Quest Journals
Journal of Research in Applied Mathematics
Volume 2~ Issue 10 (2016) pp: 10-15
ISSN(Online) : 2394-0743 ISSN (Print):2394-0735
www.questjournals.org

Research Paper

Generalized Difference Entire Sequence Spaces Defined By Musielak-Orlicz Function

Zakawat U. Siddiqui and Ado Balili
Department of Mathematics and Statistics, University Of Maiduguri, Borno State, Nigeria

Received 28 June, 2016; **A**ccepted 26 July, 2016 © The author(s) 2016. **P**ublished with open access at **www.questjournals.org**

ABSTRACT: *In this paper, we introduce generalized difference entire sequence spaces by means of Musielak-Orlicz functions and study some of their topological properties and a few inclusion relations among them.*
Keywords: *Analytic Sequence, Entire Sequences, Generalized Difference Sequences, Musielak-Orlicz Functions, Solid Sequences.*
Mathematics Subject Classification: *40A05, 40C05, 40D05*

I. INTRODUCTION

The notion of difference sequence spaces was introduced by Kizmaz [1], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. This notion was further generalized by Et [2] who defined the sequence spaces $l_\infty(\Delta^2)$, $c(\Delta^2)$ and $c_0(\Delta^2)$. Later, Et and Colak [3] defined the sequence spaces $l_\infty(\Delta^m)$, $c(\Delta^m)$ and $c_0(\Delta^m)$. Et and Esi [4], then defined the following spaces:

$$\ell_\infty(\Delta_\nu^m) = \{x = (x_k) \in \omega : (\Delta_\nu^m x_k) \in \ell_\infty\}$$
$$c(\Delta_\nu^m) = \{x = (x_k) \in \omega : (\Delta_\nu^m x_k) \in c\}$$
$$c_0(\Delta_\nu^m) = \{x = (x_k) \in \omega : (\Delta_\nu^m x_k) \in c_0\},$$

where

$$\Delta_\nu^m x = (\Delta_\nu^m x_k) = (\Delta_\nu^{m-1} x_k - \Delta_\nu^{m-1} x_{k+1}, \text{ and } \Delta_\nu^0 x_k = x_k$$

for all $k \in N$, which is equivalent to binomial representation

$$\Delta_\nu^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{+\nu i}$$

It was proved that the generalized sequence space $Z(\Delta_\nu^m)$, where $Z = \ell_\infty, c$ or c_0, is a Banach space with norm defined by

$$\|x\|_{\Delta_\nu^m} = \sum_{i=1}^{m} |x_i| + \sup|\Delta_\nu^m x_k|.$$

Taking $\nu = 1$, we get the spaces which were studied by Et and Colak [3].

Taking $m = \nu = 1$, we get the spaces which were introduced and studied by Kizmaz [1].

A complex sequence whose k^{th} term is denoted by x_k is said to be analytic if $\sup_k |x_k|^{1/k} < \infty$. The vector space of all analytic sequences will be denoted by Λ. A sequence $x = (x_k)$ is said to be entire if $\lim_{k\to\infty} |x_k|^{1/k} = 0$. The space of all entire sequences is denoted by Γ.

Orlicz function is defined as the function $M : [0, \infty) \to [0, \infty)$, which is continuous, non-decreasing and convex such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. Lindenstrauss and Tzafriri [4] used the concept of Orlicz functions to define the space

$$\ell_M = \left\{x \in \omega : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty\right\}. \tag{1.1}$$

called Orlicz sequence space, and proved that every Orlicz sequence space contains a subspace isomorphic to $\ell_p (1 \le p < \infty)$. Subsequently, different classes of sequence spaces were defined by Parashar and Choudhary [5], Mursaleen et al [6], Bektas and Altin [7], Tripathy et al [8], Rao and Subramaniam [9] and many others.

It is to be noted that if the convexity in an Orlicz function is replaced by the condition $M(x + y) \le M(x) + M(y)$, then this function is called Modulus function, defined and discussed by Ruckle [10] and Maddox [11].An Orlicz function is said to satisfy the $\Delta_2 -$ condition for all values of u if there exists a constant $k > 0$ such that $M(2u) \le kM(u)$, $u \ge 0$. In other words $M(nu) \le knM(u)$, for all values of u and $n > 1$.

