Quest Journals
Journal of Research in Applied Mathematics
Volume 9 ~ Issue 9 (2023) pp: 34-41
ISSN (Online): 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Review Paper

Iterative Algorithms of Positive Solutions For the Lotka-Volterra Competition Model

LiLi Shi
(Humanities and General Education, Chengdu Textile College)

ABSTRACT: *The Lotka-Volterra competition model is a landmark population biology model widely ap-plied in various fields such as commercial competition, stock index futures market, port container forecast, etc. In this paper, we present some conditions for the existence and uniqueness of positive solutions for the most common Lotka-Volterra model and establish iterative algorithms and error estimations. The results of this paper can be generalized to the model consisting of more than two equations.I*

KEYWORDS: *Lotka-Volterra model, Positive solution, Iterative algorithm, error estimation*

***R**eceived 14 Sep., 2023; **R**evised 27 Sep., 2023; **A**ccepted 29 Sep., 2023 © The author(s) 2023.*
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I. INTRODUCTION

The Lotka-Volterra model, independently published by Alfred Lotka and Vito Volterra in 1925 and 1926, is a classic model for describing the dynamics of preda-tor-prey interactions in biological systems. In recent decades, it has evolved into a landmark model in population biology and has made significant contributions to the study of population dynamics in ecosystems and other fields.

In recent years, many scholars at home and abroad have studied this model using various methods. It is not only applied to biological systems, but also to various other fields such as market competition, financial derivatives, energy consumption, medical science, environmental pollution, and so on. Let's just review a few recent results. Wang et al. [1], Higazy et al. [2] did fractal dimension analysis and control of Julia set and dynamical and structural study generated by fractional Lotka–Volterra models respectively. Zhang et al. [3] studied the model with Robin boundary condition. Yang et al. [4] and Wang et al. [5] considered almost periodic models respectively. Khan and Li [6] obtained the existence results of the fractional-order. Hu et al. [7] studied the traveling wave. Wang et al. [8] used a panel Lotka-Voterra method to China's manufacturing industry. Wang [9] applied gray forecast theory with the Lot-ka–Volterra competition model to explore the dynamic competition between smart TVs and flat panel TVs. Hung, Tsai, and Wu [10] attempted to develop an improved forecasting methodology for retail industry competition subject to seasonal patterns and cycles. Xiong et al. [11] used the Lotka–Volterra model to analyze the stock in-dex futures market. Marasco and Romano [12] rented a nonautonomous Lotka–Volterra model to give a scenario analysis for inter-port interactions in the Le Ha-vre–Hamburg range. Zhang et al. [13] proposed a novel gray Lotka-Volterra model for energy consumption forecasting to evaluate the impact of long-term competition and cooperation on the national energy consumption system and its development trend. W. W. Mohammed et al. [14] investigated dynamics in Lotka-Volterra based models of COVID-19. N. Brunner, S. Das and M. Starkl [15] fitted systems of generalized Lot-ka-Volterra differential equations to pollution shares and studied their dynamics.

The most common Lotka-Volterra competition model can be written as

$$\begin{cases} \dfrac{du(t)}{dt} = u(t)\left[\sigma_1(t) - b_1(t)u(t) - c_1(t)v(t)\right] := f(t,u(t),v(t)), t > 0 \\ \dfrac{dv(t)}{dt} = v(t)\left[\sigma_2(t) - b_2(t)v(t) - c_2(t)u(t)\right] := g(t,u(t),v(t)), t > 0 \to (1) \\ \qquad u(0) = u_0 > 0, v(0) = v_0 > 0. \end{cases}$$

For relevant literature, see [8]-[10], where

(i) $u(t)$ and $v(t)$ are the numbers of populations of the prey and predator species at time t.

(ii) $\frac{du(t)}{dt}$ and $\frac{dv(t)}{dt}$ represent the instantaneous growth rates of each population.

(iii) $\sigma_1(t)$ and $\sigma_2(t)$ are the parameters related to the number of each population.

(iv) $b_1(t)$ and $b_2(t)$ are the intrinsic growth parameters of each population.

(v) $c_1(t)$ and $c_2(t)$ are the interaction coefficients between the two populations.

