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Review Paper

Boundedness and Stability of Solutions of Nonautonomous Delay Differential Equations of Third Order.

Onwukwe Ijioma.
Department of Mathematics, Abia State University, Uturu, Abia State, Nigeria

ABSTRACT: *We considered the sufficient conditions for the Boundedness and stability of solutions of a third order nonlinear non-autonomous delay differential equation by the construction Lyapunov functional as a tool.*

KEYWORDS: *Lyapunov functional, Asymptotic Stability, delay differential equations, Boundedness, Third order*

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I. INTRODUCTION:

For years, researchers have concentrated on the general behaviours of differential equationsolutions, such as stability, asymptotic stability, uniform asymptotic stability, Boundedness, and uniform Boundedness.
Many physical, chemical and biological events are modelled with differential equations, see [1, 2]
The rapid work in differential equations with lag arguments in recent years and covers many areas in physics, economics and finance.
The behaviour of the solutions of differential equations with delay arguments attracts the attention of many researchers. The trajectory curve of a solution starting in a defined region does not leave this region of stability (see [3]).
In this paper, the uniform asymptotic stability of the equation of the form:

$$[g(x(t))x'(t)]''' + a(t)x''(t) + b(t)x'(t) + c(t)f(x(t-r)) = 0 \quad (1)$$

where $a(t), b(t), and\ c(t)$ *are constants*, $ab - c > 0$are satisfied and the boundedness of

$$[g(x(t))x'(t)]''' + a(t)x''(t) + b(t)x'(t) + c(t)f(x(t-r)) = p(t)\ , with\ p = 0, or\ p \neq 0 \quad (2)$$

$f(x)$is continuous function depending only on the argument shown and $g'(x), f'(x)$ exist and are continuous for all $x, f(0) = 0$.
In the work of M. O. Omeike (see [4]), the author constructed some new Lyapunov functions to examine the asymptotic stability and boundedness of nonlinear delay differential equation described by (2), $p(t, x, x'x'')$
In (see[5-12]), the authors examined the behaviour such as asymptotic stability, global asymptotic stability, global stability, boundedness, uniform boundedness in different third order nonlinear models using the Lyapunov method.
The equation discussed in equation (1), (see[13]), is a particular case of our equation (1). Inspired by the study of (2) and expanding the scope, we examined the uniform asymptotic stability of the solutions for $p(t, x, x'x'') = 0$ and boundedness of the solution to the third order nonlinear differential equation with bounded delay.
We recast (2) as:

$$[\tilde{p}(x(t))x'(t)]'' + a(t)\tilde{Q}(x((t)x'(t))' + b(t)(R(x(t))x'(t))$$
$$+c(t)f(x(t-r(t))) = \tilde{p}(t, x, x', x'') \quad (3)$$

Set $$\tau_1(t) = \frac{\tilde{p}^1(x(t))}{\tilde{p}^2(x(t))}x'(t) \quad (4)$$

$$\tau_2(t) = \frac{\tilde{Q}'(x(t))\tilde{p}(x(t)) - \tilde{Q}(x(t))\tilde{p}'(x(t))}{\tilde{p}^2(x(t))}x'(t) \quad (5)$$

And $$\tau_3(t) = \frac{\tilde{R}'(x(t))\tilde{p}(x(t)) - \tilde{R}(x(t))\tilde{p}'(x(t))}{\tilde{p}^2(x(t))}x'(t) \quad (6)$$

By conjunction, we have the following expressions: $x' = \frac{1}{\tilde{p}(x)}$, $y' = z$ *and*

$$z' = -a(t)\tau_2(t)y - \frac{a(t)\tilde{Q}(x)}{p(x)}z - \frac{b(t)\tilde{R}(x(t))y}{p(x(t))} - c(t)f(x(t))$$
$$+c(t)\int_{t-r(t)}^{t}\frac{1}{p(x)}f'(x)yd\eta + p(t,x,y,z) \quad (7)$$

Where r is a bounded delay with the region $0 \le r(t) \le \Lambda, r'(t) \le \lambda, 0 < \lambda < 1$, Λ and $\lambda > 0$, the functions a, b, c are continuously differentiable and the function $\tilde{P}, \tilde{Q}, \tilde{R}, f, \tilde{p}$ are continuous functions depending only on the above arguments.

