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Research Paper

Solutions of Ecological models by Homotopy-Perturbation Method (HPM)

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ABSTRACT: *In this paper, He's homotopy perturbation method(HPM) is used to find approximate analytical solutions of a system of nonlinear differential equations which are the mathematical models of two interacting species in population dynamics. The HPM method is straight forward, highly effective and a promising tool for obtaining the approximate analytical solution of non-linear ODE's. Three different models having commensalism as interaction between the species are considered. The functional response differs in each model. He's Homotopy-Perturbation Method (HPM) is used in the present study to find approximate solutions of these models and the solution curves are drawn using MATLAB. The solution curves obtained by HPM and R-K 4th order method are depicted in the same graph and compared.*

Keywords: *Homotopy-Perturbation Method (HPM), non-linear differential equations, commensalism, commensal, host, Monod model.*

I. INTRODUCTION

The non-linear phenomena play a crucial role in applied mathematics and science. It is one of the most stimulating areas of the research. In the research of past few decades [1-6] great progress was made in the development of methods for obtaining approximate analytical solutions for non-linear differential equations arising in various fields of Science and Engineering. It is observed that most of the methods require a tedious analysis. Comparatively, He's Homotopy Perturbation Method (HPM) require less complicated analysis and easy to find the approximate solutions of the non-linear differential equations.

The Homotopy- Perturbation Method (HPM) was initially proposed by Chinese mathematician J.H.He [2-4]. The HPM is useful to obtain exact or approximate solutions of linear and non-linear differential equations. The primary objective of this method is to approximate the actual solution from initial approximation as the homotopy parameter, say p, varies from 0 to 1. In this method a solution is expressed as a series in p which converges to the exact solution. The key feature of this HPM is that linearization or discretization or round of errors can be evaded. The approximations converge rapidly to exact solutions [11]. HPM has been used to find approximate solutions effectively, easily and accurately a large class of non-linear problems in population dynamics, and epidemic models[1,2,6,7,8,9,10].

The objective of this paper is to explore the use of HPM for obtaining approximate solutions to biological models of population dynamics having commensalism between the species. A comparative study of these solutions with the solutions obtained using 4th order Runge- Kutta method (R-K method) is presented.

In this paper three biological models are considered in which the interaction between the species is commensalism. In model-1 and model-3 the two species have limited food source and each species follow logistic growth law in the absence of the other. The model-3 is more complex than model-1 in the sense that the commensalism is characterised by a function of host species. Unlike model-1 and model-3, model-2 the host species follows logistic growth law and has limited food sources whereas the commensal species decline in the absence of host species. The commensal species survive only because of the interaction of the host species. Description of these models is given in the next section.

II. MODELS

2.1 Model-1: This model deals with interaction of two species utilizing the same limited assets for inborn development of the species. It is assumed that the interaction of the two species benefits one of the species. The species which gets benefit is called commensal species and the species which helps for the growth of the other is called host species. Suppose $N_1 = N_1(t), N_2 = N_2(t)$ represents the size of population of the commensal and host species respectively at any time 't'. The commensal species (N_1), in spite of the limitation of its natural resources, flourishes by drawing strength from the host species (N_2). Suppose a_1, a_2 respectively represent the intrinsic growth rates; a_{11}, a_{22} represent the rates of reduction due to the limitation of natural sources of the commensal and host species. a_{12} represent beneficial factor of the commensal species by interaction with host species. All these parameters assume positive values .The mathematical model governing the commensalism between two species is given by coupled non-linear differential equations,

$$\frac{dN_1}{dt} = a_1 N_1 - a_{11} N_1^2 + a_{12} N_1 N_2$$
$$\frac{dN_2}{dt} = a_2 N_2 - a_{22} N_2^2 \qquad (1)$$

Phani Kumar.etal. [13] established that the system has four equilibrium states and the co-existent state $\left(\frac{a_1 a_{22} + a_2 a_{12}}{a_{11}\, a_{22}}, \frac{a_2}{a_{22}}\right)$ is the only stable state under the assumption

(A) $a_1 a_{22} + a_2 a_{12} \neq a_2 a_{22}$.

The global stability is analysed by a suitably constructed Liapunov's function.

2.2 Model-2: This model is concerned with the interaction of the two species having commensalism between the species. Commensal species (N_1) is weak to sustain, despite of the support of the other host species (N_2). The host species has their limited food source and the species (N_1) benefits by the interaction with the host species (N_2). Mathematically this model is represented by:

$$\frac{dN_1}{dt} = -d_1 N_1 - a_{11} N_1^2 + a_{12} N_1 N_2$$
$$\frac{dN_2}{dt} = a_2 N_2 - a_{22} N_2^2 \qquad (2)$$

Here $-d_1$ represents the natural death rate of the commensal species in the absence of (N_2) and all other parameters have same meaning as mentioned in model-1. Seshagiri Rao et.al [14] studied this model extensively and concluded that (i) N_2 will sustain forever in the absence of N_1 and tend to the equilibrium point $\left(0, \frac{a_2}{a_{22}}\right)$ **(ii)** In the coexistent state the equilibrium point $\left(\frac{a_2 a_{12} - d_1 a_{11} a_{22}}{a_{11}\, a_{22}}, \frac{a_2}{a_{22}}\right)$ is stable with the condition

(B) $d_1 < \frac{a_2 a_{12}}{a_{11} a_{22}}$.

The global stability is analysed by a suitably constructed Liapunov function.

2.3 Model-3: In this model the commensalism is represented by $F(N_2)$,a function of the host species of the form $F(N_2) = \frac{\alpha N_2}{\beta + N_2}$, instead of a linear function $a_{12} N_2$ in the model discussed in section 2.1.

It is clear that $F(N_2)$ is bounded and $F(N_2) \to$ a constant $\alpha > 0$ as $N_2 \to \infty$. Further $\beta (\neq 0)$ is a parameter which signifies the strength of the commensalism. The commensalism is strong, weak or neutral according as $\beta > 0$, $\beta < 0$ or $\beta = 0$ .The ratio $K_i = \frac{a_i}{a_{ii}}, i = 1,2$ is the carrying capacity of commensal and host species.

This mathematical representation of this model is

$$\frac{dN_1}{dt} = a_1 N_1 - a_{11} N_1^2 + N_1 F(N_2)$$

$$\frac{dN_2}{dt} = a_2 N_2 - a_{22} N_2^{\,2} \qquad (3)$$

Phani Kumar and Pattabhiramacharyulu [12] established that the system has four equilibrium states and the co-existent state $\left(\bar{N}_1, \bar{N}_2\right) = \left(\frac{1}{a_{11}}\left[K_1 a_{11} + \frac{\alpha K_2}{\beta + K_2}\right], K_2\right)$ is always stable under the assumption

(C) $\qquad K_1 a_{11} + \frac{\alpha K_2}{\beta + K_2} \neq K_2 a_{22}$.

The global stability is established by adopting Liapunov second method.

III. HOMOTOPY-PERTURBATION METHOD (HPM)

The Homotopy-Perturbation Method is a combination of the classical perturbation technique and homotopy technique. To explain the basic idea of homotopy-perturbation method, consider a non-linear differential equations.

$$A(u) - f(r) = 0, \quad r \in \Omega \qquad (4)$$

subject to the boundary condition $B\left(u, \frac{\partial u}{\partial n}\right) = 0$, $r \in \Gamma$ where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function, Γ is the boundary of the domain Ω and $\frac{\partial}{\partial n}$ denotes differentiation along the normal drawn outwards from Ω .

In general the operator A be divided into a linear part *L* and a non-linear part *N*. The equation (4) can be written as

$$L(u) + N(u) - f(r) = 0 \qquad (5)$$

By the Homotopy technique [4], construct a function $v(r, p) : \Omega \times [0,1] \to \mathbb{R}$ which satisfies

$$H(v, p) = (1-p)\left[L(v) - L(u_0)\right] + p\left[A(v) - f(r)\right] = 0, \; p \in [0,1], r \in \Omega \qquad (6)$$

$$\Rightarrow H(v, p) = L(v) - L(u_0) + pL(u_0) + p\left[N(v) - f(r)\right] = 0 \qquad (7)$$

$p \in [0,1]$ is an imbedding parameter. u_0 is initial approximate solution to equation (4),which satisfies the boundary conditions.

From the equations (6), (7)

$$H(v, 0) = L(v) - L(u_0) = 0 \qquad (8)$$

and $$H(v, 1) = A(v) - f(r) = 0 \qquad (9)$$

Thus it can be observed that as the parameter p varies from zero to unity $v(r, p)$ varies from $u_0(r)$ to $u(r)$.

Assume $v(r, p)$ as a power series in p

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + p^4 v_4 + ---------- \qquad (10)$$

The limit as $p \to 1$ v approaches the solution of equation (4)

$$u = \underset{p \to 1}{Lt}\, v = v_0 + v_1 + v_2 + v_3 + v_4 + ---------- \qquad (11)$$

3.1 Solutions Of The Models By HPM

In this section we apply the HPM explained in section 3 to the system of non-linear ordinary differential equations (1), (2) and (3).