The goal of this paper is to establish the existence of positive solutions and iterative algorithms for (1). It is well known that the solutions of (1) should be positive (called to be positive solutions) and determining the existence of positive solutions requires special methods, the usual method of positive solutions for self mapping defined on a cone [16], [17] cannot used to treat (1) due to the nonlinear terms with sign-changing in (1). At the same time, some questions raise naturally: what are the conditions for the existence of positive solutions of the model? Is the solution unique? If there is such solution, how can we present iterative algorithms and error estimations?

These questions are very interesting and important for nonlinear problems. However, as far as we know, there is little study on them for the model (1). In this paper, we provide some conditions for the existence and uniqueness of positive solutions and develop iterative algorithms and error estimates for the model (1). Finally, we point out that the methods in this paper are suitable for the Lotka-Volterra models consisting of more than two equations.

II. ITERATIVE ALGORITHMS AND ERROR ESTIMATIONS FOR POSITIVE SOLUTION OF (1)SYSTEM COORDINATES

Let $T \in (0,\infty)$ is a constant. We define

$C[0,T]=\{u:u(t)$ is a continuous function defined on $[0,T]$ }, $C_+[0,T]=\{u:u(t)\in C[0,T], u(t)\geq 0, t\in[0,T]\}$.

If $u,v\in C[0,T]$ with the first-order continuous derivative and satisfy (1), then they are called to be positive solutions of (1) when $u(t)>0, v(t)>0$ $(t\in[0,T])$.

Integrating (1) from 0 to t, we obtain the following integral system:

$$\begin{cases} u(t)=\int_0^t f(s,u(s),v(s))ds+u_0, \\ v(t)=\int_0^t g(s,u(s),v(s))ds+v_0, t\geq 0. \end{cases} \rightarrow (2)$$

This means the solution of (1) is the solution of (2). Conversely, if $u,v\in C[0,T],\ u(t)>0,\ v(t)>0 (t\in[0,T])$ is the solution of (2), then by differentiating (2), we know that it satisfies (1) and also satisfies the initial condition. Therefore, (1) is equivalent to the integral system (2). Hence, we only need to study the existence, uniqueness, iterative algorithms, and error estimations of positive solutions for the integral system (2).

For (1), we make the following assumptions $(i=1,2)$:

(P_1) $b_i(t),\sigma_i(t),c_i(t)$ are all continuous on $[0,\infty)$, bounded functions and $b_i(t),\ \sigma_i(t),c_i(t):[0,\infty)\rightarrow(0,\infty)$.

(P_2) $\sigma_i=\inf\{\sigma_i(t):t\in[0,\infty)\}>0$.

Remark 1. *If $b_i(t),\sigma_i(t),c_i(t)$ are positive constant functions on $[0,\infty)$, then the conditions (P_1) - (P_2) are satisfied automatically.*

Let $c_i=\sup\{c_i(t):t\in[0,\infty)\}, b_i=\sup\{b_i(t):t\in[0,\infty)\}, \sigma=\max\{\sigma_1,\sigma_2\}$, then $0<c_i,\ b_i<\infty$ $(i=1,2)$.

We assume that u_0, v_0 satisfy

(P_3) $b_1u_0+c_1v_0<\sigma_1$.

(P_4) $b_2v_0+c_2u_0<\sigma_2$.

Then there exists a constant $\tau>0$ such that

$$b_1u_0+c_1v_0\leq\sigma_1(1-\sigma\tau), \rightarrow (3)$$
$$b_2v_0+c_2u_0\leq\sigma_2(1-\sigma\tau). \rightarrow (4)$$

Let $T=\sup\{\tau:\tau$ satisfies (3) and (4)$\}$. Then

$$b_1u_0+c_1v_0\leq\sigma_1(1-\sigma T), \rightarrow (5)$$
$$b_2v_0+c_2u_0\leq\sigma_2(1-\sigma T). \rightarrow (6)$$

Clearly, $0<T<\infty$.

Notation 1. $\Omega=\{(u,v):b_1u+c_1v\leq\sigma_1, b_2v+c_2u\leq\sigma_2, u,v\in C_+[0,T]\}$.

Obviously, Ω is nonempty since $(u_0, v_0) \in \Omega$.