The sequence space ℓ_M defined in (i) is a Banach space with the norm

*Corresponding Author: Zakawat U. Siddiqui [1]
Department of Mathematics and Statistics, University Of Maiduguri, Borno State, Nigeria

$$\|x\| = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1\right\} \tag{1.2}$$

II. BASIC DEFINITIONS

Definition 2.1: The space consisting of all sequences $x \in \omega$ such that $M\left(\frac{|x_k|^{1/k}}{\rho}\right) \to 0$ as $k \to \infty$ for some arbitrary fixed $\rho > 0$, is denoted by Γ_M, with M being a modulus function. In other words $\left\{M\left(\frac{|x_k|^{1/k}}{\rho}\right)\right\}$ is a null space. The space Γ_M is a metric space with metric

$$d(x,y) = \sup_k M\left(\frac{|x_k - y_k|^{1/k}}{\rho}\right) \text{ for all } x = (x_k),\ y = (y_k).$$

The space Γ_M is called entire sequence space defined by Orlicz functions.

Definition 2.2: A sequence space S is said to be solid or normal if whenever $\{x_k\} \in S$, the sequence $\{\alpha_k x_k\} \in S$, where $\{\alpha_k\}$ is a sequence of scalars with $|\alpha_k| \le 1$.

Definition 2.3 (see [12]): Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions, then M is called Musielak-Orlicz function.

We shall use the following inequality throughout this work. Let $\{p_k\}$ be a sequence of positive real numbers with $0 < p_k < \sup p_k = P$. Let $C = 2^{P-1}$. Then

$$|a_k + b_k|^k \le C\{|a_k|^{p_k} + |b_k|^{p_k}\} \tag{2.1}$$

where a_k and b_k are complex numbers.

In this paper we shall define a new class of sequence, which is the generalization of the sequence space given in Raj et al [13], as follows:

$$\Gamma_{\mathcal{M}}(\Delta_\nu^m,\ p,q,s) = \left\{x_k \in \Gamma(x): \frac{1}{n}\sum_{k=1}^{n} k^{-s}\left[M_q\left(\frac{q|\Delta_\nu^m x_k|^{\frac{1}{k}}}{\rho}\right)\right]^{p_k}\right\} \to 0 \text{ as } n \to \infty, \text{uniformly in } n > 0,$$

$s \ge 0$ for some $\rho > 0$.

III. MAIN RESULTS

We shall prove the following theorems in this paper.

Theorem 3.1: Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $p = (p_k)$ be a sequence of strictly positive real numbers. Then the space $\Gamma_{\mathcal{M}}(\Delta_\nu^m,\ p,q,s)$ is a linear space over the field $\mathbb{C}$ of complex numbers.

Proof. Let $x = (x_k)$, $y = (y_k) \in \Gamma_{\mathcal{M}}(\Delta_\nu^m,\ p,q,s)$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive numbers $\rho_1,\ \rho_2$ such that

$$\frac{1}{n}\sum_{k=1}^{n} k^{-s}\left[M_k\left(\frac{q|\Delta_\nu^m x_k|^{\frac{1}{k}}}{\rho_1}\right)\right]^{p_k} \to 0 \text{ as } n \to \infty \tag{3.1}$$

and

$$\frac{1}{n}\sum_{k=1}^{n} k^{-s}\left[M_k\left(\frac{q|\Delta_\nu^m x_k|^{\frac{1}{k}}}{\rho_2}\right)\right]^{p_k} \to 0 \text{ as } n \to \infty \tag{3.2}$$

In order to prove the result, we need to find ρ_3 such that

$$\frac{1}{n}\sum_{k=1}^{n} k^{-s}\left[M_k\left(\frac{q|\Delta_\nu^m(\alpha x_k + \beta y_k)|^{\frac{1}{k}}}{\rho_3}\right)\right]^{p_k} \to 0 \ as\ n \to \infty \tag{3.3}$$