3.1.1. Solution Of Model-1

Consider the equations (1) with initial approximations

$$\begin{aligned} v_{10}(t) &= N_{10}(t) = v_1(0) = c_1 \\ v_{20}(t) &= N_{20}(t) = v_2(0) = c_2 \end{aligned} \qquad (12)$$

As explained in section 3.1, the equations (1) can be written as

$$\begin{aligned} &v_1' - N_{10}' + p\left(N_{10}' - a_1 v_1 + a_{11} v_1^{\,2} - a_{12} v_1 v_2\right) = 0 \\ &v_2' - N_{20}' + p\left(N_{20}' - a_2 v_2 + a_{22} v_2^{\,2}\right) = 0 \end{aligned} \qquad (13)$$

Assume the approximate solutions of $N_1(t), N_2(t)$ as

$$v_1(t) = v_{1,0} + pv_{1,1}(t) + p^2 v_{1,2}(t) + p^3 v_{1,3}(t) + p^4 v_{1,4}(t) + ----------$$
$$v_2(t) = v_{2,0} + pv_{2,1}(t) + p^2 v_{2,2}(t) + p^3 v_{2,3}(t) + p^4 v_{2,4}(t) + ---------- \quad (14)$$

$v_{i,j}\,(i = 1,2;\, j = 1,2,3,4,........)$ are to be determined by substituting equations (12) and (14) into equation (13) and arranging the terms in the increasing powers of p

$$\left[v_{1,1}'(t) + N_{10}'(t) - a_1 v_{1,0}(t) + a_{11} v_{1,0}^2(t) - a_{12} v_{1,0}(t) v_{2,0}(t)\right] p + \left[v_{1,2}'(t) - a_1 v_{1,1}(t) + 2a_{11} v_{1,0}(t) v_{1,1}(t) - a_{12} v_{1,0}(t) v_{2,1}(t) - a_{12} v_{1,1}(t) v_{2,0}(t)\right] p^2$$
$$+\left[v_{1,3}'(t) - a_1 v_{1,2}(t) + 2a_{11} v_{1,0}(t) v_{1,2}(t) + a_{11} v_{1,1}^2(t) - a_{12} v_{1,0}(t) v_{2,2}(t) - a_{12} v_{1,1}(t) v_{2,1}(t) - a_{12} v_{1,2}(t) v_{2,0}(t)\right] p^3$$
$$+\left[v_{1,4}'(t) - a_1 v_{1,3}(t) + 2a_{11} v_{1,0}(t) v_{1,3}(t) + 2a_{11} v_{1,1}(t) v_{1,2}(t) - a_{12} v_{1,0}(t) v_{2,3}(t) - a_{12} v_{1,1}(t) v_{2,2}(t) - a_{12} v_{1,2}(t) v_{2,1}(t) - a_{12} v_{1,3}(t) v_{2,0}(t)\right] p^4 + ------\infty = 0$$
$$\left[v_{2,1}'(t) + N_{20}'(t) - a_2 v_{2,0}(t) + a_{22} v_{2,0}^2(t)\right] p + \left[v_{2,2}'(t) - a_2 v_{2,1}(t) + 2a_{22} v_{2,0}(t) v_{2,1}(t)\right] p^2$$
$$+\left[v_{2,3}'(t) - a_2 v_{2,2}(t) + 2a_{22} v_{2,0}(t) v_{2,2}(t) + a_{22} v_{2,1}^2(t)\right] p^3 + \left[v_{2,4}'(t) - a_2 v_{2,3}(t) + 2a_{22} v_{2,0}(t) v_{2,3}(t) + 2a_{22} v_{2,1}(t) v_{2,2}(t)\right] p^4 + -------- = 0$$

(15) To obtain the unknowns $v_{i,j}(t), i = 1,2;\, j = 1,2,3,4$ solve the following system of linear differential equations with the initial conditions given in (12)

From (15)

$$v_{1,1}' + N_{10}' - a_1 v_{1,0} + a_{11} v_{1,0}^2 - a_{12} v_{1,0} v_{2,0} = 0, \quad v_{1,1}(0) = 0 \quad (16)$$
$$v_{2,1}' + N_{20}' - a_2 v_{2,0} + a_{22} v_{2,0}^2 = 0, \quad v_{2,1}(0) = 0 \quad (17)$$
$$v_{1,2}' - a_1 v_{1,1} + 2a_{11} v_{1,0} v_{1,1} - a_{12} v_{1,0} v_{2,1} - a_{12} v_{1,1} v_{2,0} = 0, \quad v_{1,2}(0) = 0 \quad (18)$$
$$v_{2,2}' - a_2 v_{2,1} + 2a_{22} v_{2,0} v_{2,1} = 0, \quad v_{2,2}(0) = 0 \quad (19)$$
$$v_{1,3}' - a_1 v_{1,2} + 2a_{11} v_{1,0} v_{1,2} + a_{11} v_{1,1}^2 - a_{12} v_{1,0} v_{2,2} - a_{12} v_{1,1} v_{2,1} - a_{12} v_{1,2} v_{2,0} = 0, \quad v_{1,3}(0) = 0 \quad (20)$$
$$v_{2,3}' - a_2 v_{2,2} + 2a_{22} v_{2,0} v_{2,2} + a_{22} v_{2,1}^2 = 0, \quad v_{2,3}(0) = 0 \quad (21)$$
$$v_{1,4}' - a_1 v_{1,3} + 2a_{11} v_{1,0} v_{1,3} + 2a_{11} v_{1,1} v_{1,2} - a_{12} v_{1,0} v_{2,3} - a_{12} v_{1,1} v_{2,2} - a_{12} v_{1,2} v_{2,1} - a_{12} v_{1,3} v_{2,0} = 0, \quad v_{1,4}(0) = 0 \quad (22)$$
$$v_{2,4}' - a_2 v_{2,3} + 2a_{22} v_{2,0} v_{2,3} + 2a_{22} v_{2,1} v_{2,2} = 0, \quad v_{2,4}(0) = 0 \quad (23)$$

Solutions of the differential equations (16)-(23) are given by

$$v_{1,1}(t) = a_1 \int_0^t v_{1,0}\,dt - a_{11} \int_0^t v_{1,0}^2\,dt + a_{12} \int_0^t v_{1,0} v_{2,0}\,dt = (a_1 - a_{11} c_1 + a_{12} c_2) c_1 t \quad (24)$$

$$v_{2,1}(t) = a_2 \int_0^t v_{2,0}\,dt - a_{22} \int_0^t v_{2,0}^2\,dt = (a_2 - a_{22} c_2) c_2 t \quad (25)$$

$$v_{1,2}(t) = a_1 \int_0^t v_{1,1}\,dt - 2a_{11} \int_0^t v_{1,0} v_{1,1}\,dt + a_{12} \int_0^t v_{1,0} v_{2,1}\,dt + a_{12} \int_0^t v_{1,1} v_{2,0}\,dt$$
$$= \left[(a_1 - a_{11} c_1 + a_{12} c_2)\left(a_1 c_1 - 2a_{11} c_1^2 + a_{12} c_1 c_2\right) + a_{12}(a_2 - a_{22} c_2) c_1 c_2\right] \frac{t^2}{2} \quad (26)$$

$$v_{2,2}(t) = a_2 \int_0^t v_{2,1}\,dt - 2a_{22} \int_0^t v_{2,0} v_{2,1}\,dt = (a_2 - a_{22} c_2) c_2 (a_2 - 2a_{22} c_2) \frac{t^2}{2} \quad (27)$$

$$v_{1,3}(t) = a_1\int_0^t v_{1,2}dt - 2a_{11}\int_0^t v_{1,0}v_{1,2}dt - a_{11}\int_0^t {v_{1,1}}^2dt + a_{12}\int_0^t v_{1,0}v_{2,2}dt + a_{12}\int_0^t v_{1,1}v_{2,1}dt + a_{12}\int_0^t v_{1,2}v_{2,0}dt$$
$$= (a_1 - 2a_{11}c_1 + a_{12}c_2)\Big[(a_1 - a_{11}c_1 + a_{12}c_2)\big(a_1c_1 - 2a_{11}{c_1}^2 + a_{12}c_1c_2\big) + a_{12}(a_2 - a_{22}c_2)c_1c_2\Big]\frac{t^3}{6}$$
$$+(a_1 - a_{11}c_1 + a_{12}c_2)c_1\big[a_{12}(a_2 - 2a_{22}c_2)c_2 - a_{11}c_1(a_1 - a_{11}c_1 + a_{12}c_2)\big]\frac{t^3}{3} + a_{12}c_1c_2(a_2 - a_{22}c_2)(a_2 - 2a_{22}c_2)\frac{t^3}{6} \quad \textbf{(28)}$$

$$v_{2,3}(t) = a_2\int_0^t v_{2,2}dt - 2a_{22}\int_0^t v_{2,0}v_{2,2}dt - a_{22}\int_0^t {v_{2,1}}^2dt = (a_2 - a_{22}c_2)c_2\Big[(a_2 - 2a_{22}c_2)^2 - 2a_{22}c_2(a_2 - a_{22}c_2)\Big]\frac{t^3}{6} \quad (29)$$

$$v_{1,4}(t) = a_1\int_0^t v_{1,3}dt - 2a_{11}\int_0^t v_{1,0}v_{1,3}dt - 2a_{11}\int_0^t v_{1,1}v_{1,2}dt + a_{12}\int_0^t v_{1,0}v_{2,3}dt + a_{12}\int_0^t v_{1,1}v_{2,2}dt + a_{12}\int_0^t v_{1,2}v_{2,1}dt + a_{12}\int_0^t v_{1,3}v_{2,0}dt$$
$$= (a_1 - 2a_{11}c_1 + a_{12}c_2)\left\{\begin{matrix}(a_1 - 2a_{11}c_1 + a_{12}c_2)\Big[(a_1 - a_{11}c_1 + a_{12}c_2)\big(a_1c_1 - 2a_{11}{c_1}^2 + a_{12}c_1c_2\big) + a_{12}(a_2 - a_{22}c_2)c_1c_2\Big] \\ +2(a_1 - a_{11}c_1 + a_{12}c_2)c_1\big[a_{12}(a_2 - a_{22}c_2)c_2 - a_{11}c_1(a_1 - a_{11}c_1 + a_{12}c_2)\big] + a_{12}c_1c_2(a_2 - a_{22}c_2)(a_2 - 2a_{22}c_2)\end{matrix}\right\}\frac{t^4}{24}$$
$$+(a_1 - a_{11}c_1 + a_{12}c_2)c_1\left\{a_{12}c_2(a_2 - a_{22}c_2)(a_2 - 2a_{22}c_2) - 2a_{11}\Big[(a_1 - a_{11}c_1 + a_{12}c_2)\big(a_1c_1 - 2a_{11}{c_1}^2 + a_{12}c_1c_2\big) + a_{12}(a_2 - a_{22}c_2)c_1c_2\Big]\right\}\frac{t^4}{8}$$
$$+a_{12}c_1c_2(a_2 - a_{22}c_2)\Big[(a_2 - 2a_{22}c_2)^2 - 2a_{22}c_2(a_2 - a_{22}c_2)\Big]\frac{t^4}{24}$$
$$+a_{12}c_2(a_2 - a_{22}c_2)\Big[(a_1 - a_{11}c_1 + a_{12}c_2)\big(a_1c_1 - 2a_{11}{c_1}^2 + a_{12}c_1c_2\big) + a_{12}(a_2 - a_{22}c_2)c_1c_2\Big]\frac{t^4}{8} \quad (30)$$

$$v_{2,4}(t) = a_2\int_0^t v_{2,3}dt - 2a_{22}\int_0^t v_{2,0}v_{2,3}dt - 2a_{22}\int_0^t v_{2,1}v_{2,2}dt$$
$$= (a_2 - 2a_{22}c_2)(a_2 - a_{22}c_2)c_2\Big[(a_2 - 2a_{22}c_2)^2 - 8a_{22}c_2(a_2 - a_{22}c_2)\Big]\frac{t^4}{24} \quad (31)$$