Let $(u,v) \in \Omega$. We define

$$\begin{cases} A(u,v)(t) = \int_0^t f(s,u(s),v(s))ds + u_0, \\ B(u,v)(t) = \int_0^t g(s,u(s),v(s))ds + v_0, \to (7) \\ \qquad 0 \le t \le T. \end{cases}$$

The results of this paper are presented by the following three theorems. For the series and uniform convergence, see [18].

Theorem 1. *For* $(u,v) \in \Omega$, $(A(u,v), B(u,v)) \in \Omega$ *holds.*

Proof: Let $(u,v) \in \Omega$. Then $b_1(t)u(t) + c_1(t)v(t) \le \sigma_1$ and $b_2(t)v(t) + c_2(t)u(t) \le \sigma_2$, which imply that $A(u,v)(t) \ge 0$, $B(u,v)(t) \ge 0$, $0 \le t \le T$. By (5), (6) and (7), we have

$$\begin{aligned} & b_1 A(u(t),v(t)) + c_1 B(u(t),v(t)) \\ & = b_1\left[\int_0^t f(s,u(s),v(s))ds + u_0\right] + c_1\left[\int_0^t g(s,u(s),v(s))ds + v_0\right] \\ & \le \int_0^t \left(b_1 u(s)\sigma_1 + c_1 v(s)\sigma_2\right)ds + \left(b_1 u_0 + c_1 v_0\right) \\ & \le \sigma\int_0^t \left(b_1 u(s) + c_1 v(s)\right)ds + \left(b_1 u_0 + c_1 v_0\right) \\ & = \sigma\int_0^t \sigma_1 ds + \left(b_1 u_0 + c_1 v_0\right) \le \sigma\sigma_1 T + \left(b_1 u_0 + c_1 v_0\right) \le \sigma_1, \\ & b_2 B(u(t),v(t)) + c_2 A(u(t),v(t)) \\ & = b_2\left[\int_0^t g(s,u(s),v(s))ds + v_0\right] + c_2\left[\int_0^t f(s,u(s),v(s))ds + u_0\right] \\ & \le \int_0^t \left(b_2 v(s)\sigma_2 + c_2 u(s)\sigma_1\right)ds + \left(b_2 v_0 + c_2 u_0\right) \\ & \le \sigma\int_0^t \left(b_2 v(s) + c_2 u(s)\right)ds + \left(b_2 v_0 + c_2 u_0\right) \\ & = \sigma\int_0^t \sigma_2 ds + \left(b_2 v_0 + c_2 u_0\right) \le \sigma\sigma_2 T + \left(b_2 v_0 + c_2 u_0\right) \le \sigma_2. \end{aligned}$$

Therefore $(A(u,v), B(u,v)) \in \Omega$ and the proof is completed.

If $(u,v) \in \Omega$, it is easy to know $0 \le u(t) \le \dfrac{\sigma_1}{b_1}$ and $0 \le v(t) \le \dfrac{\sigma_1}{c_1}$.

Notation 2.

$$K = \max\left\{\frac{\sigma_1}{b_1}, \frac{\sigma_1}{c_1}\right\},$$

$$m_1 = \sigma_1 + 2b_1 K + \left(c_1 + c_2\right)K,$$

$$m_2 = \sigma_2 + 2b_2 K + \left(c_1 + c_2\right)K,$$

$$m = \max\left\{m_1, m_2\right\},$$

$$e_n = \sum_{k=0}^{n} \frac{(mT)^n}{n!}.$$

Define the iteration sequence as follows

$$u_n(t) = A\left(u_{n-1}, v_{n-1}\right)(t), n = 1,2,\ldots \to (8)$$

$$v_n(t) = B\left(u_{n-1}, v_{n-1}\right)(t), n = 1,2,\ldots \to (9)$$

$$u_0(t) = u_0, v_0(t) = v_0, t \in [0,T].$$

Theorem 2. $\left|u_n(t) - u_{n-1}(t)\right| \le 2K\dfrac{(mT)^{n-1}}{(n-1)!}$, $\left|v_n(t) - v_{n-1}(t)\right| \le 2K\dfrac{(mT)^{n-1}}{(n-1)!}$, $0 \le t \le T$, $n \ge 1$.

