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Review Paper

On Kenmotsu Manifolds

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ABSTRACT: *The goal of this research is to investigate Kenmotsu manifolds that satisfy specific criteria on the* W_2 *- curvature tensor.*

KEYWORDS: *Kenmotsu manifold, C-Bouchner curvature tensor, Weyl-conformal curvature tensor, Weyl-projective curvature tensor and Einstein manifold.*

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I. INTRODUCTION

K. Kenmotsu [1] explored a class of non-Sasakian contact Riemannian manifolds in 2020 known as Kenmotsu manifolds. In fact, Kenmotsu demonstrated that a locally Kenmotsu manifold is a warped product of a Kahlerian manifold with a warping function $f(t) = se^t$ and an interval I, where s is a non-zero constant, called $I \times_f N$. A Kenmotsu manifold is an illustration of hyperbolic space.

Pokhariya and Mishra [5], on the other hand, proposed and examined a novel curvature tensor known as the W_2 - curvature tensor in a Riemanian manifold. Pokhriyal [4]'s investigation into some of the traits of this tensor of curvature in a Sasakian manifold. Matsumoto et al... have investigated the W_2 - curvature tensor in $P-$ Sasakian and Kenmostu manifolds, respectively. [7] and U.C. De note with [9].

In the current study, we look into a few curvature criteria on Kenmotsu manifolds. Kenmotsu manifolds with $W_2 = 0$ and W_2 - semisymmetric manifolds are the subjects of our initial analysis. In aside from that, we look at Kenmotsu manifolds satisfying $\overline{B}$, $\overline{C}$ and $\overline{P}$, where $\overline{B}$ is the C-Bouchner curvature, $\overline{C}$ is the Weyl-conformal curvature and $\overline{P}$ is the tensor of the Weyl-projective curvature.

II. PRELIMINARIES

Let $\widetilde{M}$ be an almost contact metric manifold of n dimensions with structure (ϕ, ξ, η, g), where g is the Riemannian metric fulfilling g, ξ is a vector field, η is a 1-form and ϕ is a tensor field of type $(1,1)$.

$$\phi^2 = -I + \eta \otimes \xi, \ \ \eta(\xi) = 1, \ \ \eta \circ \phi = 0, \ \ \phi\xi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \ \ \ g(X, \xi) = \eta(X), \tag{2.2}$$

on M for all vector fields X, Y. If

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{2.3}$$

$$\nabla_X \xi = X - \eta(X)\xi. \tag{2.4}$$

In this case, (M, ϕ, ξ, η, g) is referred to as an almost Kenmotsu Manifold [2]. ∇ signifies the Riemannian connection of g .

In Kenmotsu manifolds, the relationships listed below hold true [2]:

$$R(X,Y)Z = \{g(X,Z)Y - g(Y,Z)X\}, \tag{2.5}$$

$$R(X,Y)\xi = \{\eta(X)Y - \eta(Y)X\}, \tag{2.6}$$

$$R(\xi, X)Y = \{\eta(Y)X - g(X,Y)\xi\}, \tag{2.7}$$

$$R(\xi, X)\xi = \{X - \eta(X)\xi\}, \tag{2.8}$$

$$S(X,\xi) = -(n-1)\eta(X), \tag{2.9}$$

$$Q\xi = -(n-1)\xi. \tag{2.10}$$

In the paper Pokhariyal and Mishra [5], the curvature tensor W_2 is defined.

$$W_2(X,Y,U,V) = R(X,Y,U,V) + \frac{1}{(n-1)}[g(X,U)S(Y,V) - g(Y,U)S(X,V)], \tag{2.11}$$

where S is a tensor of the form $(0,2)$ in the Ricci space.

Assume a Kenmotsu manifold satisfying $W_2 = 0$; in this case, (2.12) becomes true.

$$R(X,Y,U,V) = \frac{1}{(n-1)}[g(Y,U)S(X,V) - g(X,U)S(Y,V)]. \tag{2.12}$$

With the help of $X = U = \xi$ from (2.12) and (2.8), (2.9), we have

$$S(Y,V) = \alpha^2(n-1)g(Y,V). \tag{2.13}$$

An Einstein manifold is consequently M .