Let $\rho_3 = max\left(|\alpha|^{1/k}\rho_1, |\beta|^{1/k}\rho_2\right)$. Since $\mathcal{M} = (M_k)$ is non decreasing, convex and q is a semi norm, so by using inequality (2.1), we have

$$\frac{1}{n}\sum_{k=1}^{n} k^{-s}\left[M_k\left(\frac{q|\Delta_\nu^m(\alpha x_k + \beta y_k)|^{\frac{1}{k}}}{\rho_3}\right)\right]^{p_k}$$

$$\le \frac{1}{n}\sum_{k=1}^{n} k^{-s}\left[M_k\left(q\left\{\frac{\left(|\alpha|^{1/k}|\Delta_\nu^m x_k|\right)^{1/k}}{\rho_3} + \frac{\left(|\beta|^{1/k}|\Delta_\nu^m y_k|\right)^{1/k}}{\rho_3}\right\}\right)\right]^{p_k}$$

$$\le \frac{1}{n}\sum_{n=1}^{k} k^{-s}\left[M_k\left(q\left\{\frac{(|\Delta_\nu^m x_k|)^{1/k}}{\rho_1} + \frac{(|\Delta_\nu^m y_k|)^{1/k}}{\rho_2}\right\}\right)\right]^{p_k}$$

$$\le C\frac{1}{n}\sum_{n=1}^{k} k^{-s}\left[M_k\left(q\frac{|\Delta_\nu^m x_k|^{1/k}}{\rho_1}\right)\right]^{p_k} + C\frac{1}{n}\sum_{n=1}^{k} k^{-s}\left[M_k\left(q\frac{|\Delta_\nu^m y_k|^{1/k}}{\rho_2}\right)\right]^{p_k}$$

$$\to 0 \ \ as \ \ n \to \infty.$$

Thus $\alpha x + \beta y \in \Gamma_m(\Delta_\nu^m,\ p,\ q,\ s)$, showing that it is a linear space.

Theorem 3.2: Suppose $\mathcal{M}$ = (M_k) is Musielak-Orlicz function and p = (p_k) be a sequence of strictly positive real numbers. Then the space $\Gamma_{\mathcal{M}}(\Delta_\nu^m,\ p,\ q,\ s)$ is a paranormed space with the paranorm defined by

$$h(x) = inf\left\{\rho^{p_n}: \sup_{k\geq 1} k^{-s}\left[M_k\left(\frac{q|\Delta_\nu^m x_k|^{1/k}}{\rho}\right)\right]^{p_k/H} \leq 1\right\}, \text{ uniformly in n > 0, } \rho > 0,$$

where H = max $\left(1, \sup_k p_k\right)$.

Proof. Clearly $h(x) \geq 0, h(x) = h(-x)$ and $h(\theta) = 0$, where θ is the zero sequence of X. Let $x_k, y_k \in \Gamma_m(\Delta_\nu^m,\ p,\ q,\ s)$. Let $\rho_1, \rho_2 > 0$ be such that

$$\sup_{k\geq 1} k^{-s}\left[M_k\left(\frac{q|\Delta_\nu^m x_k|^{1/k}}{\rho_1}\right)\right]^{p_k/H} \leq 1$$

and,

$$\sup_{k\geq 1} k^{-s}\left[M_k\left(\frac{q|\Delta_\nu^m y_k|^{1/k}}{\rho_2}\right)\right]^{p_k/H} \leq 1.$$

Let $\rho = \rho_1 + \rho_2$, then by using Minkowski's inequality, we have

$$\sup_{k\geq 1} k^{-s}\left[M_k\left(\frac{q|\Delta_\nu^m (x_k+y_k)|^{1/k}}{\rho}\right)\right]^{p_k/H}$$

$$\leq \left(\frac{\rho_1}{\rho_1+\rho_2}\right)\sup_{k\geq 1} k^{-s}\left[M_k\left(\frac{q|\Delta_\nu^m x_k|^{\frac{1}{k}}}{\rho_1}\right)\right]^{\frac{p_k}{H}} + \left(\frac{\rho_2}{\rho_1+\rho_2}\right)\sup_{k\geq 1} k^{-s}\left[M_k\left(\frac{q|\Delta_\nu^m y_k|^{\frac{1}{k}}}{\rho_1}\right)\right]^{\frac{p_k}{H}}$$