The derivatives $\tilde{P}', \tilde{P}'', \tilde{Q}', \tilde{R}'$, *and* $f'(x)$ exist and all are continuous and$f(0) = 0$.

Consider the positive constants $a_0, b_0, c_0, r_0, a_1, b_1, c_1, \tilde{p}_1, \tilde{q}_1$ *and* $\tilde{r}_1$ such that the following assumptions hold:

A1: $0 < a_0 \le a(t), 0 < b_0 \le b(t) \le b_1, and\ 0 < c_0 \le c(t) < c_1\ \forall t \ge 0$

A2: $0 < \tilde{p}_0 \le \tilde{P}(x) \le \tilde{p}_1,\ 0 < \tilde{q}_0 \le \tilde{Q}(x) \le \tilde{q}_1, and\ 0 < r_0 \le \tilde{R}(x) \le \tilde{r}_1,\ for\ some\ x \in \mathbb{R}$

A3: $f(x) \ge \delta_0 > 0\ for\ x \ne 0\ and\ |f'(x)| \le \delta_1\ \forall x$

A4: $|\tilde{p}(t,x,y,z)| \le |e(t)|$

Our novel approach in this work is significant for the study of the qualitative behaviour of solutions of higher order (≥ 3) functional differential equations.

II. PRELIMINARIES

First we will give some definitions and the stability criteria for the general non-autonomous delay differential system.

Consider the following equations:

$$x' = f(t, x_t),\ x_t(\theta) = x(t+\theta),\ \theta \in [-r, 0], t \ge 0 \quad (7a)$$

where $f: I \times C_H \to \mathbb{R}^n$ is a continuous mapping ,

$I = [0,\infty), f(t,0), C_H := \{\phi \in (C[-r,0],\mathbb{R}^n): \|\phi\| \le H\}\ and for\ H_1 < H, \exists \mathcal{L}(H_1) > 0, with |f(t,\phi)| < L(H_1), when\ \|\phi\| < H_1$

Definition 1: An element $\psi \in C$ is in the $\omega - limit$ set of $\phi \in \Omega(\phi)$, if $x(t,0,\phi)$ is defined on $[0,+\infty)$ and there is a sequence $\{t_n\}, t_n \to \infty\ where\ x_{t_n}(\phi) \in Q\ for\ 0 \le t < \infty$ (see [14])

Definition 2:A set $Q \in C_H$is an invariant set if for any $\phi \in Q$ the solution of (8), $x(t,0,\phi)$ is defined on $[0,\infty)\ and\ x_t(\phi) \in Q\ for\ 0 \le t < \infty$

Lemma 1: $If\ \phi \in C_H$ is such that the solution $x_t(\theta)$ of (8) with $x_0(\phi) = 0$ is defined on $[0,\infty)\ and\ \|x_t(\phi)\| \le H_1 < H\ for\ t \in [0,\infty)\ then\ \Omega(\phi)$ is a nonempty, compact, invariant set and $dist\ (x_t(\phi), \Omega(\phi)) \to 0\ as\ t \to \infty$. (see[15])

Lemma 2: $Let\ V(\phi): C_H \to \mathbb{R}^n$ be a continuous functional satisfying a local Lipschitz condition $V(0) = 0$ and such that:

(*i*) $W_1(|\phi(0)|) \le V(\phi) \le W_2(\|\phi\|)\ where\ W_1(r), W_2$are wedges.

(*ii*) $V^1(t,\phi) \le 0,\ for\ \phi \in C_H$. (See [16 and 17])

Then, the zero solution of (8) is asymptotically stable and consistent with the solution, provided that the largest invariant set in$Z\ is\ Q = \{0\}$.