The approximate solution of $N_1(t), N_2(t)$ is

$$N_1(t) = \underset{p\to1}{Lt}\, v_1(t) = \sum_{k=0}^{4} v_{1,k}(t) \quad (32)$$

$$N_2(t) = \underset{p\to1}{Lt}\, v_2(t) = \sum_{k=0}^{4} v_{2,k}(t) \quad (33)$$

which yield

$$N_1(t) = c_1 + (a_1 - a_{11}c_1 + a_{12}c_2)c_1t + \Big[(a_1 - a_{11}c_1 + a_{12}c_2)\big(a_1c_1 - 2a_{11}{c_1}^2 + a_{12}c_1c_2\big) + a_{12}(a_2 - a_{22}c_2)c_1c_2\Big]\frac{t^2}{2}$$
$$+\left\{\begin{matrix}(a_1 - 2a_{11}c_1 + a_{12}c_2)\Big[(a_1 - a_{11}c_1 + a_{12}c_2)\big(a_1c_1 - 2a_{11}{c_1}^2 + a_{12}c_1c_2\big) + a_{12}(a_2 - a_{22}c_2)c_1c_2\Big] \\ +2(a_1 - a_{11}c_1 + a_{12}c_2)c_1\big[a_{12}(a_2 - 2a_{22}c_2)c_2 - a_{11}c_1(a_1 - a_{11}c_1 + a_{12}c_2)\big] + a_{12}c_1c_2(a_2 - a_{22}c_2)(a_2 - 2a_{22}c_2)\end{matrix}\right\}\frac{t^3}{6}$$
$$+\left\{\begin{matrix}(a_1 - 2a_{11}c_1 + a_{12}c_2)\left\{\begin{matrix}(a_1 - 2a_{11}c_1 + a_{12}c_2)\Big[(a_1 - a_{11}c_1 + a_{12}c_2)\big(a_1c_1 - 2a_{11}{c_1}^2 + a_{12}c_1c_2\big) + a_{12}(a_2 - a_{22}c_2)c_1c_2\Big] \\ +2(a_1 - a_{11}c_1 + a_{12}c_2)c_1\big[a_{12}(a_2 - a_{22}c_2)c_2 - a_{11}c_1(a_1 - a_{11}c_1 + a_{12}c_2)\big] + a_{12}c_1c_2(a_2 - a_{22}c_2)(a_2 - 2a_{22}c_2)\end{matrix}\right\} \\ +3(a_1 - a_{11}c_1 + a_{12}c_2)c_1\left\{a_{12}c_2(a_2 - a_{22}c_2)(a_2 - 2a_{22}c_2) - 2a_{11}\Big[(a_1 - a_{11}c_1 + a_{12}c_2)\big(a_1c_1 - 2a_{11}{c_1}^2 + a_{12}c_1c_2\big) + a_{12}(a_2 - a_{22}c_2)c_1c_2\Big]\right\} \\ +a_{12}c_1c_2(a_2 - a_{22}c_2)\Big[(a_2 - 2a_{22}c_2)^2 - 2a_{22}c_2(a_2 - a_{22}c_2)\Big] + 3a_{12}c_2(a_2 - a_{22}c_2)\Big[(a_1 - a_{11}c_1 + a_{12}c_2)\big(a_1c_1 - 2a_{11}{c_1}^2 + a_{12}c_1c_2\big) + a_{12}(a_2 - a_{22}c_2)c_1c_2\Big]\end{matrix}\right\}\frac{t^4}{24}$$

34)

$$N_2(t)=c_2+(a_2-a_{22}c_2)c_2t+(a_2-a_{22}c_2)c_2(a_2-2a_{22}c_2)\frac{t^2}{2}+(a_2-a_{22}c_2)c_2\left[(a_2-2a_{22}c_2)^2-2a_{22}c_2(a_2-a_{22}c_2)\right]\frac{t^3}{6}$$
$$+(a_2-2a_{22}c_2)(a_2-a_{22}c_2)c_2\left[(a_2-2a_{22}c_2)^2-8a_{22}c_2(a_2-a_{22}c_2)\right]\frac{t^4}{24}$$

(35)

The values of the parameters in equation(1) which satisfy condition (A) (see section 2.1) are taken as $a_1=1.15; a_{11}=0.05; a_{12}=0.2; a_2=1.78; a_{22}=0.01; N_{10}=2; N_{20}=2$

The approximate solutions of equation (1) are obtained by using equations (34) and (35) and their graphical representations are given in figure 1 and Figure 2 . In these figures solution obtained by 4th order Runge- Kutta method are also depicted.

It is observed from the figures1 and 2 that the solutions obtained by the two methods agree up to the calculated degree of accuracy.

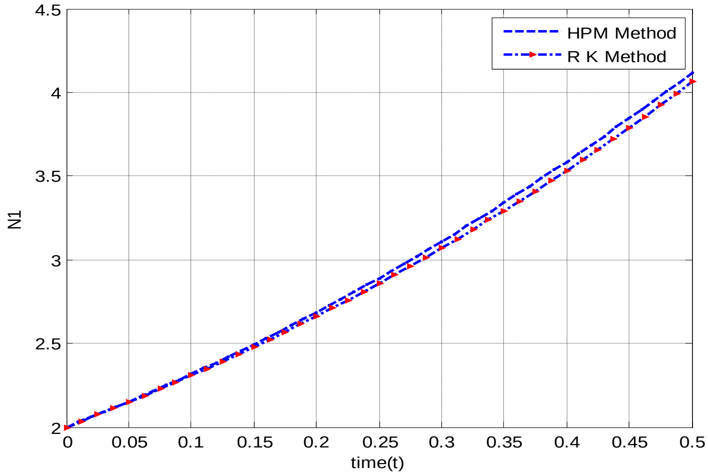

Fig.1. For the values $a_1=1.15; a_{11}=0.05; a_{12}=0.2; a_2=1.78; a_{22}=0.01; N_{10}=2; N_{20}=2$,Comparison between the solutions of N1 (Eq(1)) obtained by HPM (calculated up to 4 terms) and R-K method.

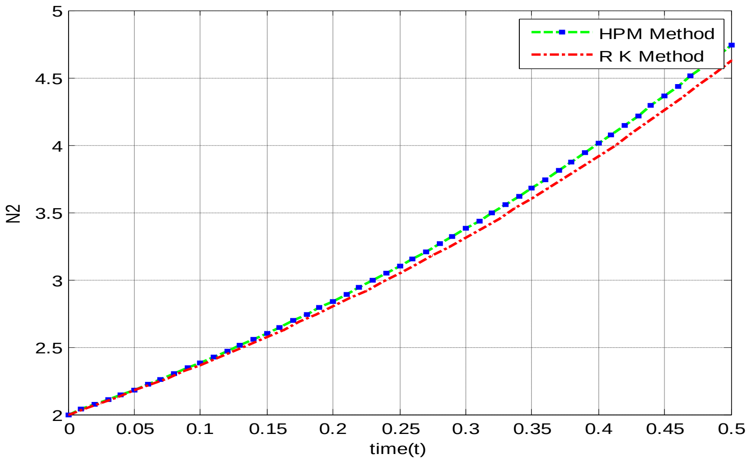

Fig.2. For the values $a_1=1.15; a_{11}=0.05; a_{12}=0.2; a_2=1.78; a_{22}=0.01; N_{10}=2; N_{20}=2$,Comparison between the solutions of N2 (Eq(1)) obtained by HPM (calculated up to 4 terms) and R-K method.

3.1.2. Solution of model-2

Consider the system of equations (2) with initial approximations

$$v_{10}(t)=N_{10}(t)=v_1(0)=c_1$$
$$v_{20}(t)=N_{20}(t)=v_2(0)=c_2 \tag{36}$$

As explained in section 3.1, the equations (2) can be written as

$$\begin{aligned} &v_1' - N_{10}' + p\left(N_{10}' - d_1 v_1 + a_{11} v_1^2 - a_{12} v_1 v_2\right) = 0 \\ &v_2' - N_{20}' + p\left(N_{20}' - a_2 v_2 + a_{22} v_2^2\right) = 0 \end{aligned} \tag{37}$$

Assume the approximate solutions of $N_1(t), N_2(t)$ as

$$\begin{aligned} &v_1(t) = v_{1,0} + p v_{1,1}(t) + p^2 v_{1,2}(t) + p^3 v_{1,3}(t) + p^4 v_{1,4}(t) + ---------- \\ &v_2(t) = v_{2,0} + p v_{2,1}(t) + p^2 v_{2,2}(t) + p^3 v_{2,3}(t) + p^4 v_{2,4}(t) + ---------- \end{aligned} \tag{38}$$

$v_{i,j}(i = 1,2; j = 1,2,3,4,........)$ are to be determined by substituting equations (36) and (38) into equation (37) and arranging the terms in the increasing powers of p .

$$\begin{aligned} &\left[v_{1,1}'(t) + N_{10}'(t) + d_1 v_{1,0}(t) + a_{11} v_{1,0}^2(t) - a_{12} v_{1,0}(t) v_{2,0}(t)\right] p + \left[v_{1,2}'(t) + d_1 v_{1,1}(t) + 2a_{11} v_{1,0}(t) v_{1,1}(t) - a_{12} v_{1,0}(t) v_{2,1}(t) - a_{12} v_{1,1}(t) v_{2,0}(t)\right] p^2 \\ &+ \left[v_{1,3}'(t) + d_1 v_{1,2}(t) + 2a_{11} v_{1,0}(t) v_{1,2}(t) + a_{11} v_{1,1}^2(t) - a_{12} v_{1,0}(t) v_{2,2}(t) - a_{12} v_{1,1}(t) v_{2,1}(t) - a_{12} v_{1,2}(t) v_{2,0}(t)\right] p^3 \\ &+ \left[v_{1,4}'(t) + d_1 v_{1,3}(t) + 2a_{11} v_{1,0}(t) v_{1,3}(t) + 2a_{11} v_{1,1}(t) v_{1,2}(t) - a_{12} v_{1,0}(t) v_{2,3}(t) - a_{12} v_{1,1}(t) v_{2,2}(t) - a_{12} v_{1,2}(t) v_{2,1}(t) - a_{12} v_{1,3}(t) v_{2,0}(t)\right] p^4 + -------- = 0 \\ &\left[v_{2,1}'(t) + N_{20}'(t) - a_2 v_{2,0}(t) + a_{22} v_{2,0}^2(t)\right] p + \left[v_{2,2}'(t) - a_2 v_{2,1}(t) + 2a_{22} v_{2,0}(t) v_{2,1}(t)\right] p^2 \\ &+ \left[v_{2,3}'(t) - a_2 v_{2,2}(t) + 2a_{22} v_{2,0}(t) v_{2,2}(t) + a_{22} v_{2,1}^2(t)\right] p^3 + \left[v_{2,4}'(t) - a_2 v_{2,3}(t) + 2a_{22} v_{2,0}(t) v_{2,3}(t) + 2a_{22} v_{2,1}(t) v_{2,2}(t)\right] p^4 + -------- = 0 \end{aligned} \tag{39}$$