Proof: From Theorem 1, $(u_n, v_n) \in \Omega$, therefore $0 \le u_n(t), v_n(t), u_{n-1}(t), v_{n-1}(t) \le K$, and consequently

$$
\begin{aligned}
&|u_{n+1}(t)-u_n(t)| \\
&\le \int_0^t \left| f\left(s,u_n(s),v_n(s)\right)-f\left(s,u_{n-1}(s),v_{n-1}(s)\right)\right| ds \\
&= \int_0^t \Big| \left[\sigma_1(s)-c_1(s)v_n(s)-b_1(s)\left(u_n(s)+u_{n-1}(s)\right)\right]\left(u_n(s)-u_{n-1}(s)\right) \\
&\quad -c_1(s)u_{n-1}(s)[\left(v_n(s)-v_{n-1}(s)\right]\Big| ds \\
&\le \int_0^t \left[\left(\sigma_1(s)+2b_1(s)K+c_1(s)K\right)|u_n(s)-u_{n-1}(s)|+c_1(s)K|v_n(s)-v_{n-1}(s)|\right] ds \\
&\le \int_0^t \left[\left(\sigma_1+2b_1K+c_1K\right)|u_n(s)-u_{n-1}(s)|+c_1K|v_n(s)-v_{n-1}(s)|\right] ds, \\
&|v_{n+1}(t)-v_n(t)| \\
&\le \int_0^t \left| g\left(s,u_n(s),v_n(s)\right)-g\left(s,u_{n-1}(s),v_{n-1}(s)\right)\right| ds \\
&= \int_0^t \Big| \left[\sigma_2(s)-c_2(s)u_n(s)-b_2(s)\left(v_n(s)+v_{n-1}(s)\right)\right]\left(v_n(s)-v_{n-1}(s)\right) \\
&\quad -c_2(s)v_{n-1}(s)\left[\left(u_n(s)-u_{n-1}(s)\right]\right| ds \\
&\le \int_0^t \left[\left(\sigma_2(s)+2b_2(s)K+c_2(s)K\right)|v_n(s)-v_{n-1}(s)|+c_2(s)K|u_n(s)-u_{n-1}(s)|\right] ds \\
&\le \int_0^t \left[\left(\sigma_2+2b_2K+c_2K\right)|v_n(s)-v_{n-1}(s)|+c_2K|u_n(s)-u_{n-1}(s)|\right] ds.
\end{aligned}
$$

From this we have

$$
\begin{aligned}
&|u_{n+1}(t)-u_n(t)|+|v_{n+1}(t)-v_n(t)| \\
&\le \int_0^t \left[m_1|u_n(s)-u_{n-1}(s)|+m_2|v_n(s)-v_{n-1}(s)|\right] ds \to (10) \\
&\le m\int_0^t \left(|u_n(s)-u_{n-1}(s)|+|v_n(s)-v_{n-1}(s)|\right) ds.
\end{aligned}
$$

Noting that $|u_n(t)-u_{n-1}(t)|\le K, |v_n(t)-v_{n-1}(t)|\le K,$ we obtain

$$|u_{n+1}(t)-u_n(t)|+|v_{n+1}(t)-v_n(t)|\le m\int_0^t 2Kds = 2Kmt. \to (11)$$

Let $e_n(t)=|u_n(t)-u_{n-1}(t)|+|v_n(t)-v_{n-1}(t)|$. By combining (10) and (11), we get

$$e_{n+1}(t)\le 2Kmt,\ e_{n+1}(t)\le m\int_0^t e_n(s)ds,\ n\ge 1.$$

By induction, we have

$$e_n(t)\le 2K\frac{(mt)^{n-1}}{(n-1)!}\le 2K\frac{(mT)^{n-1}}{(n-1)!}, 0\le t\le T, n\ge 1 \to (12)$$

(12) implies that Theorem 2 holds.

Theorem 3. *Suppose that* (P_3) - (P_4) *are satisfied, then (2) has a unique positive solution* (u_*,v_*) *in* $[0,T]$, *and* $(u_*,v_*)\in\Omega$,

$$
\begin{aligned}
|u_*(t)-u_n(t)| &\le 2K\left(e^{mT}-e_{n-1}\right), \\
|v_*(t)-v_n(t)| &\le 2K\left(e^{mT}-e_{n-1}\right),
\end{aligned}
$$

where u_n, v_n is defined by (8) and (9).