III. PRESENTATION OF MAIN RESULT

Theorem 1: Suppose that the assumptions imposed on the functions$a, b, c, \tilde{P}, \tilde{Q}, \tilde{R}\ and\ e$. Suppose that there are positive constants $\delta_0, \delta_1, \eta_1\ and\ \eta_2$ such that the following conditions aresatisfied, then:

(*i*) $\frac{\tilde{p}_1\delta_1}{r_0} < d < a_0\tilde{q}_0$

(*ii*) $c(t) \le b(t)\ \ and\ b'(t) \le c'(t) \le 0\ \ \ for\ some\ t \in [0,\infty)$

(*iii*) $\frac{1}{2}da'(t)\tilde{Q}(x) - b_0(dr_0 - \tilde{p}_1\delta_1) \le -e < 0$

(*iv*) $\int_{-\infty}^{\infty}(|\tilde{P}'(u)| + |\tilde{Q}'(u)| + |\tilde{R}'(u)|)\,du \le \eta_1 < \infty$and

(*v*) $\int_0^{\infty}|e(s)|ds \le \eta_2 < \infty$

The solution of (3) for $x(t)$ are bounded for $p(t,x,x',x'') = 0$ is asymptotically stable if

$$\Theta < \frac{2\tilde{p}_0}{\tilde{p}_1 c_1 \delta_1} min\left\{\frac{e(1-\lambda)p_0}{\tilde{p}_1(\tilde{p}_0 + d(2-\lambda))}, (a_0\tilde{q}_0 - d)\right\}$$

Theorem 2: Suppose that $a(l), b(l)\ and\ c(l)$ are continuously differentiable on $[0,\infty)$ and the following conditions are satisfied.

(C1) $h(0) = 0,\ \frac{h(x)}{x} \ge \delta_0 > 0\ \ (x \ne 0)$ and $h'(x) \le c_0\ ,\ \forall x,\ |h'(x)| \le c_1$

(C2) $g(0) = 0\ ,\frac{g(y)}{y} \ge b > 0\ \ (y \ne 0)$ and $g'(y) \le c_2\ ,\ \forall y$

(C3) $0 < \delta_1 \le c(t) \le b(t), -W \le b'(t) \le c'(t) \le 0,\ t \ge 0$
(C4) $0 \le \Delta \le a(t) \le \mathcal{L}, for\ t \ge 0$
(C5) $\frac{1}{2}a'(t) \le \delta_2 \le \delta_1(b(t) - \alpha c_1)\ ,\ \ t \ge 0$
(C6) $r(t) \le \gamma\ \ and\ \ r'(t) \le \beta,\ 0 < \beta < 1$
(C7) $\int_0^\infty |c'(t)|dt < \infty\ ,\ \ c'(t) \to 0\ \ as\ t \to \infty$

Then, the solution of

$$x_t''' + a(t)x_t'' + b(t)g\big(x_t'(t - r(t))\big) + c(t)h\big(x(t - r(t))\big) = 0 \quad (8)$$

where $0 \le r(t) \le \gamma, \gamma$ is positive value with $a(t), b(t), c(t), g(x_t')$ *and* $h(x)$ are real-valued functions which are differentiable and continuous , and $g(0) = h(0) = 0$

Set (8) using the following separate expressions:

$$P(t, x(t), x'(t), x\big(t - r(t)\big), x'\big(t - r(t)\big), x''(t) \quad (9)$$

Or by its equivalent transformation

$$x' = y\ ,\ y' = z\ ,\ z' = -h(y)z - g(y) - f(x) + \int_{t-r(t)}^{t} g'\big(y(s)\big)z(s)ds + \int_{t-r(t)}^{t} f'\big(x(s)\big)y(s)ds +$$
P(t,x,y,xt−rt, yr−rt,z) (10)

Using Theorem 1. $\Theta < \frac{2\tilde{p}_0}{\tilde{p}_1 c_1 \delta_1} min\left\{\frac{e(1-\lambda)p_0}{\tilde{p}_1(\tilde{p}_0 + d(2-\lambda))}, (a_0\tilde{q}_0 - d)\right\}$

We set $\Theta < \frac{2\tilde{p}_0}{\tilde{p}_1 c_1 \delta_1}\ min\{\Psi, (a_0 q_0 - d)\}$ (11)

With $\sigma = \left[\frac{\tilde{p}_0(e - \Theta\tilde{p}_1) - 2\Theta\tilde{p}_1 d}{(e\tilde{p}_0 + \Theta(\tilde{p}_1 d))}\right]$ (12)

Proof:

By the definition of Lyapunov functional: $V(t, x_t, y_t, z_t) = e^{-v(t)}U(t, x_t, y_t, z_t)$ (13)

Where $v(t) = \int_0^t |c'(s)|ds\ \ and\ \ \int_0^\infty |c'(t)|dt \le N < \infty$