To obtain the unknowns $v_{i,j}(t), i = 1,2; j = 1,2,3,4$ solve the following system of linear differential equations with the initial conditions given in (36)

From (39)

$$v_{1,1}' + N_{10}' + d_1 v_{1,0} + a_{11} v_{1,0}^2 - a_{12} v_{1,0} v_{2,0} = 0, \quad v_{1,1}(0) = 0 \tag{40}$$

$$v_{2,1}' + N_{20}' - a_2 v_{2,0} + a_{22} v_{2,0}^2 = 0, \quad v_{2,1}(0) = 0 \tag{41}$$

$$v_{1,2}' + d_1 v_{1,1} + 2a_{11} v_{1,0} v_{1,1} - a_{12} v_{1,0} v_{2,1} - a_{12} v_{1,1} v_{2,0} = 0, \quad v_{1,2}(0) = 0 \tag{42}$$

$$v_{2,2}' - a_2 v_{2,1} + 2a_{22} v_{2,0} v_{2,1} = 0, \quad v_{2,2}(0) = 0 \tag{43}$$

$$v_{1,3}' + d_1 v_{1,2} + 2a_{11} v_{1,0} v_{1,2} + a_{11} v_{1,1}^2 - a_{12} v_{1,0} v_{2,2} - a_{12} v_{1,1} v_{2,1} - a_{12} v_{1,2} v_{2,0} = 0, \quad v_{1,3}(0) = 0 \tag{44}$$

$$v_{2,3}' - a_2 v_{2,2} + 2a_{22} v_{2,0} v_{2,2} + a_{22} v_{2,1}^2 = 0, \quad v_{2,3}(0) = 0 \tag{45}$$

$$v_{1,4}' + d_1 v_{1,3} + 2a_{11} v_{1,0} v_{1,3} + 2a_{11} v_{1,1} v_{1,2} - a_{12} v_{1,0} v_{2,3} - a_{12} v_{1,1} v_{2,2} - a_{12} v_{1,2} v_{2,1} - a_{12} v_{1,3} v_{2,0} = 0, \quad v_{1,4}(0) = 0 \tag{46}$$

$$v_{2,4}' - a_2 v_{2,3} + 2a_{22} v_{2,0} v_{2,3} + 2a_{22} v_{2,1} v_{2,2} = 0, \quad v_{2,4}(0) = 0 \tag{47}$$

Solutions of the differential equations (40)-(47) are given by

$$v_{1,1}(t) = -d_1 \int_0^t v_{1,0} dt - a_{11} \int_0^t v_{1,0}^2 dt + a_{12} \int_0^t v_{1,0} v_{2,0} dt = \left(-d_1 - a_{11} c_1 + a_{12} c_2\right) c_1 t \tag{48}$$

$$v_{2,1}(t) = a_2 \int_0^t v_{2,0} dt - a_{22} \int_0^t v_{2,0}^2 dt = \left(a_2 - a_{22} c_2\right) c_2 t \tag{49}$$

$$\begin{aligned} v_{1,2}(t) &= -d_1 \int_0^t v_{1,1} dt - 2a_{11} \int_0^t v_{1,0} v_{1,1} dt + a_{12} \int_0^t v_{1,0} v_{2,1} dt + a_{12} \int_0^t v_{1,1} v_{2,0} dt \\ &= \left[\left(-d_1 - a_{11} c_1 + a_{12} c_2\right)\left(-d_1 c_1 - 2a_{11} c_1^2 + a_{12} c_1 c_2\right) + a_{12}\left(a_2 - a_{22} c_2\right) c_1 c_2\right] \frac{t^2}{2} \end{aligned} \tag{50}$$

$$v_{2,2}(t) = a_2 \int_0^t v_{2,1} dt - 2a_{22} \int_0^t v_{2,0} v_{2,1} dt = \left(a_2 - a_{22} c_2\right) c_2 \left(a_2 - 2a_{22} c_2\right) \frac{t^2}{2} \tag{51}$$

$$v_{1,3}(t) = -d_1\int_0^t v_{1,2}dt - 2a_{11}\int_0^t v_{1,0}v_{1,2}dt - a_{11}\int_0^t {v_{1,1}}^2dt + a_{12}\int_0^t v_{1,0}v_{2,2}dt + a_{12}\int_0^t v_{1,1}v_{2,1}dt + a_{12}\int_0^t v_{1,2}v_{2,0}dt$$

$$= (-d_1 - 2a_{11}c_1 + a_{12}c_2)\left[(-d_1 - a_{11}c_1 + a_{12}c_2)\left(-d_1c_1 - 2a_{11}{c_1}^2 + a_{12}c_1c_2\right) + a_{12}(a_2 - a_{22}c_2)c_1c_2\right]\frac{t^3}{6}$$

$$+(-d_1 - a_{11}c_1 + a_{12}c_2)c_1\left[a_{12}(a_2 - 2a_{22}c_2)c_2 - a_{11}c_1(-d_1 - a_{11}c_1 + a_{12}c_2)\right]\frac{t^3}{3} + a_{12}c_1c_2(a_2 - a_{22}c_2)(a_2 - 2a_{22}c_2)\frac{t^3}{6}$$

(52)

$$v_{2,3}(t) = a_2\int_0^t v_{2,2}dt - 2a_{22}\int_0^t v_{2,0}v_{2,2}dt - a_{22}\int_0^t {v_{2,1}}^2dt = (a_2 - a_{22}c_2)c_2\left[(a_2 - 2a_{22}c_2)^2 - 2a_{22}c_2(a_2 - a_{22}c_2)\right]\frac{t^3}{6} \qquad (53)$$

$$v_{1,4}(t) = -d_1\int_0^t v_{1,3}dt - 2a_{11}\int_0^t v_{1,0}v_{1,3}dt - 2a_{11}\int_0^t v_{1,1}v_{1,2}dt + a_{12}\int_0^t v_{1,0}v_{2,3}dt + a_{12}\int_0^t v_{1,1}v_{2,2}dt + a_{12}\int_0^t v_{1,2}v_{2,1}dt + a_{12}\int_0^t v_{1,3}v_{2,0}dt$$

$$= (-d_1 - 2a_{11}c_1 + a_{12}c_2)\left\{\begin{aligned}&(-d_1 - 2a_{11}c_1 + a_{12}c_2)\left[(-d_1 - a_{11}c_1 + a_{12}c_2)\left(-d_1c_1 - 2a_{11}{c_1}^2 + a_{12}c_1c_2\right) + a_{12}(a_2 - a_{22}c_2)c_1c_2\right]\\ &+2(-d_1 - a_{11}c_1 + a_{12}c_2)c_1\left[a_{12}(a_2 - a_{22}c_2)c_2 - a_{11}c_1(-d_1 - a_{11}c_1 + a_{12}c_2)\right] + a_{12}c_1c_2(a_2 - a_{22}c_2)(a_2 - 2a_{22}c_2)\end{aligned}\right\}\frac{t^4}{24}$$

$$+(-d_1 - a_{11}c_1 + a_{12}c_2)c_1\left\{a_{12}c_2(a_2 - a_{22}c_2)(a_2 - 2a_{22}c_2) - 2a_{11}\left[(-d_1 - a_{11}c_1 + a_{12}c_2)\left(-d_1c_1 - 2a_{11}{c_1}^2 + a_{12}c_1c_2\right) + a_{12}(a_2 - a_{22}c_2)c_1c_2\right]\right\}\frac{t^4}{8}$$

$$+a_{12}c_1c_2(a_2 - a_{22}c_2)\left[(a_2 - 2a_{22}c_2)^2 - 2a_{22}c_2(a_2 - a_{22}c_2)\right]\frac{t^4}{24}$$

$$+a_{12}c_2(a_2 - a_{22}c_2)\left[(-d_1 - a_{11}c_1 + a_{12}c_2)\left(-d_1c_1 - 2a_{11}{c_1}^2 + a_{12}c_1c_2\right) + a_{12}(a_2 - a_{22}c_2)c_1c_2\right]\frac{t^4}{8}$$

(54)

$$v_{2,4}(t) = a_2\int_0^t v_{2,3}dt - 2a_{22}\int_0^t v_{2,0}v_{2,3}dt - 2a_{22}\int_0^t v_{2,1}v_{2,2}dt$$

$$= (a_2 - 2a_{22}c_2)(a_2 - a_{22}c_2)c_2\left[(a_2 - 2a_{22}c_2)^2 - 8a_{22}c_2(a_2 - a_{22}c_2)\right]\frac{t^4}{24}$$

(55)

The approximate solution of $N_1(t), N_2(t)$ is given by

$$N_1(t) = \underset{p\to 1}{Lt}\, v_1(t) = \sum_{k=0}^{4} v_{1,k}(t) \qquad (56)$$

$$N_2(t) = \underset{p\to 1}{Lt}\, v_2(t) = \sum_{k=0}^{4} v_{2,k}(t) \qquad (57)$$

which yield

$$N_1(t) = c_1 + (-d_1 - a_{11}c_1 + a_{12}c_2)c_1t + \left[(-d_1 - a_{11}c_1 + a_{12}c_2)\left(-d_1c_1 - 2a_{11}{c_1}^2 + a_{12}c_1c_2\right) + a_{12}(a_2 - a_{22}c_2)c_1c_2\right]\frac{t^2}{2}$$

$$+\left\{\begin{aligned}&(-d_1 - 2a_{11}c_1 + a_{12}c_2)\left[(-d_1 - a_{11}c_1 + a_{12}c_2)\left(-d_1c_1 - 2a_{11}{c_1}^2 + a_{12}c_1c_2\right) + a_{12}(a_2 - a_{22}c_2)c_1c_2\right]\\ &+2(-d_1 - a_{11}c_1 + a_{12}c_2)c_1\left[a_{12}(a_2 - 2a_{22}c_2)c_2 - a_{11}c_1(-d_1 - a_{11}c_1 + a_{12}c_2)\right] + a_{12}c_1c_2(a_2 - a_{22}c_2)(a_2 - 2a_{22}c_2)\end{aligned}\right\}\frac{t^3}{6}$$