Proof: The proof is divided into two steps.

Step 1. The existence, iterative algorithms and error estimations for positive solutions.

By Theorem 2, we have

$$
\begin{aligned}
&|u_{n+p}(t)-u_n(t)|\le \sum_{k=n+1}^{n+p}|u_k(t)-u_{k-1}(t)| \\
&\le \sum_{k=n}^{n+p} e_k(t)\le 2K\sum_{k=n+1}^{n+p}\frac{(mT)^{k-1}}{(k-1)!}=2K\sum_{k=n}^{n+p-1}\frac{(mT)^k}{k!},
\end{aligned}
\to (13)
$$

$$\left|v_{n+p}(t)-v_n(t)\right| \le \sum_{k=n+1}^{n+p}\left|v_k(t)-v_{k-1}(t)\right|$$
$$\le \sum_{k=n}^{n+p} e_k(t) \le 2K\sum_{k=n+1}^{n+p}\frac{(mT)^{k-1}}{(k-1)!} = 2K\sum_{k=n}^{n+p-1}\frac{(mT)^k}{k!}, \to (14)$$

The inequalities (13)-(14) imply that $\{u_n(t)\},\{v_n(t)\}$ converge uniformly on $[0,T]$, the limits are denoted by

$$u_*(t)=\lim_{n\to\infty}u_n(t),\ v_*(t)=\lim_{n\to\infty}\{v_n(t)\}.$$

By $(u_n,v_n)\in\Omega$, we have $0\le b_1u_n+c_1v_n\le\sigma_1, 0\le b_2v_n+c_2u_n\le\sigma_2$. Letting $n\to\infty$, then we obtain $0\le b_1u_*+c_1v_*\le\sigma_1$, $0\le b_2v_*+c_2u_*\le\sigma_2$, which mean $(u_*,v_*)\in\Omega$, $u_*(t)\ge u_0$, $v_*(t)\ge v_0$, $t\in[0,T]$. Letting $n\to\infty$ in (13) and (14), we have

$$u_*=A(u_*,v_*),\ v_*=B(u_*,v_*),$$

and (Letting $p\to\infty$ in (13),(14))

$$\left|u_*(t)-u_n(t)\right|\le 2K\sum_{k=n}^{\infty}\frac{(mT)^k}{k!}=2K\left(e^{mT}-e_{n-1}\right),$$
$$\left|v_*(t)-v_n(t)\right|\le 2K\sum_{k=n}^{\infty}\frac{(mT)^k}{k!}=2K\left(e^{mT}-e_{n-1}\right).$$

Step 2. Uniqueness of positive solution.

Suppose that (2) has another positive solution (z_*,w_*) in $C[0,T]$. We divide $[0,T]$ into N equal parts such that $\frac{T}{N}\theta<1$. Let $\Delta=\frac{T}{N}, t_i=i\Delta t, i=1,2,\dots,N$ and

$$h_1(s)=\sigma_1(s)+c_1(s)v_*(s)+b_1(s)\left(u_*(s)+z_*(s)\right)+c_1(s)z_*(s),$$
$$h_2(s)=\sigma_2(s)+c_2(s)u_*(s)+b_2(s)\left(v_*(s)+w_*(s)\right)+c_2(s)w_*(s),$$
$$\theta=\max\{h_1(s)+h_2(s): s\in[0,T]\},$$
$$e_i=\max\{|u_*(t)-z_*(t)|+|v_*(t)-w_*(t)| : t\in[t_i,t_{i+1}]\}, i=0,1,2,\dots,N-1.$$

By

$$\begin{aligned}
\left|u_*(t)-z_*(t)\right| &\le \int_0^t\left|f\left(s,u_*(s),v_*(s)\right)-f\left(s,z_*(s),w_*(s)\right)\right|ds\\
&=\int_0^t\left|\left[\sigma_1(s)-c_1(s)v_*(s)-b_1(s)\left(u_*(s)+z_*(s)\right)\right]\left(u_*(s)-z_*(s)\right)-c_1(s)z_*(s)\left(v_*(s)-w_*(s)\right)\right|ds\\
&\le\int_0^t\left[\left(\sigma_1(s)+c_1(s)v_*(s)+b_1(s)\left(u_*(s)+z_*(s)\right)\right)\left|u_*(s)-z_*(s)\right|+c_1(s)z_*(s)\left|v_*(s)-w_*(s)\right|\right]ds\\
&\le\int_0^t h_1(s)\left(\left|u_*(s)-z_*(s)\right|+\left|v_*(s)-w_*(s)\right|\right)ds,
\end{aligned}$$