And $Z = v(t, x_t, y_t, z_t) = dc(t)F(x) + c(t)f(x) + \frac{b_t\tilde{R}(x)}{2\tilde{p}(x)}y^2 + \frac{1}{2}\omega^2 + \frac{d}{\tilde{p}(x)}yz + \frac{1}{2}\frac{da(t)}{2\tilde{p}^2(x)}$ (14)

Put $\Pi = dc(t)F(x) + c(t)f(x) + \frac{b_t\tilde{R}(x)}{2\tilde{p}(x)}y^2$ and

$$\Phi = \frac{1}{2}\omega^2 + \frac{d}{\tilde{p}(x)}yz + \frac{1}{2}\frac{da(t)}{2\tilde{p}^2(x)} \quad (15)$$

Such that $V = \Pi + \Phi + \lambda\int_{-r(t)}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds + \sigma\int_{-r(t)}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds\ ,\lambda, \sigma > 0$
(16)

Furthermore, $(x) = \int_0^x f(\rho)d\rho$, the Lyapunov functional is written as

$V = \Pi + \Phi + \lambda\int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds$given$\Pi = dc(t)F(x) + c(t)f(x) + \frac{b_t\tilde{R}(x)}{2\tilde{p}(x)}y^2$ and $\Phi = \frac{1}{2}\omega^2 + \frac{d}{\tilde{p}(x)}yz + \frac{1}{2}\frac{da(t)}{2\tilde{p}^2(x)}$

By (C5) argument: $\frac{d(a(t) - d)}{2\tilde{p}^2(x)} \ge \frac{d(a_0 - d)}{2\tilde{p}^2(x)} > 0$ with positive constants such that

$$\Phi \ge \delta_2 y^2 + \delta_3 z^2 \quad (17)$$

Under the assumptions of Theorem 2, and a rearrangement of (12) on Π we obtain,

$$\Pi = dc(t)F(x) + c(t)f(x) + \frac{b_t\tilde{R}(x)}{2\tilde{p}(x)}y^2\left(y + \frac{c(t)f(x)g(x)}{b_t}\right)^2 - \frac{c^2(t)\tilde{p}(x)f^2(x)}{2b(t)} \quad (18)$$
$$\ge dc(t)F(x) - \frac{c^2(t)\tilde{p}(x)f^2(x)}{2b(t)}$$
$$\ge dc(t)\left[F(x) - \frac{\tilde{p}_1 f^2(x)}{2d}\right] \ge dc(t)\int_0^x\left(1 - \frac{\tilde{p}\delta_1}{d}\right)f(u)du$$
$$\ge dc_0\left(1 - \frac{\tilde{p}\delta_1}{d}\right) > dc_0\left(1 - \frac{d}{d}\right) = 0 \quad (19)$$

By the condition of C2, there exist a positive constant ζ such that

$$V(t, x, y, z) = V \ge \zeta(x^2 + y^2 + z^2) \quad (20)$$
$$and\ the\ integral\ \int_{t+s}^{t} y^2(\varpi)d\varpi > 0\ and\ \zeta = min\{\delta_2 y^2 + \delta_3 z^2\} \quad (21)$$
$$\tau(t) = \int_0^t |\theta(s)|ds = \int_{\phi_1}^{\phi_2}\frac{|\tilde{p}(u)|}{\tilde{p}^2(u)}du < \infty\quad where\ \phi_1(t) = min\{x_0, x_t\}\ and\ \phi_2(t) = min\{x_0, x_t\}$$

Then, $W_1(|\phi(0)|) \le U(t, \phi) > 0\ \ and\ U(t, \phi) \le W_2(\|\phi\|)$ (22)

Using the results of (see[18,19 and 20]), it follows that $Z(t, x, y, z) = dc'(t)Z_1(t, x, y, z) \le 0$

And by Schwartz inequality where $|\delta e| \le \frac{1}{2}(\delta^2 + e^2)$ then from (13), we obtain

$$\frac{dV}{dt}(t, x_t, y_t, z_t) = \exp(-v(t))\left\{\frac{dU}{dt}(t, x_t, y_t, z_t) - |c'(t)|U(t, x_t, y_t, z_t)\right\} \quad (23)$$

Using (15), (20) and (21), and further transformations, then,

$\frac{dV}{dt} + |c'(t)|U(t, x_t, y_t, z_t) \le -\zeta(x^2 + y^2 + z^2) + \zeta(\delta_2 y^2 + \delta_3 z^2)$ (24)

Given that (11) and (12) $> \Theta$ then the zero solution of (8) is asymptotically stable. For the complete proof, $b(t) and\, c(t)$ are non-decreasing functions on the interval $[0, \infty)$ and continuous on the interval.

