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Research Paper

The Effect of Mode of Taxation and Transaction Costs On Stochastic Power Utility Maximization of an Insurance Company's Wealth With Consumption and Dividends, Under Proportional Reinsurance.

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ABSTRACT: *This paper studies the effect of taxation and transaction costs on optimal investment problem of power utility maximization of an insurance company's wealth when, dividends, consumption, and reinsurance are involved. The associated H-J-B equation in optimization problem was established using Ito's lemma. The insurance company's surplus process was approximated by a Brownian motion with drift. By solving the H-J-B equations optimal strategies were explicitly derived for cases where transaction costs and taxes were charged on the investment in the risky asset only and the entire investments. Also a relationship was established for the investment in the risky asset for both cases.*
Keywords: *Consumption, dividends, H-J-B Equation, insurance company, power utility, reinsurance, taxes, transaction costs. 2000 Subject AMS Classification: 91B80, 91G30*

I. INTRODUCTION

Utility maximization has been an important issue in optimal investment problems ever since Merton [1] proposed the stochastic control approach to study the investment problem for the first time in mathematical finance and has drawn great attention in recent years. Pliska [2], Karatzas [3] adapted the martingale approach to investment problems of utility maximization. Zhang [4] investigated the utility maximization problem in an incomplete market using the martingale approach.

Investment and reinsurance are two important ways for insurers to balance their profit and risk. Recently, the problem of optimal investment and/or reinsurance for insurers has been extensively studied in the literature. For example, in the framework of utility maximization, Bai and Guo [5], Cao and Wan [6], Irgens and Paulsen [7], and Liang et al. [8] discuss optimal proportional reinsurance-investment problems and Asmussen et al. [9], Gu et al. [10], and Zhang et al. [11] consider optimal excess-of-loss reinsurance-investment problems; in the framework of mean-variance, Bai and Zhang [12], Bauerle [13], Li et al. [14], Zeng et al. [15], and Zeng and Li [16] study optimal investment and/or reinsurance problem.

Browne [17] considered a portfolio problem in continuous time where the objective of the investor or money manager was to exceed the performance of a given stochastic benchmark, as is often the case of institutional money management. Unlike in our project where the benchmark is a fixed point, the benchmark in his paper was a stochastic process that needed not be a perfectly correlated with the investment opportunities and the market is in a sense incomplete. He solved a variety of investment problems related to the achievement of goals for example he finds an optimal investment strategy that maximizes the probability that the return of the investor's portfolio beats the return of the benchmark by a given percentage. He also considered objectives related to the minimization of the expected time until the investor beats the benchmark. The problem of maximizing the expected discounted reward of outperforming the benchmark as well as minimizing the discounted penalty paid upon being outperformed was discussed.

Castillo and Parrocha [18] unlike in our model, considered an insurance business with a fixed amount available for investment in a portfolio consisting of one non-risky asset and one risky asset. They presented the Hamilton-Jacobi-

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Bellman (HJB) equation and demonstrated its use in finding the optimal investment strategy based on some given criteria.

The objective of the resulting control problem was to determine the investment strategy that minimized infinite ruin probability. The existence of a solution to the resulting HJB equation was then shown by verification theorem. A numerical algorithm is also given for analysis.

Hipp and Plum [19] modeled the risk process of an insurance company as a compound poison process unlike ours where the risk process is modeled by Brownian motion with drift. In their paper, they applied stochastic control to answer the following question: if an insurer has the possibility of investing part of his surplus into a risky asset, what is the optimal strategy to minimize the probability of ruin. They observed that the probability of ruin of the risk process can beminimized by a suitable choice of an investment strategy by a capital market index.

The optimal strategy was computed using the Bellman equation. They also proved the existence of a smooth solution and a verification theorem and give explicit solutions in some cases with exponential claim distributions as well as numerical results in a case with Pareto claim size distribution. It was observed that the optimal amount invested cannot be bounded for that last case.

Promislow and Young [20] extended the work of Browne [17] and Schmidli [21] in which they minimized the probability of ruin of an insurer facing a claim process modeled by a Brownian motion with drift.