Re-inserting (2.12) into (2.13) yields the following

$$R(X,Y,U,V) = \alpha^2[g(Y,U)g(X,V) - g(X,U)g(Y,V)]. \tag{2.14}$$

Corollary 2.1. *Due to the fact that a Kenmotsu manifold satisfying* $W_2 = 0$ is a *space with constant curvature* -1*, it is local isometric to the hyperbolic space.*

Definition 2.1. *If a Kenmotsu manifold with* W_2*-semisymmetry is satisfied*

$$R(X,Y)\cdot W_2 = 0, \tag{2.15}$$

where $R(X,Y)$ *is the tensor algebra derivation for each point on the manifold for the tangent vectors* X *and* Y.

The condition can be easily shown to hold for the Kenmotsu manifold's W_2-curvature tensor.

$$\eta(W_2(X,Y)Z) = 0. \tag{2.16}$$

Theorem 2.1. The Kenmotsu M manifolds that make up an Einstein manifold are W_2-semisymmetric.

Proof. Since $R(X,Y)\cdot W_2 = 0$, we have

$$\begin{aligned} &R(X,Y)W_2(U,V)Z - W_2(R(X,Y)U,V)Z \\ &-W_2(U,R(X,Y)V)Z - W_2(U,V)R(X,Y)Z = 0. \end{aligned} \tag{2.17}$$

By inserting $X=\xi$ in (2.17) and taking the inner product with ξ, we may obtain.

$$\begin{aligned} &g(R(\xi,Y)W_2(U,V)Z,\xi) - g(W_2(R(\xi,Y)U,V)Z,\xi) \\ &- g(W_2(U,R(\xi,Y)V)Z,\xi) - g(W_2(U,V)R(\xi,Y)Z,\xi) = 0. \end{aligned} \tag{2.18}$$

We arrive to (2.7) in reference (2.18).

$$\begin{aligned} &- g(Y,W_2(U,V)Z) - \eta(W_2(U,V)Z)\eta(Y) + g(Y,U)\eta(W_2(\xi,V)Z) \\ &- \eta(U)\eta(W_2(Y,V)Z) + g(Y,V)\eta(W_2(U,\xi)Z) - \eta(V)\eta(W_2(U,Y)Z) \\ &+ g(Y,Z)\eta(W_2(U,V)\xi) - \eta(Z)\eta(W_2(U,V)Y) = 0. \end{aligned} \tag{2.19}$$

We obtain when we insert (2.16) into reference (2.19)

$$\alpha^2 W_2(U,V,Z,Y) = 0. \tag{2.20}$$

When [(2.11) and (2.20)] are taken into account, it is evident that

$$R(U,V,Z,Y) = \frac{1}{(n-1)}[g(V,Z)S(U,Y) - g(U,Z)S(V,Y)]. \tag{2.21}$$

A contract (2.21), which we have

$$S(V,Z) = (1-n)g(V,Z). \tag{2.22}$$

In light of (2.12) and (2.23) once more, we obtain

$$R(U,V,Z,Y) = [g(U,Z)g(V,Y) - g(V,Z)g(U,Y)]. \tag{2.23}$$

Corollary 2.2. *The hyperbolic space has a constant curvature of* -1 *and is locally isometric to a* W_2-semisymmetric *Kenmotsu manifold.*

III. ENGAGING KENMOTSU MANIFOLDS WITH $\overline{B}(X.Y)\cdot W_2 = 0$

According to the C-Bouchner curvature tensor's definition, $\overline{B}$ [6]

$$\begin{aligned}\overline{B}(X,Y)Z &= R(X,Y)Z + \frac{1}{(n+3)}[S(X,Z)Y - S(Y,Z)X + g(X,Z)QY - g(Y,Z)QX \\ &+ g(\phi X,Z)\phi Y - S(\phi Y,Z)\phi X + g(\phi X,Z)Q\phi Y - g(\phi Y,Z)Q\phi X + 2S(\phi X,Y)\phi Z \\ &+ 2g(\phi X,Y)Q\phi Z - S(X,Z)\eta(Y)\xi + S(Y,Z)\eta(X)\xi - \eta(X)\eta(Z)QY + \eta(Y)\eta(Z)QX] \\ &- \frac{(p+n-1)}{(n+3)}[g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X + 2g(\phi X,Y)\phi Z] - \frac{(p-4)}{(n+3)}[g(X,Z)Y \\ &- g(Y,Z)X] + \frac{p}{(n+3)}[g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X].\end{aligned} \tag{3.1}$$

The reduction of $X = \xi$ in (3.1) using (2.1), (2.7), (2.9) and (2.10),

$$\overline{B}(\xi,Y)Z = K[\eta(Z)Y - g(Y,Z)\xi], \tag{3.2}$$

where $K = [1 - \frac{(n-1)}{(n+3)} - \frac{(p-4)}{(n+3)} + \frac{p}{(n+3)}]$.