$$\begin{aligned}
\left|v_*(t)-w_*(t)\right| &\le \int_0^t\left|g\left(s,u_*(s),v_*(s)\right)-g\left(s,z_*(s),w_*(s)\right)\right|ds\\
&=\int_0^t\left|\left[\sigma_2(s)-c_2(s)u_*(s)-b_2(s)\left(v_*(s)+w_*(s)\right)\right]\left(v_*(s)-w_*(s)\right)-c_2(s)w_*(s)\left(u_*(s)-z_*(s)\right)\right|ds\\
&\le\int_0^t\left[\left(\sigma_2(s)+c_2(s)u_*(s)+b_2(s)\left(v_*(s)+w_*(s)\right)\right)\left|v_*(s)-w_*(s)\right|+c_2(s)w_*(s)\left|u_*(s)-z_*(s)\right|\right]ds\\
&\le\int_0^t h_2(s)\left(\left|u_*(s)-z_*(s)\right|+\left|v_*(s)-w_*(s)\right|\right)ds,
\end{aligned}$$

We have

$$\begin{aligned}
&\left|u_*(t)-z_*(t)\right|+\left|v_*(t)-w_*(t)\right|\\
&\le\int_0^t\left(h_1(s)+h_2(s)\right)\left(\left|u_*(s)-z_*(s)\right|+\left|v_*(s)-w_*(s)\right|\right)ds\\
&\le\int_0^t\theta e_0\,ds=e_0\theta\Delta t, t\in[0,t_1].
\end{aligned}$$

This means that $e_0\le\int_0^t\theta e_0\,ds\le e_0\theta\Delta t=e_0\theta\frac{T}{N}$, but $\theta\frac{T}{N}<1$ implies $e_0=0$, hence

$$u_*(t)=z_*(t),\ v_*(t)=w_*(t),\ t\in[0,t_1].$$

Repeating this process on $[t_i, t_{i+1}](i=1,2,3,\ldots,N-1)$, we obtain

$$\begin{aligned}&|u_*(t)-z_*(t)|+|v_*(t)-w_*(t)|\\ &\le \int_{t_i}^{t}\left(h_1(s)+h_2(s)\right)\left(|u_*(s)-z_*(s)|+|v_*(s)-w_*(s)|\right)ds\\ &\le \int_{t_i}^{t}\theta e_i ds \le e_i\frac{T}{N}\theta, t\in[t_i,t_{i+1}],\end{aligned}$$

thus $e_i=0$, $t\in[t_i,t_{i+1}]$ $(i=1,2,3,\ldots,N-1)$. Therefore, $u_*(t)=z_*(t)$, $v_*(t)=w_*(t)$, $t\in[0,T]$.

The proof is completed.

Corollary 1. *If $b_i(t)$, $\sigma_i(t)$, $c_i(t)$ are constants and (P_3) - (P_4) hold, then the conclusions of Theorem 3 hold.*

Remark 2. For the following system of form

$$\begin{cases}\dfrac{du(t)}{dt}=u(t)\left[\sigma_1^*(t)-b_1u(t)-c_1(t)v(t)\right]-d_1(t)u(t),\\ \dfrac{dv(t)}{dt}=v(t)\left[\sigma_2^*(t)-b_2v(t)-c_2(t)u(t)\right]-d_2(t)v(t),\\ \qquad u(0)=u_0>0, v(0)=v_0>0, t>0.\end{cases}$$

Let $\sigma_i(t)=\sigma_i^*(t)-d_i(t)$, Then we can rewrite the above model as (1). As long as $\sigma_i(t)$ satisfies the condition (P_2), we can use Theorem 3 to construct its iterative solution and perform error estimations.

Remark 3. The iterative sequences $\{u_n(t)\}$, and $\{v_n(t)\}$, defined by (8) and (9), $u_*(t)$ and $\{v_*(t)\}$ in Theorem 3 have the following properties

(1) $u_n(t)$ and $v_n(t)$ are increasing on $[0,T]$. So are $u_*(t)$ and $v_*(t)$.