They Consider Two Controls To Minimize The Probability Of Ruin;

1. Investing in a risky asset (constrained and the non-constrained cases)

2. Purchasing quota-share reinsurance.

They obtained an analytic expression for the minimum probability of ruin and their corresponding optimal controls.

Osu et. al. [22],Osu et. al. [23] and, Ihedioha and Osu [24], discussed the problem of utility maximization with consumption and transaction.

In all the papers reviewed none discussed the effect of the mode of taxation and transaction costs which is the aim of this study.

This paper proceeds as follows. In Section 2, we formulate the optimal investment problem. In section 3 derives the explicit optimal investment strategy for the power utility function. Section 4, concludes the paper.

II. MODEL FORMULATION AND THE MODEL

Let the claim process $k\ (t)$ of an insurance company be described

$$dK(t) = adt - bdZ^1(t) \qquad (1)$$

Where *'a'* and *'b'* are positive constants and $Z^1(t)$ a standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, P)$.

Assuming the premium rate is

$$C = (1+\theta)a \qquad (2)$$

with security risk premium $\theta > 0$. The surplus process of the insurance company using (1) is given by;

$$dR(t) = Cdt - dK(t) = a\theta d(T) + b\ dZ^1(t) \qquad (3)$$

The insurance company is permitted company is permitted to purchase proportional reinsurance to risk to reduce risk and pays reinsurance premium at the rate $(1+\eta)a\ p(t)$ continuously, where $\eta > \theta > 0$ is the security risk of the reinsurer and $p\ (t)$ is is the proportion reinsured at time, t. Then, the surplus of the insurance company is given by;

$$dR(t) = \big(\theta - \eta p(t)\big)adt + b(1-p(t))dZ^1(t) \qquad (4)$$

The insurance company invests her surplus in a market consisting of two assets; a risky asset and a risk free asset.

Let the prices of the risk free asset and risky asset be denoted by $B(t)$ and $S(t)$ respectively, and then the equation governing the dynamics of the risk free asset is given by;

$$dB(t) = rB(t)dt, \qquad (5)$$

and the risky asset by the stochastic differential equation;

$$dS(t) = S(t)[\mu dt + \beta dZ^2(t)], \qquad (6)$$

Where r, μ, and β are constraints. μandβ denote the appreciation rate and the volatility of the risky asset respectively. $Z^2(t)$is another standard Brownian motion and$Z^1(t)$ and $Z^2(t)$are allowed to correlate with correlation coefficient σ. That is

$$Cov\,(Z^1(t)\,, Z^2(t)) = \sigma t \qquad (7)$$

The insurance company holds the risky asset as long as

$$\mu > r \qquad (8)$$

Let $\pi(t)$ be the Naira amount invested in the risky asset at time $t$and the remaining amount $[V(t) - \pi t$ be the Naira amount invested in the risk free asset, where *V(t)* is the surplus process of the insurance company (the company's total investment on both assets).

Suppose the rate of taxes in the financial market is $'\alpha'$, 'b' rate of dividend income, and $'\lambda'$ rate of transaction costs which consists of fees and stamp duties etc.

Two cases shall be considered:

1. The case where transaction costs and taxes are charged only on the risky investment.

2. The case where transaction costs and taxes are charged on the total investment of the insurance company.

Assumptions

The following assumptions are made;

(a). The insurance company makes intermediate consumption decision on admissible consumption space which satisfies

$$\int_0^t |C(s)|ds < \infty, \forall t \in [0, T] \qquad (9)$$

(b). Consumption is made through risk free account only.

(c). Dividends are paid on the investment in the risky asset only.

Therefore, corresponding to the trading strategy $\pi(t)$ and initial capital V_0, the wealth process of the insurance company follows the dynamics:

Case 1: When transaction cost and taxes are charged only on risky investment.

$$dV^\pi(t) = \pi(t)\frac{dS(t)}{S(t)} + [V(t) - \pi(t)]\frac{dB(t)}{B(t)} + b\pi(t)dt - (\alpha + \lambda)\pi(t)dt - C(t)dt + dR(t), \qquad (10)$$

where $C(t)dt$ is the consumption rate.