$$\leq 1.$$

Hence

$$h(x+y) \leq inf\left\{(\rho_1+\rho_2)^{p_m/H} : \sup_{k\geq 1} k^{-s}\left[M_k\left(\frac{q|\Delta_\nu^m (x_k+y_k)|^{1/k}}{\rho_1+\rho_2}\right)\right]^{p_k/H} \leq 1, \rho_1, \rho_2 > 0,\ m \in \mathbb{N}\right\}$$

$$\leq\ inf\left\{(\rho_1)^{p_m/H} : \sup_{k\geq 1} k^{-s}\left[M_k\left(\frac{q|\Delta_\nu^m (x_k+y_k)|^{1/k}}{\rho_1}\right)\right]^{p_k/H} \leq 1, \rho_1 > 0,\ m \in \mathbb{N}\right\} +$$

$$+inf\left\{(\rho_2)^{p_m/H} : \sup_{k\geq 1} k^{-s}\left[M_k\left(\frac{q|\Delta_\nu^m (x_k+y_k)|^{1/k}}{\rho_2}\right)\right]^{p_k/H} \leq 1, \rho_2 > 0,\ m \in \mathbb{N}\right\}$$

Thus we have $h(x+y) \leq h(x) + h(y)$. Hence h satisfies the triangle inequality. Now,

$$h(\lambda x) =\ inf\left\{(\rho)^{p_m/H} : \sup_{k\geq 1} k^{-s}\left[M_k\left(\frac{q|\lambda\Delta_\nu^m (x_k)|^{1/k}}{\rho}\right)\right]^{p_k/H} \leq 1, \rho > 0, m \in \mathbb{N}\right\}$$

$$= inf\left\{(r|\lambda|)^{p_m/H} : \sup_{k\geq 1} k^{-s}\left[M_k\left(\frac{q|\Delta_\nu^m (x_k)|^{1/k}}{r}\right)\right]^{p_k/H} \leq 1, r > 0, m \in \mathbb{N}\right\}$$

where $r = \frac{\rho}{|\lambda|}$. Hence $\Gamma_m(\Delta_\nu^m,\ p,\ q,\ s)$ is a paranormed space.

Theorem 3.3: Let $\mathcal{M}' = (M_k')$ and $M'' = (M_k'')$ be two Musielak-Orlicz functions.Then

$$\Gamma_{\mathcal{M}'}(\Delta_\nu^m,\ p,\ q,\ s) \cap \Gamma_{\mathcal{M}''}(\Delta_\nu^m,\ p,\ q,\ s) \subseteq \Gamma_{\mathcal{M}'+\mathcal{M}''}(\Delta_\nu^m,\ p,\ q,\ s).$$

Proof. Let $x \in \Gamma_{\mathcal{M}'}(\Delta_\nu^m,\ p,\ q,\ s) \cap \Gamma_{\mathcal{M}''}(\Delta_\nu^m,\ p,\ q,\ s)$. Then there exists ρ_1 and ρ_2 such that

$$\frac{1}{n}\sum_{k=1}^{n} k^{-s}\left[M_k\left(\frac{q|\Delta_\nu^m x_k|^{\frac{1}{k}}}{\rho_1}\right)\right]^{p_k} \to 0 \ as\ n \to \infty \tag{3.4}$$

and

$$\frac{1}{n}\sum_{k=1}^{n} k^{-s}\left[M_k\left(\frac{q|\Delta_\nu^m x_k|^{\frac{1}{k}}}{\rho_2}\right)\right]^{p_k} \to 0 \ as\ n \to \infty \tag{3.5}$$