$$+\left\{\begin{aligned}&(-d_1 - 2a_{11}c_1 + a_{12}c_2)\left\{\begin{aligned}&(-d_1 - 2a_{11}c_1 + a_{12}c_2)\left[(-d_1 - a_{11}c_1 + a_{12}c_2)\left(-d_1c_1 - 2a_{11}{c_1}^2 + a_{12}c_1c_2\right) + a_{12}(a_2 - a_{22}c_2)c_1c_2\right]\\ &+2(-d_1 - a_{11}c_1 + a_{12}c_2)c_1\left[a_{12}(a_2 - a_{22}c_2)c_2 - a_{11}c_1(-d_1 - a_{11}c_1 + a_{12}c_2)\right] + a_{12}c_1c_2(a_2 - a_{22}c_2)(a_2 - 2a_{22}c_2)\end{aligned}\right\}\\ &+3(-d_1 - a_{11}c_1 + a_{12}c_2)c_1\left\{a_{12}c_2(a_2 - a_{22}c_2)(a_2 - 2a_{22}c_2) - 2a_{11}\left[(-d_1 - a_{11}c_1 + a_{12}c_2)\left(-d_1c_1 - 2a_{11}{c_1}^2 + a_{12}c_1c_2\right) + a_{12}(a_2 - a_{22}c_2)c_1c_2\right]\right\}\\ &+a_{12}c_1c_2(a_2 - a_{22}c_2)\left[(a_2 - 2a_{22}c_2)^2 - 2a_{22}c_2(a_2 - a_{22}c_2)\right] + 3a_{12}c_2(a_2 - a_{22}c_2)\left[(-d_1 - a_{11}c_1 + a_{12}c_2)\left(-d_1c_1 - 2a_{11}{c_1}^2 + a_{12}c_1c_2\right) + a_{12}(a_2 - a_{22}c_2)c_1c_2\right]\end{aligned}\right\}\frac{t^4}{24}$$

(58)

$$N_2(t)=c_2+(a_2-a_{22}c_2)c_2t+(a_2-a_{22}c_2)c_2(a_2-2a_{22}c_2)\frac{t^2}{2}+(a_2-a_{22}c_2)c_2\left[(a_2-2a_{22}c_2)^2-2a_{22}c_2(a_2-a_{22}c_2)\right]\frac{t^3}{6}$$
$$+(a_2-2a_{22}c_2)(a_2-a_{22}c_2)c_2\left[(a_2-2a_{22}c_2)^2-8a_{22}c_2(a_2-a_{22}c_2)\right]\frac{t^4}{24}$$

(59)

The values of the parameters in equation(2) which satisfy condition (B) (see section 2.2) are taken as $d_1=0.4; a_{11}=0.1; a_{12}=0.007; a_2=1.3; a_{22}=0.15; N_{10}=1; N_{20}=1$

The approximate solutions of equation (2) are obtained by using equations (58) and (59) and their graphical representations are given in figure 3 and Figure 4 . In these figures solution obtained by 4th order Runge- Kutta method are also depicted.

It is observed from the figures3 and 4 that the solutions obtained by the two methods agree up to the calculated degree of accuracy.

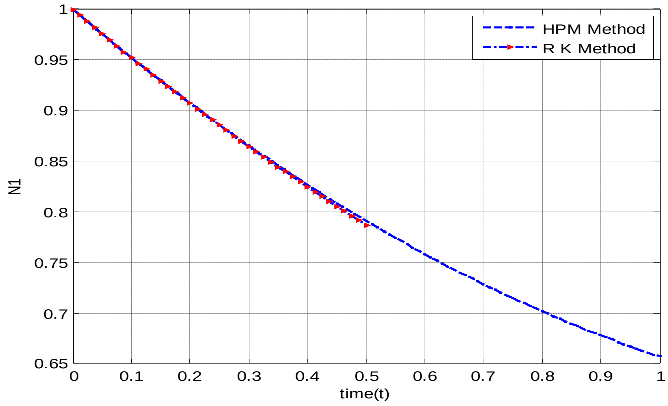

Fig.3. For the values $d_1=0.4; a_{11}=0.1; a_{12}=0.007; a_2=1.3; a_{22}=0.15; N_{10}=1; N_{20}=1$,Comparison between the solutions of N1 (Eq(2)) obtained by HPM (calculated up to 4 terms) and R-K method

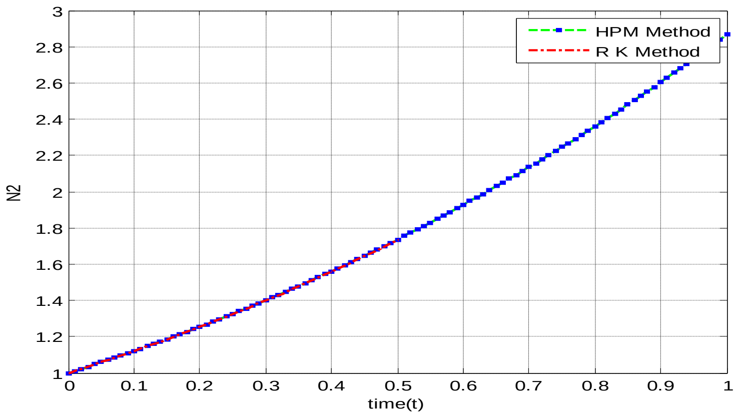

Fig.4. For the values $d_1=0.4; a_{11}=0.1; a_{12}=0.007; a_2=1.3; a_{22}=0.15; N_{10}=1; N_{20}=1$,Comparison between the solutions of N2 (Eq(2)) obtained by HPM (calculated up to 4 terms) and R-K method

3.1.3 Solution of Model-3

Consider the system of equations (3) with initial approximations

$$v_{10}(t)=N_{10}(t)=v_1(0)=c_1$$
$$v_{20}(t)=N_{20}(t)=v_2(0)=c_2 \qquad (60)$$

As explained in section 3.1, the equations (3) can be written as

$$v_1'-N_{10}'+p\left(N_{10}'-a_1v_1+a_{11}v_1^2-\frac{\alpha v_1v_2}{\beta+v_2}\right)=0$$

$$v_2' - N_{20}' + p\left(N_{20}' - a_2 v_2 + a_{22} v_2^2\right) = 0 \tag{61}$$

Assume the approximate solutions of solutions of $N_1(t), N_2(t)$ as

$$\begin{aligned} v_1(t) &= v_{1,0} + p v_{1,1}(t) + p^2 v_{1,2}(t) + p^3 v_{1,3}(t) + p^4 v_{1,4}(t) + ---------- \\ v_2(t) &= v_{2,0} + p v_{2,1}(t) + p^2 v_{2,2}(t) + p^3 v_{2,3}(t) + p^4 v_{2,4}(t) + ---------- \end{aligned} \tag{62}$$

$v_{i,j}\,(i = 1,2;\ j = 1,2,3,4,........)$ are to be determined by substituting equations (60) and (62) into equation (61) and arranging the terms in the increasing powers of p .

$$\left\{ v_{1,1}'(t) + N_{10}'(t) - a_1 v_{1,0}(t) + a_{11} v_{1,0}^2(t) - \frac{\alpha}{\beta}\left[v_{1,0}(t) v_{2,0}(t) - \frac{1}{\beta} v_{1,0}(t) v_{2,0}^2(t) + \frac{1}{\beta^2} v_{1,0}(t) v_{2,0}^3(t) - \frac{1}{\beta^3} v_{1,0}(t) v_{2,0}^4(t) \right] \right\} p$$

$$+\left\{ v_{1,2}'(t) - a_1 v_{1,1}(t) + 2a_{11} v_{1,0}(t) v_{1,1}(t) - \frac{\alpha}{\beta}\left[\begin{array}{l} v_{1,0}(t) v_{2,1}(t) + v_{1,1}(t) v_{2,0}(t) - \frac{1}{\beta}\left(2v_{1,0}(t) v_{2,0}(t) v_{2,1}(t) + v_{1,1}(t) v_{2,0}^2(t)\right) \\ + \frac{1}{\beta^2}\left(3v_{1,0}(t) v_{2,1}(t) v_{2,0}^2(t) + v_{1,1}(t) v_{2,0}^3(t)\right) - \frac{1}{\beta^3}\left(4v_{1,0}(t) v_{2,0}^3(t) v_{2,1}(t) + v_{1,1}(t) v_{2,0}^4(t)\right) \end{array} \right] \right\} p^2$$

$$+\left\{ v_{1,3}'(t) - a_1 v_{1,2}(t) + 2a_{11} v_{1,0}(t) v_{1,2}(t) + a_{11} v_{1,1}^2(t) - \frac{\alpha}{\beta}\left[\begin{array}{l} v_{1,0}(t) v_{2,2}(t) + v_{1,1}(t) v_{2,1}(t) + v_{1,2}(t) v_{2,0}(t) - \frac{1}{\beta}\left(\begin{array}{l} 2v_{1,0}(t) v_{2,0}(t) v_{2,2}(t) + v_{1,0}(t) v_{2,1}^2(t) \\ +2v_{1,1}(t) v_{2,0}(t) v_{2,1}(t) + v_{1,2}(t) v_{2,0}^2(t) \end{array} \right) \\ + \frac{1}{\beta^2}\left(\begin{array}{l} 3v_{1,0}(t) v_{2,2}(t) v_{2,0}^2(t) + 3v_{1,0}(t) v_{2,0}(t) v_{2,1}^2(t) \\ +3v_{1,1}(t) v_{2,1}(t) v_{2,0}^2(t) + v_{1,2}(t) v_{2,0}^3(t) \end{array} \right) - \frac{1}{\beta^3}\left(\begin{array}{l} 4v_{1,0}(t) v_{2,0}^3(t) v_{2,2}(t) + 6v_{1,0}(t) v_{2,1}^2(t) v_{2,0}^2(t) \\ +4v_{1,1}(t) v_{2,0}^3(t) v_{2,1}(t) + v_{1,2}(t) v_{2,0}^4(t) \end{array} \right) \end{array} \right] \right\} p^3$$