Substituting for $\frac{dS(t)}{S(t)}, \frac{dB(t)}{B(t)}$ and $dR(t)$ in (10) using (4), (5), and (6) obtained,

$$dV^\pi(t) = \pi(t)[\mu dt + \beta dZ^2(t)] + [V(t) - \pi(t)]rdt + b\pi(t)dt - (\alpha + \lambda)\pi(t)dt - C(t)dt + [\theta - \eta p(t)]adt + b\big(1 - p(t)\big)dZ^1(t). \qquad (11)$$

Further (11) simplifies to

$$dV^\pi(t) = \{[(\mu + b) - (\alpha + \lambda + r)]\pi(t) + rv(t) + [\theta - \eta p(t)]a - c(t)\}dt + \beta\pi(t)dZ^2(t) + b\big(1 - p(t)\big)dZ^1(t). \qquad (12)$$

Case 2: When transaction cost and taxes are changed on the entire investment.

$$dV^\pi(t) = \pi(t)\frac{dS(t)}{S(t)} + [V(t) - \pi(t)]\frac{dB(t)}{B(t)} + b\pi(t)dt - (\alpha + \lambda)V(t)dt - C(t)dt + dR(t)). \qquad (13)$$

This simplifies to

$$dV^\pi(t) = \{[\mu + b) - r]\pi(t) + [r - (\alpha + \lambda)]V(t) + [\theta - \eta p(t)]a - C(t)\}dt + \beta\pi(t)dZ^2(t) + b(1 - p(t))dZ^1(t). \qquad (14)$$

Since $Z^1(t)$ and $Z^2(t)$ are correlating standard Brownian motions with correlation coefficient 'σ', and applying the rule;

$$dZ^1(t).dZ^1(t) = dZ^2(t).dZ^2(t) = dt\,, dZ^2(t).dZ^1(t) = dt.dt = 0, \qquad (15)$$

the quadratic variations of (12) and (14) is;

$$< dV^\pi(t) >= \{[\pi(t)\beta]^2 + [b\big(1 - p(t)\big]^2 + 2\sigma b\beta\big(1 - p(t)\big)\pi(t)\big\}dt. \qquad (16)$$

Therefore, the insurance company's problem can now be written as;

$$H(V, t; T) = \text{Max}_{\pi(t)}\, E\left[U\big(V^\pi(t)\big)\right]|V(0) = v, \qquad (17)$$

subject to;

$$dV^\pi(t) = \{[(\mu + b) - (\alpha + \lambda + r)]\pi(t) + rv(t) + [\theta - \eta p(t)]a - C(t)\}dt + \beta\pi(t)dZ^2(t) + b\big(1 - p(t)\big)dZ^1(t),$$

in the case where transaction costs and taxes are charged **only** on the risky investment, and,

$$H(V, t; T) = \text{Max}_{\pi(t)}\, E\left[U\big(V^\pi(t)\big)\right]|V(0) = v,$$

subject to;

$$dV^{\pi}(t) = \{[\mu + b) - r]\pi(t) + [r - (\alpha + \lambda)]V(t) + [\theta - \eta p(t)]a \;\; C(t)\}dt$$
$$+\beta\pi(t)dZ^2(t) + b(1 - p(t))dZ^1(t),$$

in the case where transaction costs and taxes are charged on the insurance company's total investment (both risky and risk free)

III. THE OPTIMIZATION

This section provides the optimization program for the insurance company's problem. In this work, the optimization problem considered is that based on power utility function.

Suppose the insurance company has power utility preference, the Arrow-Pratt measure of relative risk aversion (RRA) is defined by;

$$R(V) = \frac{U''(V)}{U'(V)}, \qquad (18)$$

where$V$ is the wealth level of the company. Considered, in this case is the utility function of the type;

$$U(V) = \frac{V^{1-\zeta}}{1-\zeta}. \qquad (19)$$

This power utility function has constant relative risk averse parameter ζ.

