, \\ 1 & k = 0, \end{cases}$$

the matrix $\{b_{n,k}\}_{n,k=0}^{\infty}$ of (2.15)-(2.17) turns out to be

$$\begin{bmatrix} a_0 & \lambda & 0 & \lambda & 0 & \lambda & \dots \\ 0 & a_0 & 2\lambda & 0 & 2\lambda & 0 & \dots \\ 0 & 0 & a_0 & 2\lambda & 0 & 2\lambda & \dots \\ 0 & 0 & 0 & a_0 & 2\lambda & 0 & \dots \\ 0 & 0 & 0 & 0 & a_0 & 2\lambda & \dots \\ 0 & 0 & 0 & 0 & 0 & a_0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

However, if *a(t)=1*, instead of the system with the previous matrix, it is suitable to solve the system indicated in the following

Theorem 3.1 *Assume a(t)=1 and let* $\{e_k\}_{k=0}^{\infty}$ *be the sequence of the Fourier coefficients with respect to the first kind Chebyshev polynomials of the known function e(t) in* (2.2).
Then the infinite system

$$2c_0 + 2\lambda c_1 - c_2 = 2e_0 - e_2,$$

$$c_k + 2\lambda c_{k+1} - c_{k+2} = e_k - e_{k+2}, \quad k = 1,2,\dots \qquad (3.1)$$

has as solution the Fourier coefficients $\{c_k\}_{k=0}^{\infty}$ *of the solution y(t) of* (2.2).

Proof. In view of (2.7) and (2.8), the equation (2.2) gives

$$\sum_{k=0}^{\infty} c_k T_k(t) + \lambda \sum_{k=0}^{\infty} c_{k+1} U_k(t) = e(t), -1 < t < 1. \qquad (3.2)$$

We will determine the series expansion by the second kind Chebyshev polynomials of the functions on the left and right sides of (3.2). By using (2.4) and (2.12) in (3.2), we have

$$c_0 U_0(t) + \frac{1}{2} c_1 U_1(t) + \frac{1}{2} \sum_{k=2}^{\infty} c_k [U_k(t) - U_{k-2}] + \lambda \sum_{k=0}^{\infty} c_{k+1} U_k(t)$$
$$= e_0 U_0(t) + \frac{1}{2} e_1 U_1(t) + \frac{1}{2} \sum_{k=2}^{\infty} e_k [U_k(t) - U_{k-2}(t)], \quad -1 < t < 1.$$

Finally, ordering with respect to the second kind Chebyshev polynomials

$$\left(c_0 + \lambda c_1 - \frac{1}{2} c_2\right) U_0(t) + \sum_{k=1}^{\propto} \left(\frac{1}{2} c_k + \lambda c_{k+1} - \frac{1}{2} c_{k+2}\right) U_k(t)$$
$$= \left(e_0 - \frac{1}{2} e_2\right) U_0(t) + \frac{1}{2} \sum_{k=1}^{\infty} (e_k - e_{k+2}) U_k(t), -1 < t < 1.$$

Making equal the Fourier coefficients with respect to Chebychev polynomials of the functions on the left and right sides of the previous identity, we deduce (3.1).■

The system (3.1), if it is solved using the Schimidt theory, has the advantage that the matrix $\{\alpha_{n,k}\}_{n,k=0}^{\infty}$ defined in the previous section takes the form

$$\begin{bmatrix} 5+4\lambda^2 & 0 & -1 & 0 & 0 & 0 & \cdots \\ 0 & 2+4\lambda^2 & 0 & -1 & 0 & 0 & \cdots \\ -1 & 0 & 2+4\lambda^2 & 0 & -1 & 0 & \cdots \\ 0 & -1 & 0 & 2+4\lambda^2 & 0 & -1 & \cdots \\ 0 & 0 & -1 & 0 & 2+4\lambda^2 & 0 & \cdots \\ 0 & 0 & 0 & -1 & 0 & 2+4\lambda^2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

Finally we remark that if the known functions *a(t)* and *e(t)* in (2.2) are polynomials, then the solution *y(t)* is itself a polynomial and the system (2.15)-(2.17) becomes a finite system.

References

[1]. Bantsuri,R.,Solution of an infinite system of algebraic equations, J. Math. Sciences 193, (2013) 374 - 377,https://doi.org/10.1007/s10958-013-1463-x.
[2]. Cheney,E.W., Introduction to Approximation Theory, McGraw-Hill Book Co., New York, 1966.
[3]. Fares,A., Ayad,A., Application of the infinite matrix theory to the solvability of a system of differential equations, J. Nonlinear Sci. and Appl. 5 (2012) 439-447, http://dx.doi.org/10.22436/jnsa.005.06.04.
[4]. Faddeev,D.K., Faddeeva,V.N. Computational Methods of Linear Algebra, W. H. Freeman & Co., S. Francisco, 1963
[5]. Gomilko,A.M., Koval'chuk,V.F. Asymptotics of solutions of an infinite system of linear algebraic equations in potential theory, Ukrainian Math. J. 53 (2001), 1419-1432, https://doi.org/10.1023/A:1014354322297.
[6]. Hamel,P.G. Integralgleichungen, Einfuhrung in Lehre und Gebrauch, Springer, Berlin, Gottingen, Heidelberg, 1919
[7]. Ivanov,V.V. The Theory of Approximate Methods and Their Application to the Numerical Solution of Singular Integral Equations, Noordhoff International Publishing Leyden, 1976
[8]. Luke,Y.L. The Special Functions and their Approximations, Acad. Press, New York, London, 1969
[9]. Tricomi,F.G. Integral Equations, Interscience Publishers, Inc. New York, 1968