In a Kenmotsu manifold, it's possible that

$$\overline{B}(X,Y).W_2 = 0. \tag{3.3}$$

This being the case,

$$\begin{aligned}&\overline{B}(X,Y)W_2(U,V)Z - W_2(\overline{B}(X,Y)U,V)Z \\ &-W_2(U,\overline{B}(X,Y)V)Z - W_2(U,V)\overline{B}(X,Y)Z = 0.\end{aligned} \tag{3.4}$$

When we use (3.4) to enter $X = \xi$ and extract the inner product, we obtain

$$\begin{aligned}&g(\overline{B}(\xi,Y)W_2(U,V)Z,\xi) - g(W_2(\overline{B}(\xi,Y)U,V)Z,\xi) \\ &- g(W_2(U,\overline{B}(\xi,Y)V)Z,\xi) - g(W_2(U,V)\overline{B}(\xi,Y)Z,\xi) = 0.\end{aligned} \tag{3.5}$$

Utilizing (3.2) in (3.5), as in our example,

$$\begin{aligned}0 &= K_1\eta(Y)\eta(W_2(U,V)Z) - K_1 g(Y,W_2(U,V)Z) - K_1\eta(U)\eta(W_2(Y,V)Z) \\ &+ K_1 g(Y,U)\eta(W_2(\xi,V)Z) - K_1\eta(V)\eta(W_2(U,Y)Z) + K_1 g(Y,V)\eta(W_2(U,\xi)Z) \\ &- K_1\eta(Z)\eta(W_2(U,V)Y) + K_1 g(Y,Z)\eta(W_2(U,V)\xi).\end{aligned} \tag{3.6}$$

When we enter (2.16), (2.11) and (3.6), we get

$$K_1 g(Y,W_2(U,V)Z) = 0. \tag{3.7}$$

By using [(2.1), (2.8), (2.9) and (2.11)] and this provides us with $U = Z = \xi$.

$$S(Y,V) = (1-n)g(V,Y). \tag{3.8}$$

As a result, the following can be said:

Theorem 3.2. *Einstein manifolds are defined as* M *satisfying* $\overline{B}(X.Y)\cdot W_2 = 0.$

IV. ENGAGING KENMOTSU MANIFOLDS WITH $\overline{C}(X.Y)\cdot W_2 = 0$

According to what is written in the Weyl-conformal curvature tensor's definition [3], $\overline{C}$

$$\begin{aligned}\overline{C}(X.Y)Z &= R(X,Y)Z - \frac{1}{(n-2)}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ &- g(X,Z)QY] + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y].\end{aligned} \tag{4.1}$$

Using (2.1), (2.7), (2.9), (2.10) and (2.11), $X = \xi$ in (4.1) is reduced.

$$\overline{C}(\xi,Y)Z = K_1' g(Y,Z)\xi + K_2'\eta(Z)Y + K_3' S(Y,Z)\xi, \tag{4.2}$$

where $K_1' = [1 - 2\frac{(n-1)}{(n-2)} - \frac{r}{(n-1)(n-2)}]$, $K_2' = [-1 + \frac{(n-1)}{(n-2)} + \frac{r}{(n-1)(n-2)}]$ and $K_3' = [\frac{-1}{(n-2)}]$.

Assume that a Kenmotsu manifold exists.

$$\overline{C}(X,Y).W_2 = 0. \tag{4.3}$$

This being the case,

$$\begin{aligned}&\overline{C}(X,Y)W_2(U,V)Z - W_2(\overline{C}(X,Y)U,V)Z \\ &- W_2(U,\overline{C}(X,Y)V)Z - W_2(U,V)\overline{C}(X,Y)Z = 0.\end{aligned} \tag{4.4}$$

With the reference equation's $X = \xi$ formula and the inner product of ξ, we obtain

$$\begin{aligned}&g(\overline{C}(\xi,Y)W_2(U,V)Z,\xi) - g(W_2(\overline{C}(\xi,Y)U,V)Z,\xi) \\ &- g(W_2(U,\overline{C}(\xi,Y)V)Z,\xi) - g(W_2(U,V)\overline{C}(\xi,Y)Z,\xi) = 0.\end{aligned} \tag{4.5}$$