The aim in this work is to give explicit solutions the insurance company's problems. Considering the assumptions given, the insurance company's problem becomes;

$$H(V,t;T) = \text{Max}_{\pi(t)} E\left[\int_0^T e^{-\vartheta_\tau}\frac{C^{1-\zeta}(\tau)}{1-\zeta}d\tau + e^{-\vartheta_T}\frac{V^{1-\zeta}}{1-\zeta}\right], \zeta \neq 1, \qquad (20)$$

subject to;

$$dV^{\pi}(t) = \{[(\mu + b) - (\alpha + \lambda + r)]\pi(t) + rv(t) + [\theta - \eta p(t)]a - C(t)\}dt$$
$$+\beta\pi(t)dZ^2(t) + b\big(1 - p(t)\big)dZ^1(t),$$

in the case where transaction costs and taxes are charged **only** on the risky investment, and,
subject to;

$$dV^{\pi}(t) = \{[\mu + b) - r]\pi(t) + [r - (\alpha + \lambda)]V(t) + [\theta - \eta p(t)]a - \qquad C(t)\}dt +$$
$$\beta\pi(t)dZ^2(t) + b(1 - p(t))dZ^1(t),$$

in the case where transaction costs and taxes are charged on the insurance company's total investment (both risky and risk free).

Case 1:

The following theorem gives the optimization of the insurance company wealth:

Theorem1:

The optimal policy that maximizes the power utility of an insurance company at the terminal time T, when transaction costs and taxes are charged on the risky investment**only**, is to invest at each time $t \leq T$,

$$\pi_R^*(t) = \frac{\sigma b(1 - p(t))}{\beta} + \frac{[(\mu + b) - (\alpha + \lambda + r)]}{\zeta\beta^2}$$

with optimal reinsured proportion;

$$p^*{}_R(\text{t}) = \left[1 + \frac{\alpha\beta\pi\,(t)}{b}\right] - \frac{\eta a V\,(t)}{\zeta\beta^2},$$

optimal consumption;

$$C_R{}^*(t) = V\left\{e^{(1-n)[q'(T-t)+\frac{s'}{2}(T^2-t^2)]}\left[(1-n)z\int_t^T e^{(1-n)[q'(T-\tau)+\frac{s'}{2}(T^2-\tau^2)]}d\tau + 1\right]\right\}^{\frac{1}{\zeta-1}},$$

and optimal value function;

$$H(V,t;T) = \frac{V^{1-\zeta}}{1-\zeta}\left\{e^{(1-n)[q'[T-t]+\frac{s'}{2}[T^2-t^2]}\left[(1-n)z\int_t^T e^{(1-n)[q'[T-\tau]+\frac{s'}{2}[T^2-\tau^2]}d\tau + 1\right]\right\}^{\frac{\zeta}{1-\xi}}.$$

Proof:

The derivation of the Hamilton-Jacobi-Bellman (H-J-B) partial differential equation starts with the Bellmanequation thus;

$$H(V,t;T) = \text{Max}_{\pi(t)} E\left\{\frac{C^{1-\zeta}}{1-\zeta} + \frac{1}{1+\vartheta}E[H(V',\; t + \Delta t;\; T)]\right\}, \qquad (21)$$

where V'denotes the wealth of the insurance company at time, $t + \Delta t$.

The actual utility and the time interval of length $\Delta t\ is\ \frac{C^{1-\zeta}}{1-\zeta}\ \Delta t$, and the counting over this time interval is $\frac{1}{1+\vartheta}\Delta t$, where $v > 0$.