$$+\left\{ v_{1,4}'(t) - a_1 v_{1,3}(t) + 2a_{11} v_{1,0}(t) v_{1,3}(t) + 2a_{11} v_{1,1}(t) v_{1,2}(t) - \frac{\alpha}{\beta}\left[\begin{array}{l} v_{1,0}(t) v_{2,3}(t) + v_{1,1}(t) v_{2,2}(t) + v_{1,2}(t) v_{2,1}(t) + v_{1,3}(t) v_{2,0}(t) \\ - \frac{1}{\beta}\left(\begin{array}{l} 2v_{1,0}(t) v_{2,0}(t) v_{2,3}(t) + 2v_{1,0}(t) v_{2,1}(t) v_{2,2}(t) + 2v_{1,1}(t) v_{2,0}(t) v_{2,2}(t) \\ + v_{1,1}(t) v_{2,1}^2(t) + 2v_{1,2}(t) v_{2,0}(t) v_{2,1}(t) + v_{1,3}(t) v_{2,0}^2(t) \end{array} \right) \\ + \frac{1}{\beta^2}\left(\begin{array}{l} 3v_{1,0}(t) v_{2,3}(t) v_{2,0}^2(t) + 6v_{1,0}(t) v_{2,0}(t) v_{2,1}(t) v_{2,2}(t) + v_{1,0}(t) v_{2,1}^3(t) \\ +3v_{1,1}(t) v_{2,2}(t) v_{2,0}^2(t) + 3v_{1,1}(t) v_{2,0}(t) v_{2,1}^2(t) + 3v_{1,2}(t) v_{2,1}(t) v_{2,0}^2(t) + v_{1,3}(t) v_{2,0}^3(t) \end{array} \right) \\ - \frac{1}{\beta^3}\left(\begin{array}{l} 4v_{1,0}(t) v_{2,0}^3(t) v_{2,3}(t) + 12v_{1,0}(t) v_{2,1}(t) v_{2,2}(t) v_{2,0}^2(t) + 4v_{1,0}(t) v_{2,1}^3(t) v_{2,0}(t) \\ +4v_{1,1}(t) v_{2,0}^3(t) v_{2,2}(t) + 6v_{1,1}(t) v_{2,1}^2(t) v_{2,0}^2(t) + 4v_{1,2}(t) v_{2,0}^3(t) v_{2,1}(t) + v_{1,3}(t) v_{2,0}^4(t) \end{array} \right) \end{array} \right] \right\} p^4$$

$$+ ---------------------- = 0$$

$$\begin{aligned} &\left[v_{2,1}'(t) + N_{20}'(t) - a_2 v_{2,0}(t) + a_{22} v_{2,0}^2(t)\right] p + \left[v_{2,2}'(t) - a_2 v_{2,1}(t) + 2a_{22} v_{2,0}(t) v_{2,1}(t)\right] p^2 \\ &+ \left[v_{2,3}'(t) - a_2 v_{2,2}(t) + 2a_{22} v_{2,0}(t) v_{2,2}(t) + a_{22} v_{2,1}^2(t)\right] p^3 + \left[v_{2,4}'(t) - a_2 v_{2,3}(t) + 2a_{22} v_{2,0}(t) v_{2,3}(t) + 2a_{22} v_{2,1}(t) v_{2,2}(t)\right] p^4 + ------- = 0 \end{aligned} \tag{63}$$

To obtain the unknowns $v_{i,j}(t), i = 1,2;\ j = 1,2,3,4$ solve the following system of linear differential equations with the initial conditions given in (36)

From (63)

$$v_{1,1}'(t) + N_{10}'(t) - a_1 v_{1,0}(t) + a_{11} v_{1,0}^2(t) - \frac{\alpha}{\beta}\left[v_{1,0}(t) v_{2,0}(t) - \frac{1}{\beta} v_{1,0}(t) v_{2,0}^2(t) + \frac{1}{\beta^2} v_{1,0}(t) v_{2,0}^3(t) - \frac{1}{\beta^3} v_{1,0}(t) v_{2,0}^4(t) \right] = 0,\ v_{1,1}(0) = 0 \tag{64}$$

$$v_{2,1}' + N_{20}' - a_2 v_{2,0} + a_{22} v_{2,0}^2 = 0,\ \ v_{2,1}(0) = 0 \tag{65}$$

$$v_{1,2}'(t) - a_1 v_{1,1}(t) + 2a_{11} v_{1,0}(t) v_{1,1}(t) - \frac{\alpha}{\beta}\left[\begin{array}{l} v_{1,0}(t) v_{2,1}(t) + v_{1,1}(t) v_{2,0}(t) - \frac{1}{\beta}\left(2v_{1,0}(t) v_{2,0}(t) v_{2,1}(t) + v_{1,1}(t) v_{2,0}^2(t)\right) \\ + \frac{1}{\beta^2}\left(3v_{1,0}(t) v_{2,1}(t) v_{2,0}^2(t) + v_{1,1}(t) v_{2,0}^3(t)\right) - \frac{1}{\beta^3}\left(4v_{1,0}(t) v_{2,0}^3(t) v_{2,1}(t) + v_{1,1}(t) v_{2,0}^4(t)\right) \end{array} \right] = 0,\ v_{1,2}(0) = 0 \tag{66}$$

$$v_{2,2}' - a_2 v_{2,1} + 2a_{22} v_{2,0} v_{2,1} = 0,\ \ \ v_{2,2}(0) = 0 \tag{67}$$

$$v_{1,3}'(t)-a_1v_{1,2}(t)+2a_{11}v_{1,0}(t)v_{1,2}(t)+a_{11}v_{1,1}^2(t)-\frac{\alpha}{\beta}\left[\begin{array}{l} v_{1,0}(t)v_{2,2}(t)+v_{1,1}(t)v_{2,1}(t)+v_{1,2}(t)v_{2,0}(t)-\frac{1}{\beta}\left(\begin{array}{l}2v_{1,0}(t)v_{2,0}(t)v_{2,2}(t)+v_{1,0}(t)v_{2,1}^2(t)\\ +2v_{1,1}(t)v_{2,0}(t)v_{2,1}(t)+v_{1,2}(t)v_{2,0}^2(t)\end{array}\right)\\ +\frac{1}{\beta^2}\left(\begin{array}{l}3v_{1,0}(t)v_{2,2}(t)v_{2,0}^2(t)+3v_{1,0}(t)v_{2,0}(t)v_{2,1}^2(t)\\ +3v_{1,1}(t)v_{2,1}(t)v_{2,0}^2(t)+v_{1,2}(t)v_{2,0}^3(t)\end{array}\right)-\frac{1}{\beta^3}\left(\begin{array}{l}4v_{1,0}(t)v_{2,0}^3(t)v_{2,2}(t)+6v_{1,0}(t)v_{2,1}^2(t)v_{2,0}^2(t)\\ +4v_{1,1}(t)v_{2,0}^3(t)v_{2,1}(t)+v_{1,2}(t)v_{2,0}^4(t)\end{array}\right)\end{array}\right]=0,\ \ v_{1,3}(0)=0$$

(68)

$$v_{2,3}'-a_2v_{2,2}+2a_{22}v_{2,0}v_{2,2}+a_{22}v_{2,1}^2=0,\quad v_{2,3}(0)=0 \tag{69}$$

$$v_{1,4}'(t)-a_1v_{1,3}(t)+2a_{11}v_{1,0}(t)v_{1,3}(t)+2a_{11}v_{1,1}(t)v_{1,2}(t)-\frac{\alpha}{\beta}\left[\begin{array}{l} v_{1,0}(t)v_{2,3}(t)+v_{1,1}(t)v_{2,2}(t)+v_{1,2}(t)v_{2,1}(t)+v_{1,3}(t)v_{2,0}(t)\\ -\frac{1}{\beta}\left(\begin{array}{l}2v_{1,0}(t)v_{2,0}(t)v_{2,3}(t)+2v_{1,0}(t)v_{2,1}(t)v_{2,2}(t)+2v_{1,1}(t)v_{2,0}(t)v_{2,2}(t)\\ +v_{1,1}(t)v_{2,1}^2(t)+2v_{1,2}(t)v_{2,0}(t)v_{2,1}(t)+v_{1,3}(t)v_{2,0}^2(t)\end{array}\right)\\ +\frac{1}{\beta^2}\left(\begin{array}{l}3v_{1,0}(t)v_{2,3}(t)v_{2,0}^2(t)+6v_{1,0}(t)v_{2,0}(t)v_{2,1}(t)v_{2,2}(t)+v_{1,0}(t)v_{2,1}^3(t)\\ +3v_{1,1}(t)v_{2,2}(t)v_{2,0}^2(t)+3v_{1,1}(t)v_{2,0}(t)v_{2,1}^2(t)+3v_{1,2}(t)v_{2,1}(t)v_{2,0}^2(t)+v_{1,3}(t)v_{2,0}^3(t)\end{array}\right)\\ -\frac{1}{\beta^3}\left(\begin{array}{l}4v_{1,0}(t)v_{2,0}^3(t)v_{2,3}(t)+12v_{1,0}(t)v_{2,1}(t)v_{2,2}(t)v_{2,0}^2(t)+4v_{1,0}(t)v_{2,1}^3(t)v_{2,0}(t)\\ +4v_{1,1}(t)v_{2,0}^3(t)v_{2,2}(t)+6v_{1,1}(t)v_{2,1}^2(t)v_{2,0}^2(t)+4v_{1,2}(t)v_{2,0}^3(t)v_{2,1}(t)+v_{1,3}(t)v_{2,0}^4(t)\end{array}\right)\end{array}\right]=0,\ \ v_{1,4}(0)=0 \tag{70}$$

$$v_{2,4}'-a_2v_{2,3}+2a_{22}v_{2,0}v_{2,3}+2a_{22}v_{2,1}v_{2,2}=0,\quad v_{2,4}(0)=0 \tag{71}$$

