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Review Paper

A Method For Deriving a Generating Function For an Arbitrary Polynomial

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ABSTRACT: *Creating a generating function for a polynomial is sometimes very troublesome as there seems not to be a general method available. Each polynomial will usually require a specific way. A procedure is needed to ease creating a generating function for a new polynomial. A method is derived for this purpose appearing to be universal and independent of the details of the polynomial. A second method is given applying to more complex polynomial types being also universal, except for the power's index function. 1*
KEYWORDS: *Generating functions for polynomials, applied mathematics, methods for analysis*

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I. INTRODUCTION

Various polynomials are studied in approximation theories, numerical analysis, initial value problems and special function theories. Generating functions are often required for further analysis. Commonly they are derived case by case as it appears that there is no universal formula available. Rather simple examples of creating generating functions specifically can be found in [1], [2]. The literature is abundant in this respect as every important polynomial has already its generating function derived. In the following is developed a widely applicable method. The more general version with an arbitrary index function is derived next. Then the method is subjected to elementary, but important, sample cases showing the simplicity of its application.

II. THE METHOD FOR DERIVING A GENERATING FUNCTION

II.1 THE LINEAR INDEX POLYNOMIAL

Instead of treating the polynomial itself, it is more useful to process the difference and then return back to the polynomial. An arbitrary polynomial is the following with

$$x \in R$$

defined as

$$z_j(x) = \sum_{n=n_0}^{j} b_n x^n$$

$$b_n \in R$$

being a constant in x but a function of the index. This model will apply to those cases where the index function of the polynomial is linear. One can assume that if j approaches infinity with zj being convergent, a function

$$z_\infty(x) = z(x) \qquad (2)$$

may exist. The lower end of the index is assumed to be n0. Later one can use the relation

$$z_{n_0}(x) = b_{n_0} x^{n_0} \qquad (3)$$

The polynomial is always finite for finite N and finite x. As long as the argument is within the allowed range, which is not needed at this time, the infinite polynomial or series may be convergent as well. The polynomial is expected to be specified to at least the index N + 1 and N is not required to go to infinity. The polynomial needs to have a rule by which its terms are calculated and the index function can be specified. Therefore, a polynomial of random coefficients can not be treated with this method. Also polynomials which grow without limit while the number of terms is increasing, are likely outside of this method. A difference of the polynomial is

$$\Delta z_j(x) = b_{j+1} x^{j+1} \qquad (4)$$

Both sides of this can be multiplied by

$$\cdot\ t^{j+1}$$

and summed

$$\sum_{j=n_0}^{N} \Delta z_j(x) t^{j+1} = \sum_{j=n_0}^{N} b_{j+1} (tx)^{j+1} \qquad (5)$$

with

$|t| < 1$ (6)

If N goes to infinity then the range of x must be valid for convergence. One can use the known result for partial summation [4]

$$\sum_{k=n_0}^{N} y_k \Delta z_k = |_{n_0}^{N+1} y_k z_k - \sum_{k=n_0}^{N} z_{k+1} \Delta y_k \qquad (7)$$

to develop the left side of (5) further.

$$\sum_{j=n_0}^{N} \Delta z_j(x) t^{j+1} = |_{n_0}^{N+1} z_j(x) t^{j+1} - \sum_{j=n_0}^{N} z_{j+1}(x) \Delta t^{j+1} \qquad (8)$$

This is opened up as

$$\sum_{j=n_0}^{N} \Delta z_j(x)t^{j+1} = z_{N+1}(x)t^{N+2} - z_{n_0}(x)t^{n_0+1} - (t-1)\sum_{j=n_0}^{N} z_{j+1}(x)t^{j+1} \quad (9)$$

On the right side of (5) the index can be changed a little to find out

$$\sum_{j=n_0}^{N+1} b_j(tx)^j - b_{n_0}(xt)^{n_0} = z_{N+1}(xt) - b_{n_0}(xt)^{n_0} \quad (10)$$

Combining all terms leads to

$$z_{N+1}(xt) - b_{n_0}(xt)^{n_0} = z_{N+1}(x)t^{N+2} - b_{n_0}x^{n_0}t^{n_0+1} - (t-1)\sum_{k=n_0+1}^{N+1} z_k(x)t^k = \quad (11)$$

$$z_{N+1}(x)t^{N+2} - b_{n_0}x^{n_0}t^{n_0+1} - (t-1)\sum_{k=n_0}^{N+1} z_k(x)t^k + (t-1)b_{n_0}(xt)^{n_0} \quad (12)$$

giving

$$z_{N+1}(xt) = z_{N+1}(x)t^{N+2} - (t-1)\sum_{k=n_0}^{N+1} z_k(x)t^k \quad (13)$$

The final finite form is

$$\sum_{k=n_0}^{N+1} z_k(x)t^k = \frac{z_{N+1}(xt)}{1-t} - \frac{z_{N+1}(x)t^{N+2}}{1-t} \quad (14)$$

Usually the generating function is specified when N goes to infinity as

$$\sum_{k=n_0}^{\infty} z_k(x)t^k = \frac{z_\infty(xt)}{1-t} = \frac{z(xt)}{1-t} \quad (15)$$

with the assumption that

$$z_N(x)$$

converges and

$$t^N$$

will approach zero since |t| < 1. These two equations form the method in the case of polynomials with a linear index function. They are accurate, not approximations.