Re writing equation (21) gives;

$$H(V,t;T)\ =\ Max_{\pi(t)}\left\{\frac{C^{1-\zeta}}{1-\zeta}\Delta t+\ \frac{1}{1+\vartheta\Delta t}t^2[H(V',t+\Delta t;\ T)]\right\} \qquad (22)$$

The multiplication of both sides of (22) by the factor $1+\vartheta\Delta t$ and rearranging gives;

$$\vartheta H\Delta t = \text{Max}_{\pi(t)}\left\{\frac{c^{1-\zeta}}{1-\zeta}\Delta t(1+\vartheta\Delta t)+E[\Delta H]\right\} \qquad (23)$$

The division of (23) by the factor $1+\vartheta\nabla t$ and re-arranging obtained;

$$\vartheta H = \text{Max}_{\pi(t)}\left\{\frac{c^{1-\zeta}}{1-\zeta}+\ \frac{1}{dt}\ E[dH]\right\} \qquad (24)$$

Ito's lemma (Nie, [25]) which states that;

$$dH = \frac{\partial H}{\partial t}dt+\frac{\partial H}{\partial V}dV+\ \frac{1}{2}\frac{\partial^2 H}{\partial v^2}(dV)^2. \qquad (25)$$

Substituting in (25), the Ito's lemma, for $dV^{\pi}(t)$and $< dV^{\pi}(t) >$ using equations (12) and (16), obtains the stochastic differential equation (S.D.E),

$$\frac{C^{1-\zeta}}{1-\zeta}+\ H_t+H_V\{[(\mu+b)-(\alpha+\lambda+r)]\pi(t)+rv(t)+(\theta-\zeta p(t)]a-c(t)\}+$$
$$\frac{1}{2}Hvv\{[\pi(t)\beta]^2+[b\big(1-p(t)]^2+2\sigma b\beta\big(1-p(t)\big)\pi(t)\}=0 \qquad (26a)$$

where

$$E\big(dZ^1(t)\big)=E\big(dz^2(t)\big)=0 \qquad (26b)$$

The application of the first order condition on (26a) with respect to consumption in yields the optimal consumption as;

$$C^{-\zeta}(t)-H_V=0, \qquad (27)$$

which simplifies to

$$C^*(t)=H_V^{-1/\zeta}\ or\ C^*(t)=\ \frac{1}{\sqrt[\zeta]{H_V}}. \qquad (28)$$

Substituting (28) into (26a) gives;

$$H_t+\ \frac{\zeta}{1-\zeta}H_V{}^{1-\frac{1}{\zeta}}+H_V\ \{[(\mu+b)-(\alpha+\lambda+r)]\pi(t)+rV(t)+[\theta-\eta p(t)]a\}+$$
$$\frac{H_{VV}}{2}\{[\pi(t)\beta]^2+[b\big(1-p(t)\big)]^2+2\sigma b\beta(1-p(t))\pi(t)\}=0. \qquad (29)$$

Application of the first order condition to (29) with respect to $\pi(t)$ obtains the equation;

$$H_V[(\mu+b)-(\alpha+\lambda+r)]+H_{VV}[\pi(t)\beta^2+\ \sigma b\beta(1-p(t))]=0. \qquad (30)$$

This simplifies to obtaining the optimal investment in the risky asset as;

$$\pi_R{}^*(t)=\ \frac{\sigma b\,(1-p(t))}{\beta}-\frac{[(\mu+b)-(\lambda+\alpha+r)]H_V}{\beta^2 H_{vv}}. \qquad (31)$$

Also, differentiating (29) with respect to $p(t)$ gives;

$$-\eta aH_V - b^2\big(1-p(t)\big)H_{VV} - \alpha b\beta\pi(t)H_{VV} \qquad (32)$$

and simplifies to;

$$b^2H_{VV}p(t)=\ \eta aH_v\ +\ b^2H_{vv}+\ \sigma bp\pi(t)H_{vv}$$

from which the optimum reinsured proportion of the company' s wealth equals;

$$p_R{}^*(t)=\ 1+\ \frac{\sigma\beta\pi\ (t)}{b}+\frac{\eta a\,H_V}{bH_{VV}} \qquad (33)$$