Let $\rho = min(\frac{1}{\rho_1}, \frac{1}{\rho_2})$. Then we have

$$\frac{1}{n}\sum_{k=1}^{n} k^{-s}\left[(M_k' + M_k'')\left(\frac{q|\Delta_\nu^m x_k|^{\frac{1}{k}}}{\rho}\right)\right]^{p_k}$$

$$\leq K\frac{1}{n}\sum_{k=1}^{n} k^{-s}\left[M_k'\left(\frac{q|\Delta_\nu^m x_k|^{\frac{1}{k}}}{\rho_1}\right)\right]^{p_k} + K\frac{1}{n}\sum_{k=1}^{n} k^{-s}\left[M_k''\left(\frac{q|\Delta_\nu^m x_k|^{\frac{1}{k}}}{\rho_2}\right)\right]^{p_k}$$

$$\to 0\ , as\ n\ \to \infty$$

by (3.4) and (3.5). Then

$$\frac{1}{n}\sum_{k=1}^{n} k^{-s}\left[(M_k' + M_k'')\left(\frac{q|\Delta_\nu^m x_k|^{\frac{1}{k}}}{\rho}\right)\right]^{p_k} \to 0\ as\ n\ \to \infty.$$

Therefore $x \in \Gamma_{\mathcal{M}'+\mathcal{M}''}(\Delta_\nu^m,\ p,\ q,\ s)$.

Theorem 3.4 Let $m \geq 1$. Then we have

$$\Gamma_{\mathcal{M}}(\Delta_v^{m-1}, p, q, s) \subseteq \Gamma_{\mathcal{M}}(\Delta_v^m, p, q, s)$$

Proof. Let $x \in \Gamma_{\mathcal{M}}(\Delta_v^{m-1},\ p,\ q,\ s)$. Then we have

$$\frac{1}{n}\sum_{k=1}^{n} k^{-s}\left[M_k\left(\frac{q|\Delta_v^{m-1}x_k|^{\frac{1}{k}}}{\rho}\right)\right]^{p_k} \to 0 \ as\ n \to \infty \text{ for some } \rho > 0.$$

Since $\mathcal{M} = (M_k)$ is non decreasing, convex function and q is seminorm, then we have

$$\frac{1}{n}\sum_{k=1}^{n} k^{-s}\left[M_k\left(\frac{q|\Delta_v^{m}x_k|^{\frac{1}{k}}}{\rho}\right)\right]^{p_k} \leq \frac{1}{n}\sum_{k=1}^{n} k^{-s}\left[M_k\left(\frac{q|\Delta_v^{m-1}x_k - \Delta_v^{m-1}x_{k+1}|^{\frac{1}{k}}}{\rho}\right)\right]^{p_k}$$

$$\leq K\{\frac{1}{n}\sum_{k=1}^{n} k^{-s}\left[M_k\left(\frac{q|\Delta_v^{m-1}x_k|^{\frac{1}{k}}}{\rho}\right)\right]^{p_k} + \frac{1}{n}\sum_{k=1}^{n} k^{-s}\left[M_k\left(\frac{q|\Delta_v^{m-1}x_{k+1}|^{\frac{1}{k}}}{\rho}\right)\right]^{p_k}\}$$

$\to 0\ as\ n \to \infty$.

Therefore,

$$\frac{1}{n}\sum_{k=1}^{n} k^{-s}\left[(M_k\left(\frac{q|\Delta_v^{m}x_k|^{\frac{1}{k}}}{\rho}\right)\right]^{p_k} \to 0\ as\ n \to \infty.$$

Hence $x \epsilon\ \Gamma_{\mathcal{M}}(\Delta_v^m,\ p,\ q,\ s)$. This completes the proof.

Theorem 3.5: Suppose $\frac{1}{n}\sum_{k=1}^{n} k^{-s}\left[(M_k\left(\frac{q|\Delta_v^{m}x_k|^{\frac{1}{k}}}{\rho}\right)\right]^{p_k} \leq |x_k|^{\frac{1}{k}}$, then $\Gamma \subset \Gamma_{\mathcal{M}}(\Delta_v^m, p, q, s)$.

Proof. Let $x \in \Gamma$. Then we have

$$|x_k|^{\frac{1}{k}} \to 0\ as\ k \to \infty \qquad (3.6)$$

But $\frac{1}{n}\sum_{k=1}^{n} k^{-s}\left[(M_k\left(\frac{q|\Delta_v^{m}x_k|^{\frac{1}{k}}}{\rho}\right)\right]^{p_k} \leq |x_k|^{\frac{1}{k}}$ by our assumption, it implies that

$$\frac{1}{n}\sum_{k=1}^{n} k^{-s}\left[(M_k\left(\frac{q|\Delta_v^{m}x_k|^{\frac{1}{k}}}{\rho}\right)\right]^{p_k} \to 0\ as\ n \to \infty$$

by (3.6).