(2) $u_0\le u_*(t)\le u_0e^{\int_0^t\sigma_1(s)ds}$ and $v_0\le v_*(t)\le v_0e^{\int_0^t\sigma_2(s)ds}$ on $[0,T]$.

In fact, since $u_{n-1}, v_{n-1}\in\Omega$, we see

$$\begin{aligned}\sigma_1(t)-b_1(t)u_{n-1}(t)-c_1(t)v_{n-1}(t)\ge 0,\\ \sigma_2(t)-b_2(t)v_{n-1}(t)-c_2(t)u_{n-1}(t)\ge 0,\end{aligned}$$

for $t\in[0,T]$. From

$$\begin{cases}\dfrac{du_n(t)}{dt}=u_{n-1}(t)\left[\sigma_1(t)-b_1(t)u_{n-1}(t)-c_1(t)v_{n-1}(t)\right],\\ \dfrac{dv_n(t)}{dt}=v_{n-1}(t)\left[\sigma_2(t)-b_2(t)v_{n-1}(t)-c_2(t)u_{n-1}(t)\right].\end{cases}$$

we know $\dfrac{du_n(t)}{dt}\ge 0$ and $\dfrac{dv_n(t)}{dt}\ge 0$ on $[0,T]$ and $u_n(t)$ and $v_n(t)$ are increasing on $[0,T]$, which imply that $u_*(t)$ and $v_*(t)$ have same property.

The increasing of $u_*(t)$ and $v_*(t)$ show $u_0\le u_*(t)$ and $v_0\le v_*(t)$ on $[0,T]$. On the other hand, we have

$$\begin{cases}\dfrac{du_*(t)}{dt}=u_*(t)\left[\sigma_1(t)-b_1(t)u_*(t)-c_1(t)v_*(t)\right],\\ \dfrac{dv_*(t)}{dt}=v_*(t)\left[\sigma_2(t)-b_2(t)v_*(t)-c_2(t)u_*(t)\right],\\ \qquad u(0)=u_0>0,\ v(0)=v_0>0,\end{cases}$$

we know $\dfrac{du_*(t)}{dt}\le\sigma_1(t)u_*(t)$ and $\dfrac{dv_*(t)}{dt}\le\sigma_2(t)v_*(t)$ on $[0,T]$ and $u_*(t)\le u_0e^{\int_0^t\sigma_1(s)ds}$ and $v_*(t)\le v_0e^{\int_0^t\sigma_2(s)ds}$ on $[0,T]$ by the Gronwall inequality [19].

Remark 4. If $b_i(t),\sigma_i(t),c_i(t)$ are all normal numbers, then $\{u_n(t)\}$ and $\{v_n(t)\}$ can be expressed in polynomial form

$$u_n(t)=\sum_{i=0}^{2^n-1}\alpha_i^{(n)}t^i \ and\, v_n(t)=\sum_{i=0}^{2^n-1}\beta_i^{(n)}t^i \ , t\in[0,T], n\geq 1.$$

In fact, through (7)-(9), it is easy to know that since $b_i(t),\sigma_i(t),c_i(t)$ are normal numbers, u_1 and v_1 are linear polynomials, u_2 and v_2 are cubic polynomials. By induction, it can be concluded that $u_n(t)=\Sigma_{i=0}^{2^n-1}\alpha_i^{(n)}t^i$ and $v_n(t)=\Sigma_{i=0}^{2^n-1}\beta_i^{(n)}t^i$, $t\in[0,T], n\geq 1$.

III. CONCLUSION AND EXTENSION

For the Lotka-Volterra model (1), this paper provides some conditions for the existence and uniqueness of positive solutions, the iterative algorithms and error estimations. To the best of our knowledge, there is little study on this aspect, and this study fills this gap.