II.2 THE ARBITRARY INDEX POLYNOMIAL

To solve the cases with more involved index functions, one will allow a similar structure in the model to begin with as in the preceding section

$$z_j(x) = \sum_{n=n_0}^{j} b_n x^{c(n)} \tag{16}$$

$$b_n \in R$$

is a constant depending on the index but not on x. It is assumed that if j approaches infinity with zj being convergent, a function

$$z_\infty(x) = z(x) \tag{17}$$

may exist. The lower end of the index is n0 and therefore

$$z_{n_0}(x) = b_{n_0} x^{c(n_0)} \tag{18}$$

The index function c(n) must be such as to maintain the polynomial finite when j goes to infinity. Following analogous steps as in the preceding section, one will get

$$\sum_{k=n_0}^{N+1} z_k(x)(t^{c(k)} - t^{c(k+1)}) = z_{N+1}(xt) - z_{N+1}(x)t^{c(N+2)} \tag{19}$$

If N goes to infinity the small term on the right vanishes

$$\sum_{k=n_0}^{\infty} z_k(x)(t^{c(k)} - t^{c(k+1)}) = z_\infty(xt) = z(xt) \tag{20}$$

with the assumption that

$$z_N(x)$$

converges to z(x) and

$$t^{c(N)}$$

will approach zero. These two equations represent the method for more general index function cases. The sum expressions will often simplify and the difference term can be pulled out from the sum. These equations are again accurate with no approximate assumptions.

III APPLICATIONS OF THE METHOD

III.1 THE SIMPLE LOGARITHMIC POLYNOMIAL

In the following the logarithmic polynomial is used as an example of a hard case to solve. The elementary logarithmic polynomial having an argument

$$x \in R$$

is defined as

$$l_j(x) = \sum_{n=1}^{j} \frac{(-1)^{n+1} x^n}{n} \qquad (21)$$

As long as the argument is small

$-1 < x \leq 1$ (22)

it approaches a limit

$$j \to \infty, l_j \to l(x) = ln(1+x) \qquad (23)$$

This is the common series expansion of the logarithm function in a limited interval. The polynomial seems to resist known methods, [1], [2], [3]. The polynomial does not have any useful connections to other polynomials and derivatives and recursion relations are useless. Differentiation brings out functions of a different type and no recursion seems possible along that route. This is only little studied at the moment, [6] [7]. One can now immediately use the method (15) to obtain the generating function of the logarithmic polynomial

$$\sum_{k=1}^{\infty} l_k(x) t^k = \frac{ln(1+xt)}{1-t} \qquad (24)$$

$$-1 < t < 1 \qquad (25)$$

III.2 THE MORE ADVANCED LOGARITHMIC POLYNOMIAL

The next logarithmic polynomial is similar to the previous one. The polynomial having an argument $x \in R$ is defined as

$$L_j(x) = \sum_{n=1}^{j} \frac{x^{2n-1}}{2n-1} \qquad (26)$$

While the argument is small

$-1 < x < 1$ (27)

the polynomial will approach the limit

$$j \to \infty, L_j \to L(x) = \frac{1}{2} ln \frac{(1+x)}{(1-x)} \qquad (28)$$

This is a common logarithmic function. One can now use the method (20) to get the generating function

$$\sum_{k=1}^{\infty} L_k(x)t^{2k-1} = \frac{1}{2}\frac{ln(\frac{1+xt}{1-xt})}{1-t^2} \quad (29)$$

$$-1 < t < 1 \quad (30)$$

III.3 THE RECIRPOCAL BINOMIAL POLYNOMIAL

The reciprocal binomial polynomial has an argument x ∈ R and is defined as

$$r_j(x) = \sum_{n=0}^{j} (-x)^n \quad (31)$$

This is always finite as long as the argument is small

$|x| < 1$ (32)

and it will meet the limit

$$j \to \infty, r_j(x) \to r(x) = \frac{1}{(1+x)} \quad (33)$$

This is the series expansion of the reciprocal binomial function in a limited interval. On can use the result (15) to get the generating function immediately

$$\sum_{k=0}^{\infty} r_k(x)t^k = \frac{r(xt)}{1-t} = \frac{1}{(1-t)(1+xt)} \quad (34)$$

$$-1 < t < 1 \quad (35)$$

IV DISCUSSION

Usually deriving a generating function for a polynomial may be a tedious task. Sometimes complicated recursion formulas are required and/or specific differential equations to be solved to reach the goal as in [5].

Here is developed a method for expressing the generating function of an arbitrary polynomial with a linear index. It is made by finding out the difference of the polynomial and then starting the usual process of writing down the common way of generating functions. The difference is next removed by using the method of summation by parts. The result is expressed in equations (14) and (15). Equations (19) and (20) are derived in the same manner and offer a solution to more complex index behavior.

The interesting feature of these results is the lack of the details of the polynomial in the resulting generating function. Only the index behavior of the powers is what matters. This means that this method is rather universal. The method used in the derivation process can be generalized for solving more complicated cases.

The method presented gives an accurate expression, not just a formal presentation. The generating function is a true function with a converging series with a range of validity. This fact allows further use of it in analysis. In addition, the method applies to complex variables and complex functions as well.

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