Using the conjecture that the value function $H(V,t;T)$ is linear to $\frac{V^{1-\zeta}}{1-\zeta}$ and takes the form;

$$H(V,t;T)=\ \frac{V^{1-\zeta}}{1-\zeta}\ Q(t;T) \qquad (34)$$

such that at the terminal time T,

$$Q(T;T) = 1. \quad (35)$$

obtains;

$$\left. H_t = \frac{V^{1-\zeta}}{1-\zeta} Q'(t,T);\ H_V = V^{-\zeta} Q(t,T);\ H_{VV} = -\zeta V^{1-\zeta} Q(t,T) \right\} \quad (36)$$

Therefore; the optimal consumption becomes;

$$C_R{}^*(t) = (V^{-\zeta} Q(t,T))^{-\frac{1}{\zeta}} = VQ^{-\frac{1}{\zeta}}(t,T),$$

$$= V\left\{e^{(1-n)[q'(T-t)+\frac{s'}{2}(T^2-t^2)]}\left[(1-n)z\int_t^T e^{(1-n)[q'(T-\tau)+\frac{s'}{2}(T^2-\tau^2)]}d\tau + 1\right]\right\}^{\frac{1}{\zeta-1}} \quad (37)$$

and optimal investment in the risky asset;

$$\pi^*{}_R(t) = \frac{\sigma b\,(1-p(t))}{\beta} + \frac{[(\mu+b)-(\alpha+\lambda+r)]}{\zeta\beta^2} V \quad (38)$$

which is both horizon and wealth dependent if $(\mu + b) > (\mu + \lambda + r)$ and purely horizon dependent if $(\mu + b{=}\mu{+}\lambda{+}r$.

Also, the optimal reinsured proportion of the company's wealth is;

$$p_R{}^*(t) = \left[\frac{\alpha\beta\pi\,(t)}{b} + 1\right] + \frac{\eta a\, V^{-\zeta} Q(t,T)}{-b^2\zeta v^{-1-\zeta} Q(t,T)}, \quad (39)$$

which simplifies to;

$$p_R{}^*(t) = \left[\frac{\alpha\beta\pi\,(t)}{b} + 1\right] - \frac{\eta a}{b^2\zeta} V \quad (40)$$

which is also dependent on horizon and wealth at hand.
Applying (36) to (29) obtains the ordinary differential equation;

$$\frac{\zeta}{1-\zeta}[Q(t;T)V^{-\zeta}]^{1-\frac{1}{\zeta}} + \frac{V^{1-\zeta}}{1-\zeta} Q'(t,T) + Q(t,T)V^{-\varepsilon}\{[(\mu+b)-(\alpha+\lambda+r)]\pi(t) + rV(t)$$
$$+[\theta - \eta p(t)a]\} + \frac{1}{2}[-\zeta V^{-1-\zeta} Q(t,T)]\{[\pi(t)\beta]^2 + [b(1-p(t)]^2 + 2\sigma b\beta(1-p(t))\pi(t))\} = 0\,. \quad (41)$$

On simplification equation (41) becomes;

$$[Q(t;T)V^{-\zeta}]^{1-\frac{1}{\zeta}} + \frac{1}{\zeta} Q'(t,T) + \frac{1-\zeta}{\zeta} V^{-1} Q(t,T)\{[(\mu+b)-(\alpha+\lambda+r)]\pi(t) + rV(t) + [\theta-\eta p(t)]a\}$$
$$- \frac{(1-\zeta)V^{-2}}{2} Q(t,T)\{[\pi(t)\beta]^2[b(1-p(t))]^2 + 2\sigma b\beta(1-p(t))\pi(t)\} = 0 \quad (42)$$