Finally, we consider the Lotka-Volterra model consisting of more than two equations $(L-V)_n$:

$$\begin{cases} \dfrac{du_1(t)}{dt}=u_1(t)\left[\sigma_1(t)-a_{11}(t)u_1(t)-a_{12}(t)u_2(t)-\cdots-a_{1n}(t)u_n(t)\right]:=f_1\left(t,u_1(t),u_2(t),\cdots,u_n(t)\right), \\ \dfrac{du_2(t)}{dt}=u_2(t)\left[\sigma_2(t)-a_{21}(t)u_1(t)-a_{22}(t)u_2(t)-\cdots-a_{2n}(t)u_n(t)\right]:=f_2\left(t,u_1(t),u_2(t),\cdots,u_n(t)\right), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\ldots\ldots \\ \dfrac{du_n(t)}{dt}=u_n(t)\left[\sigma_n(t)-a_{n1}(t)u_1(t)-a_{n2}(t)u_2(t)-\cdots-a_{nn}(t)u_n(t)\right]:=f_n\left(t,u_1(t),u_2(t),\cdots,u_n(t)\right), \\ \qquad\qquad u_i(0)=u_i^{(0)}>0, i=1,2,\cdots,n, t>0. \end{cases}$$

If $\sigma_i(t)$, $a_{ij}(t)(i,j=1,2,\ldots n)$ satisfy (P_1) and (P_2), then the same results as Theorem 3 hold. We provide a framework of main results.

Let

$$\sigma_i=\inf\{\sigma_i(t):t\in[0,\infty)\}(i=1,2,\ldots,n),$$
$$a_{ij}=\sup\left\{a_{ij}(t):t\in[0,\infty)\},i,j=1,2,\ldots,n\right\},$$
$$\sigma=\max\left\{\sigma_i,i=1,2,\ldots,n\right\},$$

Then $0<\sigma<\infty, 0<a_{ij}<\infty(i,j=1,2,\ldots,n)$.

We assume that $u_i^{(0)}$ satisfies

$$\sum_{j=1}^{n}a_{ij}u_j^{(0)}<\sigma_i, i=1,2,\ldots,n.$$

Then there must be $\tau>0$ such that

$$\sum_{j=1}^{n}a_{ij}u_j^{(0)}\leq\sigma_i(1-\sigma\tau), i=1,2,\ldots,n.$$

Let

$$T=\sup\left\{\tau:\sum_{j=1}^{n}a_{ij}u_j^{(0)}\leq\sigma_i(1-\sigma\tau), i=1,2,\ldots,n\right\}.$$

Then $0<T<\infty$.

Notation 3. $\Omega=\left\{\left(u_1,u_2,\ldots,u_n\right):\sum_{j=1}^{n}a_{ij}u_j(t)\leq\sigma_i, u_i\in C_+[0,T]\right\}$,

$$\begin{aligned} &A_i\left(u_1,u_2,\ldots,u_n\right)(t) \\ &=\int_0^t f_i\left(s,u_1(s),u_2(s),\ldots,u_n(s)\right)ds+u_i^{(0)}, \left(u_1,u_2,\ldots,u_n\right)\in\Omega, i=1,2,\ldots,n, \end{aligned}$$
$$u_i^{(k)}(t)=A_i\left(u_1^{(k-1)},u_2^{(k-1)},\ldots,u_n^{(k-1)}\right)(t), k=1,2,\ldots\to(15)$$
$$(u_1^{(0)}(t),u_2^{(0)}(t),\ldots u_n^{(0)}(t))=\left(u_1^{(0)},u_2^{(0)},\ldots,u_n^{(0)}\right).$$

We can prove

Theorem 4. *For* $(u_1,u_2,\ldots,u_n)\in\Omega$, *we have*

$$(A_1(u_1,u_2,\ldots,u_n),\ldots,A_n(u_1,u_2,\ldots,u_n))\in\Omega.$$

Theorem 5. $\{(u_1^{(k)}(t),u_2^{(k)}(t),\ldots,u_n^{(k)}(t))\}$ *converge uniformly on* $[0,T]$.

We denote the limit of $\{u_i^{(k)}(t)\}$ by $u_*^{(i)}(t)=\lim_{k\to\infty}u_i^{(k)}(t), i=1,2,\dots,n$. Letting $k\to\infty$ in (15), we know

Theorem 6. $(u_*^{(1)},u_*^{(2)},...,u_*^{(n)})$ *is a unique positive solution of* $(L-V)_n$.

We may establish error estimations similar to Theorem 3 in section II, all details including the proof of Theorem 4 to Theorem 6 are omitted due to the duplication of most work.

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