Solutions of the differential equations (64)-(71) are given by

$$v_{1,1}(t)=a_1\int_0^t v_{1,0}dt-a_{11}\int_0^t v_{1,0}^2dt+\frac{\alpha}{\beta}\left[\int_0^t v_{1,0}v_{2,0}dt-\frac{1}{\beta}\int_0^t v_{1,0}v_{2,0}^2dt+\frac{1}{\beta^2}\int_0^t v_{1,0}v_{2,0}^3dt-\frac{1}{\beta^3}\int_0^t v_{1,0}v_{2,0}^4(t)\right]$$
$$=c_1\left(a_1-a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)t \tag{72}$$

$$v_{2,1}(t)=a_2\int_0^t v_{2,0}dt-a_{22}\int_0^t v_{2,0}^2dt=(a_2-a_{22}c_2)c_2t \tag{73}$$

$$v_{1,2}(t)=a_1\int_0^t v_{1,1}dt-2a_{11}\int_0^t v_{1,0}v_{1,1}dt+\frac{\alpha}{\beta}\left[\begin{array}{l}\int_0^t v_{1,0}v_{2,1}dt+\int_0^t v_{1,1}v_{2,0}dt-\frac{1}{\beta}\left(2\int_0^t v_{1,0}v_{2,0}v_{2,1}dt+\int_0^t v_{1,1}v_{2,0}^2dt\right)\\ +\frac{1}{\beta^2}\left(3\int_0^t v_{1,0}v_{2,1}v_{2,0}^2dt+\int_0^t v_{1,1}v_{2,0}^3dt\right)-\frac{1}{\beta^3}\left(4\int_0^t v_{1,0}v_{2,0}^3v_{2,1}dt+\int_0^t v_{1,1}v_{2,0}^4dt\right)\end{array}\right]$$
$$=\left[c_1\left(a_1-a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)\left(a_1-2a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)+\frac{\alpha}{\beta}c_1c_2V_2(a_2-a_{22}c_2)\right]\frac{t^2}{2} \tag{74}$$

$$v_{2,2}(t)=a_2\int_0^t v_{2,1}dt-2a_{22}\int_0^t v_{2,0}v_{2,1}dt=(a_2-a_{22}c_2)c_2(a_2-2a_{22}c_2)\frac{t^2}{2} \tag{75}$$

$$v_{1,3}(t)=a_1\int_0^t v_{1,2}dt-2\int_0^t a_{11}v_{1,0}v_{1,2}dt-\int_0^t a_{11}v_{1,1}^2dt+\frac{\alpha}{\beta}\left[\begin{array}{l}\int_0^t v_{1,0}v_{2,2}dt+\int_0^t v_{1,1}v_{2,1}dt+\int_0^t v_{1,2}v_{2,0}dt-\frac{1}{\beta}\left(\begin{array}{l}2\int_0^t v_{1,0}v_{2,0}v_{2,2}dt+\int_0^t v_{1,0}v_{2,1}^2dt\\ +2\int_0^t v_{1,1}v_{2,0}v_{2,1}dt+\int_0^t v_{1,2}v_{2,0}^2dt\end{array}\right)\\ +\frac{1}{\beta^2}\left(\begin{array}{l}3\int_0^t v_{1,0}v_{2,2}v_{2,0}^2dt+3\int_0^t v_{1,0}v_{2,0}v_{2,1}^2dt\\ +3\int_0^t v_{1,1}v_{2,1}v_{2,0}^2dt+\int_0^t v_{1,2}v_{2,0}^3dt\end{array}\right)-\frac{1}{\beta^3}\left(\begin{array}{l}4\int_0^t v_{1,0}v_{2,0}^3v_{2,2}dt+6\int_0^t v_{1,0}v_{2,1}^2v_{2,0}^2dt\\ +4\int_0^t v_{1,1}v_{2,0}^3v_{2,1}dt+\int_0^t v_{1,2}v_{2,0}^4dt\end{array}\right)\end{array}\right]$$
$$=\left\{\left(a_1-2a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)\left[c_1\left(a_1-a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)\left(a_1-2a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)+\frac{\alpha}{\beta}c_1c_2V_2(a_2-a_{22}c_2)\right]\right\}\frac{t^3}{6}-a_{11}\left[c_1^2\left(a_1-a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)^2\right]\frac{t^3}{3}$$
$$+\left[\frac{\alpha}{\beta}c_1c_2V_2(a_2-a_{22}c_2)(a_2-2a_{22}c_2)\right]\frac{t^3}{6}+\left[\frac{\alpha}{\beta}c_1c_2V_2(a_2-a_{22}c_2)\left(a_1-a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)\right]\frac{t^3}{3}+\left[\frac{\alpha}{\beta^2}c_1c_2^2V_3(a_2-a_{22}c_2)^2\right]\frac{t^3}{3} \tag{76}$$

$$v_{2,3}(t)=a_2\int_0^t v_{2,2}dt-2a_{22}\int_0^t v_{2,0}v_{2,2}dt-a_{22}\int_0^t {v_{2,1}}^2dt=(a_2-a_{22}c_2)c_2\left[(a_2-2a_{22}c_2)^2-2a_{22}c_2(a_2-a_{22}c_2)\right]\frac{t^3}{6} \qquad (77)$$

$$v_{1,4}(t)=\int_0^t a_1v_{1,3}dt-2\int_0^t a_{11}v_{1,0}v_{1,3}dt-2\int_0^t a_{11}v_{1,1}v_{1,2}dt+\frac{\alpha}{\beta}\left[\begin{array}{l}\int_0^t v_{1,0}v_{2,3}dt+v\int_0^t {}_{1,1}v_{2,2}dt+\int_0^t v_{1,2}v_{2,1}dt+\int_0^t v_{1,3}v_{2,0}dt\\ -\frac{1}{\beta}\left(\begin{array}{l}2\int_0^t v_{1,0}v_{2,0}v_{2,3}dt+2\int_0^t v_{1,0}v_{2,1}v_{2,2}dt+2\int_0^t v_{1,1}v_{2,0}v_{2,2}dt\\ +\int_0^t v_{1,1}{v_{2,1}}^2dt+2\int_0^t v_{1,2}v_{2,0}v_{2,1}dt+\int_0^t v_{1,3}{v_{2,0}}^2dt\end{array}\right)\\ +\frac{1}{\beta^2}\left(\begin{array}{l}3\int_0^t v_{1,0}v_{2,3}{v_{2,0}}^2dt+6\int_0^t v_{1,0}v_{2,0}v_{2,1}v_{2,2}dt+\int_0^t v_{1,0}{v_{2,1}}^3dt\\ +3\int_0^t v_{1,1}v_{2,2}{v_{2,0}}^2dt+3\int_0^t v_{1,1}v_{2,0}{v_{2,1}}^2dt+3\int_0^t v_{1,2}v_{2,1}{v_{2,0}}^2dt+\int_0^t v_{1,3}{v_{2,0}}^3dt\end{array}\right)\\ -\frac{1}{\beta^3}\left(\begin{array}{l}4\int_0^t v_{1,0}{v_{2,0}}^3v_{2,3}dt+12\int_0^t v_{1,0}v_{2,1}v_{2,2}{v_{2,0}}^2dt+4\int_0^t v_{1,0}{v_{2,1}}^3v_{2,0}dt\\ +4\int_0^t v_{1,1}{v_{2,0}}^3v_{2,2}dt+6\int_0^t v_{1,1}{v_{2,1}}^2{v_{2,0}}^2dt+4\int_0^t v_{1,2}{v_{2,0}}^3v_{2,1}dt+\int_0^t v_{1,3}{v_{2,0}}^4dt\end{array}\right)\end{array}\right]$$

$$\begin{aligned}&=\left(a_1-2a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)\left\{\begin{array}{l}\left(a_1-2a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)\left[c_1\left(a_1-a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)\left(a_1-2a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)+\frac{\alpha}{\beta}c_1c_2V_2(a_2-a_{22}c_2)\right]\\ -2a_{11}{c_1}^2\left(a_1-a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)^2+\frac{\alpha}{\beta}c_1c_2V_2(a_2-a_{22}c_2)(a_2-2a_{22}c_2)\\ +2\frac{\alpha}{\beta}c_1V_2\left(a_1-a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)(a_2-a_{22}c_2)c_2+2\frac{\alpha}{\beta^2}c_1{c_2}^2V_3(a_2-a_{22}c_2)^2\end{array}\right\}\frac{t^4}{24}\\
&-2a_{11}c_1\left(a_1-a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)\left[c_1\left(a_1-a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)\left(a_1-2a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)+\frac{\alpha}{\beta}c_1c_2V_2(a_2-a_{22}c_2)\right]\frac{t^4}{8}\\
&+\frac{\alpha}{\beta}c_1c_2V_2(a_2-a_{22}c_2)\left[(a_2-2a_{22}c_2)^2-2a_{22}c_2(a_2-a_{22}c_2)\right]\frac{t^4}{24}+\frac{\alpha}{\beta}c_1V_2\left(a_1-a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)c_2(a_2-a_{22}c_2)(a_2-2a_{22}c_2)\frac{t^4}{8}\\
&+\frac{\alpha}{\beta}V_2\left[c_1\left(a_1-a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)\left(a_1-2a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)+\frac{\alpha}{\beta}c_1c_2V_2(a_2-a_{22}c_2)\right]c_2(a_2-a_{22}c_2)\frac{t^4}{8}\\
&+\frac{\alpha}{\beta^2}2c_1V_3(a_2-a_{22}c_2)^2{c_2}^2(a_2-2a_{22}c_2)\frac{t^4}{8}+\frac{\alpha}{\beta^2}c_1V_3\left(a_1-a_{11}c_1+\frac{\alpha}{\beta}c_2V_1\right)(a_2-a_{22}c_2)^2{c_2}^2\frac{t^4}{4}\\
&+\frac{\alpha}{\beta^3}c_1\left(1-\frac{4}{\beta}c_2\right){c_2}^3(a_2-a_{22}c_2)^3\frac{t^4}{4}\end{aligned}$$

(78)

$$\begin{aligned}v_{2,4}(t)&=a_2\int_0^t v_{2,3}dt-2a_{22}\int_0^t v_{2,0}v_{2,3}dt-2a_{22}\int_0^t v_{2,1}v_{2,2}dt\\ &=(a_2-2a_{22}c_2)(a_2-a_{22}c_2)c_2\left[(a_2-2a_{22}c_2)^2-8a_{22}c_2(a_2-a_{22}c_2)\right]\frac{t^4}{24}\end{aligned} \qquad (79)$$

where $V_1=1-\frac{1}{\beta}c_2+\frac{1}{\beta^2}{c_2}^2-\frac{1}{\beta^3}{c_2}^3$; $V_2=1-\frac{1}{\beta}2c_2+\frac{1}{\beta^2}3{c_2}^2-\frac{1}{\beta^3}4{c_2}^3$; $V_3=-1+\frac{1}{\beta}3c_2-\frac{1}{\beta^2}6{c_2}^2$