We obtain (2.16), (4.2) in (4.5)

$$0 = K_2^{'} g(Y,W_2(U,V)Z) + K_3^{'} S(Y,W_2(U,V)Z) - K_2^{'} g(Y,U)\eta(W_2(\xi,V)Z)$$
$$- K_3^{'} S(Y,U)\eta(W_2(\xi,V)Z) - K_2^{'} g(Y,V)\eta(W_2(U,\xi)Z) - K_3^{'} S(Y,V)\eta(W_2(U,\xi)Z) \quad (4.6)$$
$$- K_2^{'} g(Y,Z)\eta(W_2(U,V)\xi) - K_3^{'} S(Y,Z)\eta(W_2(U,V)\xi).$$

We get (2.16) by substituting it into (4.6).

$$K_2^{'} g(Y,W_2(U,V)Z) + K_3^{'} S(Y,W_2(U,V)Z) = 0. \quad (4.7)$$

Given $U = Z = \xi$ and the references (2.1), (2.8), (2.9) and (2.11), we have

$$S(V,QY) = (1-n)S(V,Y). \quad (4.8)$$

As evidence for

$$QY = (1-n)Y. \quad (4.9)$$

That produces

$$S(Y,V) = (1-n)g(V,Y). \quad (4.10)$$

As a result, the following can be said.

Theorem 4.3. *An Einstein manifold is a* M *Kenmotsu manifold that satisfies the* $\overline{C}(X.Y)\cdot W_2 = 0$ *definition.*

V. ENGAGING KENMOTSU MANIFOLDS WITH $\overline{P}(X.Y)\cdot W_2 = 0$

As stated in [8], the Weyl-projective curvature tensor $\overline{P}$ is defined.

$$\overline{P}(X,Y)Z = R(X,Y)Z - \frac{1}{(n-1)}[S(Y,Z)X - S(X,Z)Y]. \quad (5.1)$$

Reducing $X = \xi$ in (5.1) using (2.7), (2.9) and (2.11) results in

$$\overline{P}(\xi,Y)Z = -g(Y,Z)\xi - \frac{1}{(n-1)}S(Y,Z)\xi. \quad (5.2)$$

Now look at the Kenmotsu manifold satisfying.

$$\overline{P}(X,Y).W_2 = 0. \quad (5.3)$$

This instance demonstrates that

$$\overline{P}(X,Y)W_2(U,V)Z - W_2(\overline{P}(X,Y)U,V)Z$$
$$- W_2(U,\overline{P}(X,Y)V)Z - W_2(U,V)\overline{P}(X,Y)Z = 0. \quad (5.4)$$

By entering $X = \xi$ into (5.4) and taking the inner product, we obtain

$$\begin{aligned} & g(\overline{P}(\xi,Y)W_2(U,V)Z,\xi) - g(W_2(\overline{P}(\xi,Y)U,V)Z,\xi) \\ & - g(W_2(U,\overline{P}(\xi,Y)V)Z,\xi) - g(W_2(U,V)\overline{P}(\xi,Y)Z,\xi) = 0. \end{aligned} \tag{5.5}$$

By utilizing (5.2) in (5.5), we obtain

$$\begin{aligned} 0 &= g(Y,W_2(U,V)Z) - \frac{1}{(n-1)}S(Y,W_2(U,V)Z) + g(Y,U)\eta(W_2(\xi,V)Z) \\ &+ \frac{1}{(n-1)}S(Y,U)\eta(W_2(\xi,V)Z) + g(Y,V)\eta(W_2(U,\xi)Z) + \frac{1}{(n-1)}S(Y,V)\eta(W_2(U,\xi)Z) \\ &+ g(Y,Z)\eta(W_2(U,V)\xi) + \frac{1}{(n-1)}S(Y,Z)\eta(W_2(U,V)\xi). \end{aligned} \tag{5.6}$$

We get (2.1), (2.6), (2.7), (2.9), (2.11) when we place them in (5.6).

$$-\alpha^2 g(Y,W_2(U,V)Z) + \frac{1}{(n-1)}S(Y,W_2(U,V)Z) = 0. \tag{5.7}$$

With the aid of (2.11) and (2.8), (2.9) and $U = Z = \xi$, we have

$$S(V,QY) = (1-n)S(V,Y). \tag{5.8}$$

This is to say,

$$QY = (1-n)Y. \tag{5.9}$$

Which outcome

$$S(Y,V) = (1-n)g(V,Y). \tag{5.10}$$

As a result, we may state that

Theorem 5.4. *Kenmotsu manifolds with Einstein manifolds satisfy the equation* $\overline{P}(X.Y)\cdot W_2 = 0.$

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