Then, $x \in \Gamma_{\mathcal{M}}(\Delta_v^m, p, q, s)$ and hence $\Gamma \subset \Gamma_{\mathcal{M}}(\Delta_v^m, p, q, s)$.

Theorem 3.6: $\Gamma_{\mathcal{M}}(\Delta_v^m, p, q, s)$ is solid

Proof. Let $|x_k| \leq |y_k|$ and $y = (y_k) \in \Gamma_{\mathcal{M}}(\Delta_v^m, p, q, s)$.

Since $\mathcal{M} = (M_k)$ is non decreasing, it implies that

$$\frac{1}{n}\sum_{k=1}^{n} k^{-s}\left[(M_k\left(\frac{q|\Delta_v^{m}x_k|^{\frac{1}{k}}}{\rho}\right)\right]^{p_k} \leq \frac{1}{n}\sum_{k=1}^{n} k^{-s}\left[(M_k\left(\frac{q|\Delta_v^{m}y_k|^{\frac{1}{k}}}{\rho}\right)\right]^{p_k}$$

Since $y \in \Gamma_{\mathcal{M}}(\Delta_v^m, p, q, s)$. Therefore,

$$\frac{1}{n}\sum_{k=1}^{n} k^{-s}\left[(M_k\left(\frac{q|\Delta_v^{m}y_k|^{\frac{1}{k}}}{\rho}\right)\right]^{p_k} \to 0\ as\ n \to \infty$$

and

$$\frac{1}{n}\sum_{k=1}^{n} k^{-s}\left[(M_k\left(\frac{q|\Delta_v^{m}x_k|^{\frac{1}{k}}}{\rho}\right)\right]^{p_k} \to 0\ as\ n \to \infty$$

Therefore $x = (x_k) \in \Gamma_{\mathcal{M}}(\Delta_v^m, p, q, s)$. Hence the result.

Theorem 3.7: (i) Let $0 < infp_k \leq p_k \leq 1$. Then $\Gamma_{\mathcal{M}}(\Delta_v^m, p, q, s) \subset \Gamma_{\mathcal{M}}(\Delta_v^m,\ q, s)$

(ii) Let $1 \leq p_k \leq supp_k < \infty$. Then $\Gamma_{\mathcal{M}}(\Delta_v^m, q, s) \subset \Gamma_{\mathcal{M}}(\Delta_v^m, p, q, s)$

Proof. (i) Let $x \in \Gamma_{\mathcal{M}}(\Delta_v^m, p, q, s)$. Then

$$\frac{1}{n}\sum_{k=1}^{n} k^{-s}\left[(M_k\left(\frac{q|\Delta_v^{m}x_k|^{\frac{1}{k}}}{\rho}\right)\right]^{p_k} \to 0\ as\ n \to \infty \qquad (3.7)$$

Since $0 < infp_k \leq p_k \leq 1$.

$$\frac{1}{n}\sum_{k=1}^{n} k^{-s}\left[(M_k\left(\frac{q|\Delta_v^{m}x_k|^{\frac{1}{k}}}{\rho}\right)\right] \leq \frac{1}{n}\sum_{k=1}^{n} k^{-s}\left[(M_k\left(\frac{q|\Delta_v^{m}x_k|^{\frac{1}{k}}}{\rho}\right)\right]^{p_k} \to 0\ as\ n \to \infty \qquad (3.8)$$

From (3.7) and (3.8) it follows that, $x \in \Gamma_{\mathcal{M}}(\Delta_v^m,\ q, s)$

Thus $\Gamma_{\mathcal{M}}(\Delta_v^m, p, q, s) \subset \Gamma_{\mathcal{M}}(\Delta_v^m,\ q, s)$