The approximate solution of $N_1(t), N_2(t)$ is given by (by considering the first 4 terms)

$$N_1(t)=\underset{p\to 1}{Lt}\,v_1(t)=\sum_{k=0}^{4}v_{1,k}(t) \qquad (80)$$

$$N_2(t)=\underset{p\to 1}{Lt}\,v_2(t)=\sum_{k=0}^{4}v_{2,k}(t) \qquad (81)$$

which yield

$$N_1(t) = c_1 + c_1\left(a_1 - a_{11}c_1 + \frac{\alpha}{\beta}c_2V_1\right)t + \left[c_1\left(a_1 - a_{11}c_1 + \frac{\alpha}{\beta}c_2V_1\right)\left(a_1 - 2a_{11}c_1 + \frac{\alpha}{\beta}c_2V_1\right) + \frac{\alpha}{\beta}c_1c_2V_2\left(a_2 - a_{22}c_2\right)\right]\frac{t^2}{2}$$

$$+\left\{\begin{array}{l}\left(a_1 - 2a_{11}c_1 + \frac{\alpha}{\beta}c_2V_1\right)\left[c_1\left(a_1 - a_{11}c_1 + \frac{\alpha}{\beta}c_2V_1\right)\left(a_1 - 2a_{11}c_1 + \frac{\alpha}{\beta}c_2V_1\right) + \frac{\alpha}{\beta}c_1c_2V_2\left(a_2 - a_{22}c_2\right)\right] - 2a_{11}c_1^2\left(a_1 - a_{11}c_1 + \frac{\alpha}{\beta}c_2V_1\right)^2 \\ + \frac{\alpha}{\beta}c_1c_2V_2\left(a_2 - a_{22}c_2\right)\left(a_2 - 2a_{22}c_2\right) + 2\frac{\alpha}{\beta}c_1c_2V_2\left(a_2 - a_{22}c_2\right)\left(a_1 - a_{11}c_1 + \frac{\alpha}{\beta}c_2V_1\right) + 2\frac{\alpha}{\beta^2}c_1c_2^2V_3\left(a_2 - a_{22}c_2\right)^2\end{array}\right\}\frac{t^3}{6}$$

$$+\left(a_1 - 2a_{11}c_1 + \frac{\alpha}{\beta}c_2V_1\right)\left\{\begin{array}{l}\left(a_1 - 2a_{11}c_1 + \frac{\alpha}{\beta}c_2V_1\right)\left[c_1\left(a_1 - a_{11}c_1 + \frac{\alpha}{\beta}c_2V_1\right)\left(a_1 - 2a_{11}c_1 + \frac{\alpha}{\beta}c_2V_1\right) + \frac{\alpha}{\beta}c_1c_2V_2\left(a_2 - a_{22}c_2\right)\right] \\ -2a_{11}c_1^2\left(a_1 - a_{11}c_1 + \frac{\alpha}{\beta}c_2V_1\right)^2 + \frac{\alpha}{\beta}c_1c_2V_2\left(a_2 - a_{22}c_2\right)\left(a_2 - 2a_{22}c_2\right) + 2\frac{\alpha}{\beta}c_1V_2\left(a_1 - a_{11}c_1 + \frac{\alpha}{\beta}c_2V_1\right)\left(a_2 - a_{22}c_2\right)c_2 + 2\frac{\alpha}{\beta^2}c_1c_2^2V_3\left(a_2 - a_{22}c_2\right)^2 \\ -6a_{11}c_1\left(a_1 - a_{11}c_1 + \frac{\alpha}{\beta}c_2V_1\right)\left[c_1\left(a_1 - a_{11}c_1 + \frac{\alpha}{\beta}c_2V_1\right)\left(a_1 - 2a_{11}c_1 + \frac{\alpha}{\beta}c_2V_1\right) + \frac{\alpha}{\beta}c_1c_2V_2\left(a_2 - a_{22}c_2\right)\right] \\ + \frac{\alpha}{\beta}c_1c_2V_2\left(a_2 - a_{22}c_2\right)\left[\left(a_2 - 2a_{22}c_2\right)^2 - 2a_{22}c_2\left(a_2 - a_{22}c_2\right)\right] + 3\frac{\alpha}{\beta}c_1V_2\left(a_1 - a_{11}c_1 + \frac{\alpha}{\beta}c_2V_1\right)c_2\left(a_2 - a_{22}c_2\right)\left(a_2 - 2a_{22}c_2\right) \\ +3\frac{\alpha}{\beta}V_2\left[c_1\left(a_1 - a_{11}c_1 + \frac{\alpha}{\beta}c_2V_1\right)\left(a_1 - 2a_{11}c_1 + \frac{\alpha}{\beta}c_2V_1\right) + \frac{\alpha}{\beta}c_1c_2V_2\left(a_2 - a_{22}c_2\right)\right]c_2\left(a_2 - a_{22}c_2\right) + 6\frac{\alpha}{\beta^2}c_1V_3\left(a_2 - a_{22}c_2\right)^2c_2^2\left(a_2 - 2a_{22}c_2\right) \\ 6\frac{\alpha}{\beta^2}c_1V_3\left(a_1 - a_{11}c_1 + \frac{\alpha}{\beta}c_2V_1\right)\left(a_2 - a_{22}c_2\right)^2c_2^2 + 6\frac{\alpha}{\beta^3}c_1\left(1 - \frac{4}{\beta}c_2\right)c_2^3\left(a_2 - a_{22}c_2\right)^3\end{array}\right\}\frac{t^4}{24}$$

(82)

$$N_2(t) = c_2 + \left(a_2 - a_{22}c_2\right)c_2t + \left(a_2 - a_{22}c_2\right)c_2\left(a_2 - 2a_{22}c_2\right)\frac{t^2}{2} + \left(a_2 - a_{22}c_2\right)c_2\left[\left(a_2 - 2a_{22}c_2\right)^2 - 2a_{22}c_2\left(a_2 - a_{22}c_2\right)\right]\frac{t^3}{6}$$

$$+\left(a_2 - 2a_{22}c_2\right)\left(a_2 - a_{22}c_2\right)c_2\left[\left(a_2 - 2a_{22}c_2\right)^2 - 8a_{22}c_2\left(a_2 - a_{22}c_2\right)\right]\frac{t^4}{24}$$

(83) The values of the parameters in equation(3) which satisfy condition (C) (see section 2.2) are taken as $a_1 = 1; a_{11} = 2.4; p = 2; q = 3.1; a_2 = 0.03; a_{22} = 0.65; N_{10} = 1; N_{20} = 1$

The approximate solutions of equation (3) are obtained by using equations (82) and (83) and their graphical representations are given in figure 5 and Figure 6 . In these figures solution obtained by 4^{th} order Runge- Kutta method are also depicted.

It is observed from the figures5 and 6 that the solutions obtained by the two methods agree up to the calculated degree of accuracy.

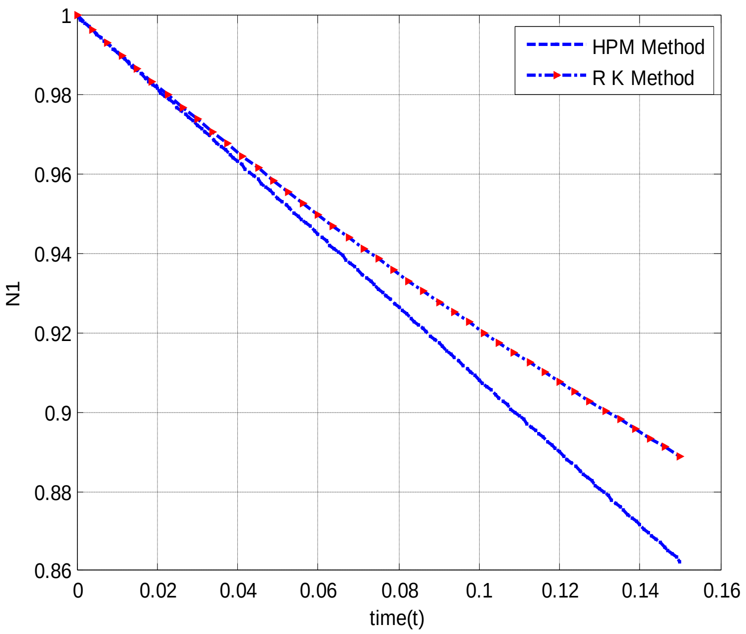

Fig.5. For the values $a_1 = 1; a_{11} = 2.4; p = 2; q = 3.1; a_2 = 0.03; a_{22} = 0.65; N_{10} = 1; N_{20} = 1$,Comparison between the solutions of N1 (Eq(3)) obtained by HPM (calculated up to 4 terms) and R-K method

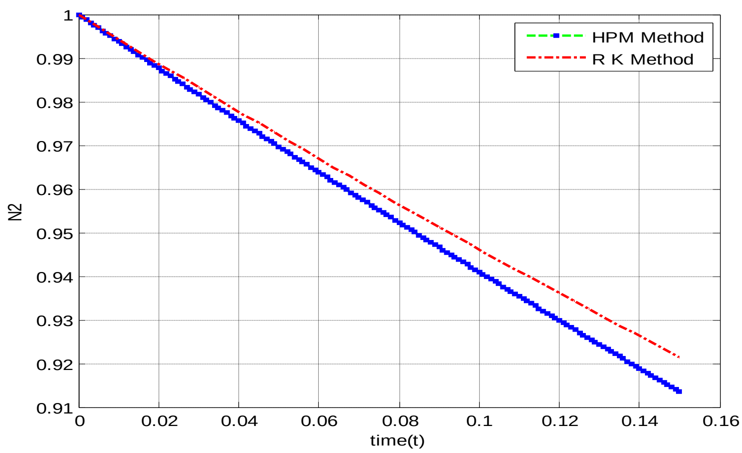

Fig.5. For the values $a_1 = 1; a_{11} = 2.4; p = 2; q = 3.1; a_2 = 0.03; a_{22} = 0.65; N_{10} = 1; N_{20} = 1$,Comparison between the solutions of N2 (Eq(3)) obtained by HPM (calculated up to 4 terms) and R-K method

IV. CONCLUSIONS AND RESULTS.

In this paper using He's homotopy perturbation method, approximate analytical solutions of a system of nonlinear differential equations which represent the interaction between two species are evaluated. Three different models of population dynamics with commensalism between the species are considered and their approximate solutions (first four terms of the series solutions, see sections 3.2.1-3.2.3) are obtained. The solutions of the population models are also evaluated by R-K method of 4^{th} order and compared with that of the solutions obtained by utilising HPM.

In Figs.1and 2 the solution curves of the mathematical model of population dynamics (equation (1)) obtained by the HPM and R-K methods are depicted. It can be observed in Fig 1&2 that there is a very close approximation between the solutions for N1 (Commensal population) and N2 (Host population) in the time interval [0, 0.45] by using only 4 terms of the series given by Eq. (11), which indicates that the speed of convergence of HPM is very fast. A better approximate analytical solution for $t \geq 0.45$ can be achieved by adding more terms to the series in Eq. (11).

In the similar way for the other two models (Eqns(2)&(3)) are also depicted in figs 3,4,5and 6 . The graphs indicate that the approximate solution curves obtained HPM (by considering the first four terms in the series) agree with the solution curves obtained by R-K method of 4^{th} order

Figs.3 and 4 indicate that the solution curves of Eq.(2) by HPM for N1 and N2 almost identical with the solution curves by R-K method in the time interval [0, 0.5].

Figs.5 and 6 demonstrate that the solution curves by four-term HPM and R-K method of the system in Eq. (3) are indistinguishable in the time interval [0, 0.38]

These numerical solutions are obtained by using ode45, an ordinary differential equation solver of *MATLAB* .

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