Also, $0 \le \delta_1 \le c(t) \le b(t) \le W,\ \lim_{t\to\infty} c(t) = c_0,\quad \lim_{t\to\infty} b(t) = b_{0,}$
(25)

$\Longrightarrow\ \delta_1 \le c_0 \le b_0 \le W$ (26)

Subsequently, (23) :$\frac{dV}{dt}(t, x_t, y_t, z_t) \le \zeta_1 \exp(-v(t))\,(x^2 + y^2 + z^2)\ for\ \zeta_1 > 0$

$$\le V^1(t, \phi) < 0 \quad for\ \phi \in C_H$$

It follows therefore, the Lyapunov functional $V(t, x_t, y_t, z_t) = e^{-v(t)}U(t, x_t, y_t, z_t)$ satisfies all the conditions of Theorem 2.

IV. ILLUSTRATION

Consider the third-order nonlinear non-autonomous delay differential equation

$$x''' + \left(\frac{1}{4}sin\theta + \frac{5}{4}\right)x'' + \left(1 + \frac{1}{\theta^2 + 2}\right)\{2x'(\theta - r(\theta) + (sin\theta)x'(\theta - r(\theta))\}$$

$$+\frac{1}{28}\left(\frac{1}{4} + \frac{1}{\theta^2+3}\right)x(\theta - r(\theta) = 0 \quad (27)$$

Using the equivalent system form:

$x' = y\ ,\ y' = z$ and

$$z' = -\left(\frac{1}{4}sin\theta + \frac{5}{4}\right)z - \left(1 + \frac{1}{\theta^2 + 2}\right)(2y + siny) + \left(1 + \frac{1}{\theta^2 + 2}\right)\int_{\theta - r(\theta)}^{\theta}\{2 + cosy(s)\}z(s)ds$$

$$-\frac{1}{28}\left(\frac{1}{4} + \frac{1}{\theta^2+3}\right)x + \frac{1}{28}\left(\frac{1}{4} + \frac{1}{\theta^2+3}\right)\int_{\theta - r(\theta)}^{\theta} y(s)ds \quad (28)$$

It follows that:

$\Theta = \frac{1}{4} \le a(\theta) = \frac{1}{4}sin\theta + \frac{5}{4} \le \frac{3}{2} = W, \frac{1}{2}a'(\theta) = \frac{1}{8}cos\theta \le \frac{1}{8} = \delta \quad ,\ \forall\theta \ge 0$

And $\delta_1 = \frac{1}{4} \le c(\theta) = \frac{1}{4} + \frac{1}{\theta^2+3} \le \frac{7}{12} \quad ,\ -\frac{3}{2} < c'(\theta) = -\frac{2\theta}{(\theta^2+3)^2} \le 0$

$\Rightarrow$ from (27) , (28) and (C2), we have

$g(y) - 2y - siny = 0$

$\frac{g(y)}{g} - 2 - \frac{siny}{y} - 1 \ge b = 0\ \ (y \ne 0)$

$y'(y) - 3 \le c_1 \qquad \forall\, y$

$h(x) = \frac{1}{28}x\ , h(0) = 0$

$\frac{h(x)}{x} - \frac{1}{28} - \delta_0 > 0$

$h'(x) - \frac{1}{28} \le \frac{1}{14} \quad ,\ \forall\, x$

By conjuncture;

$\int_0^\infty |c'(t)|dt = \frac{1}{3} < \infty \quad and\quad \lim_{t\to\infty} c'(t) = 0$

$\Rightarrow$the system is asymptotically stable and $\frac{1}{3}$ is the region of stability.

V. CONCLUSION

The problem of nonlinear system of delay differential equation is significant and valuable in many scientific areas and can be applied in control theory, information theory and other similar areas.

We established sufficient conditions and obtained the asymptotic stability of the zero solution under variable delay. More importantly is the application of Lyapunov functional technique and showed the stability of the delay system over time lag.

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