(ii) Let $p_k \geq 1$ for each k and $supp_k < \infty$ and let $x \in \Gamma_{\mathcal{M}}(\Delta_\nu^m, q, s)$ then

$$\frac{1}{n}\sum_{k=1}^{n}\left[(M_k\left(\frac{q|\Delta_\nu^m x_k|^{\frac{1}{k}}}{\rho}\right)\right] \to 0 \; as \; n \to \infty \tag{3.9}$$

Since $1 \leq p_k \leq supp_k < \infty$, we have

$$\frac{1}{n}\sum_{k=1}^{n} k^{-s}\left[(M_k\left(\frac{q|\Delta_\nu^m x_k|^{\frac{1}{k}}}{\rho}\right)\right]^{p_k} \leq \frac{1}{n}\sum_{k=1}^{n} k^{-s}\left[(M_k\left(\frac{q|\Delta_\nu^m x_k|^{\frac{1}{k}}}{\rho}\right)\right]$$

$$\Rightarrow \frac{1}{n}\sum_{k=1}^{n} k^{-s}\left[(M_k\left(\frac{q|\Delta_\nu^m x_k|^{\frac{1}{k}}}{\rho}\right)\right]^{p_k} \to 0 \; as \; n \to \infty$$

This implies that $x \in \Gamma_{\mathcal{M}}(\Delta_\nu^m, p, q, s)$.Therefore, $\Gamma_{\mathcal{M}}(\Delta_\nu^m, q, s) \subset \Gamma_{\mathcal{M}}(\Delta_\nu^m, p, q, s)$.

Theorem 3.8 Let $0 \leq p_k \leq r_k$ and let $(\frac{r_k}{p_k})$ be bounded then $\Gamma_{\mathcal{M}}(\Delta_\nu^m, r, q, s) \subset \Gamma_{\mathcal{M}}(\Delta_\nu^m, p\, q, s)$.

Proof. Let $x \in \Gamma_{\mathcal{M}}(\Delta_\nu^m, r, q, s)$. Then

$$\frac{1}{n}\sum_{k=1}^{n} k^{-s}\left[(M_k\left(\frac{q|\Delta_\nu^m x_k|^{\frac{1}{k}}}{\rho}\right)\right]^{r_k} \to 0 \; as \; n \to \infty \tag{3.10}$$

Let $t_k = \frac{1}{n}\sum_{k=1}^{n} k^{-s}\left[(M_k\left(\frac{q|\Delta_\nu^m x_k|^{\frac{1}{k}}}{\rho}\right)\right]^{r_k}$ and $\lambda_k = \frac{p_k}{r_k}$, since $p_k \leq r_k$, we have $0 \leq \lambda_k \leq 1$. Take $0 < \lambda < \lambda_k$. Define

$$u_k = \begin{cases} t_k, & if\ t_k \geq 1 \\ 0, & if\ t_k < 1 \end{cases}$$

and,

$$v_k = \begin{cases} 0, & if\ t_k \geq 1 \\ t_k, & if\ t_k < 1 \end{cases}$$

$t_k = u_k + v_k$, $\quad {t_k}^{\lambda_k} = {u_k}^{\lambda_k} + {v_k}^{\lambda_k}$. It follows that ${u_k}^{\lambda_k} \leq u_k \leq t_k$, ${v_k}^{\lambda_k} \leq {v_k}^{\lambda}$. Since, ${t_k}^{\lambda_k} = {u_k}^{\lambda_k} + {v_k}^{\lambda_k}$, then ${t_k}^{\lambda_k} \leq t_k + {v_k}^{\lambda}$.Now

$$\frac{1}{n}\sum_{k=1}^{n} k^{-s}\left[(M_k\left(\frac{q|\Delta_\nu^m x_k|^{\frac{1}{k}}}{\rho}\right))^{r_k}\right]^{\lambda_k} \leq \frac{1}{n}\sum_{k=1}^{n} k^{-s}\left[(M_k\left(\frac{q|\Delta_\nu^m x_k|^{\frac{1}{k}}}{\rho}\right)\right]^{r_k}$$

$$\Rightarrow \frac{1}{n}\sum_{k=1}^{n} k^{-s}\left[(M_k\left(\frac{q|\Delta_\nu^m x_k|^{\frac{1}{k}}}{\rho}\right))^{r_k}\right]^{\frac{p_k}{r_k}} \leq \frac{1}{n}\sum_{k=1}^{n} k^{-s}\left[(M_k\left(\frac{q|\Delta_\nu^m x_k|^{\frac{1}{k}}}{\rho}\right)\right]^{r_k}$$

$$\Rightarrow \frac{1}{n}\sum_{k=1}^{n} k^{-s}\left[(M_k\left(\frac{q|\Delta_\nu^m x_k|^{\frac{1}{k}}}{\rho}\right))^{r_k}\right]^{p_k} \leq \frac{1}{n}\sum_{k=1}^{n} k^{-s}\left[(M_k\left(\frac{q|\Delta_\nu^m x_k|^{\frac{1}{k}}}{\rho}\right)\right]^{r_k}$$

But

$$\frac{1}{n}\sum_{k=1}^{n} k^{-s}\left[(M_k\left(\frac{q|\Delta_\nu^m x_k|^{\frac{1}{k}}}{\rho}\right)\right]^{r_k} \to 0 \; as \; n \to \infty$$

by (3.10). Therefore,

$$\frac{1}{n}\sum_{k=1}^{n} k^{-s}\left[(M_k\left(\frac{q|\Delta_\nu^m x_k|^{\frac{1}{k}}}{\rho}\right))^{r_k}\right]^{p_k} \to 0 \; as \; n \to \infty$$

Hence $x \in \Gamma_{\mathcal{M}}(\Delta_\nu^m, p, q, s)$. Thus we get

$$\Gamma_{\mathcal{M}}(\Delta_\nu^m, r, q, s) \subset \Gamma_{\mathcal{M}}(\Delta_\nu^m, p, q, s)$$

IV. CONCLUSION

We observe that the difference sequence space $x \in \Gamma_{\mathcal{M}}(\Delta_\nu^m, p, q, s)$ is not only a linear space but also a paranormed space when the given sequence p = (p_k) contains strictly positive terms. Further the space is also solid. Moreover, the intersection of the spaces defined by two Musielak-Orlicz functions is identical with the space defined by the addition of the two given functions.

REFERENCES

[1]. H. Kizmaz, On certain Sequence spaces, Canada Math. Bull. 24 (2) (1981), 169-176.
[2]. M. Et, On some difference sequence spaces, Doga-Tr. J. Math. 17 (1993), 18-24.
[3]. M. Et, and R. Colak, On generalized difference sequence spaces, Soochow J. Math. 21 (4) (1995), 377-386.
[4]. J. Lindenstrauss and L. Tzafriri, On Orlicz sequence spaces, Israel J. Math. 10 (1971), 379-390.
[5]. S. D. Parashar and B. Choudhary, Sequence spaces defined by Orlicz function, Indian J. Pure Appl. Math. (1994), 419-428.

[6]. M. Mursaleen, M.A Khan and Qamaruddin, Difference sequence spaces defined by Orlicz functions, Demonstration Math. (1999), 145-150.
[7]. C. Bektas and Y. Altin, The sequence space $L_M(p,q,s)$ on semi normed spaces, Indian J. Pure Appl. Math. (2003), 529-534.
[8]. B. C. Tripathy, M. Et and Y. Altin, Generalized difference sequence spaces defined by Orlicz functions in a locally convex spaces, J. Analysis and Application, (2003), 175-192.
[9]. K. C. Rao and N. Subramanian, The Orlicz space of entire sequences, Int. J. Math. Sci.,(2004), 3755-3764.
[10]. W. H. Ruckle, FK Spaces in which the sequence of coordinate vectors is bounded, Canada J. Math., (1973), 937-978.
[11]. I. J. Maddox, Sequence spaces defined by a modulus, Math. Proc. Camb. Philos. Soc., 100 (1978), 161-166.
[12]. J. Musielak, Orlicz spaces and Modular spaces, Lecture Notes in Mathematics, (1983), 1034.
[13]. K. Raj, S. K. Sharma and A. Gupta, Entire sequence spaces defined by Musielak Orlicz function, Int. J. of Mathematical sciences and Application, 1 (2) (2011), 954-960.