This further reduces to;

$$Q^n(t,T) + qQ'(t,T) + (r+st)Q(t,T) = 0 \quad (43a)$$

where

$$n = 1 - \frac{1}{\zeta}, q = \frac{1}{\zeta}\ and\ (r+st) = \{\frac{1-\zeta}{\zeta} V^{-1}\{[(\mu+b)-(\alpha+\lambda+r)\pi(t) + rV(t) + [\theta-\eta p(t)]a\}$$
$$-\frac{(1-\zeta)}{2} V^{-2}\{[\pi(t)b]^2 + [b(1-p(t)]^2 + 2\sigma b\beta(1-p(t))\pi(t)\}. \quad (43b)$$

the division of (43a) by Q^n and simplifying yields;

$$Q^{-n}(t;T)\frac{dQ\,(t,T)}{dt} + \frac{(r+st)}{q} Q^{1-n}(t,T) = -\frac{1}{q}, \quad (44)$$

and further simplification leads to;

$$Q^{-n}(t,T)\frac{Q(t,T)}{dt} + (r^1 + s^1 t)\varphi^{1-n} = z, \quad (45a)$$

where,

$$r' = \frac{r}{q}, s' = \frac{s}{q}\ and\ z = \frac{-1}{q}. \quad (45b)$$

Equation (45a) becomes;

$$\frac{dA(t,T)}{dt} + (1-n)(r' + s't)A(t,T) = (1-n)z, \quad (46a)$$

where

$$Q^{-n}(t,T) = A(t,T) \quad (46b)$$

The following theorem helps to solve the above ordinary differential equation (46a).

Theorem 2:

If $y(t)$ and $u(t)$ are continous functions in the interval $I = (t,T)$, then the general solution of $\frac{dA(t,T)}{dt} + y(t)A(t,T) = u(t)$in the interval $I = (t,T)$ is given by

$$A(t,T) = e^{-\int_t^T y(\tau)d\tau'}[\int_t^T q(t)e^{\int_t^T y(s)ds}ds + f] \qquad (47)$$

(Myint, [26]).
Therefore, the solution to (46a) is

$$A(t,T) = e^{(1-n)[q'(T-t)+\frac{s'}{2}[T^2-t^2]}[\int_t^T (1-n)ze^{\int_t^T (1-n)(q^1+s^1k)dk}d\tau + f] \qquad (48)$$

Applying the boundary conditions;

$$f = A(T,T) = 1. \qquad (49)$$

Therefore;

$$A(t,T) = \left\{e^{(1-n)[q'(T-t)+\frac{s'}{2}(T^2-t^2)]}\left[(1-n)z\int_t^T e^{(1-n)[q'(T-\tau)+\frac{s'}{2}(T^2-\tau^2)]}d\tau + 1\right]\right\}, \quad (50)$$

and

$$Q(t,T) = \left\{e^{(1-n)[q'(T-t)+\frac{s'}{2}(T^2-t^2)]}\left[(1-n)z\int_t^T e^{(1-n)[q'(T-\tau)+\frac{s'}{2}(T^2-\tau^2)]}d\tau + 1\right]\right\}^{\frac{-1}{n}}. \quad (51)$$

Substituting (51) into (34) obtains the optimal value function for the insurance company's problem as;

$$H(V,t,T) = \frac{V^{1-\zeta}}{1-\zeta}\left\{e^{(1-n)[q'(T-t)+\frac{s'}{2}(T^2-t^2)]}\left[(1-n)z\int_t^T e^{(1-n)[q'(T-\tau)+\frac{s'}{2}(T^2-\tau^2)]}d\tau + 1\right]\right\}^{\frac{-1}{n}}. \quad (52)$$

Using (43b) and replacing the value of n gives the optimal value function for the insurance company's problem as;

$$H(V,t,T) = \frac{V^{1-\zeta}}{1-\zeta}\left\{e^{(1-n)[q'(T-t)+\frac{s'}{2}(T^2-t^2)]}\left[(1-n)z\int_t^T e^{(1-n)[q'(T-\tau)+\frac{s'}{2}(T^2-\tau^2)]}d\tau + 1\right]\right\}^{\frac{\zeta}{1-\zeta}}. \quad (53)$$

Case 2: When transactions cost and taxes are charged on the insurance company's total investment. The theorem below follows.

Theorem 3: The optimal policy that maximizes the expected power utility of an insurance company's wealth at the terminal time T, when transaction costs and taxes are charged on the company's **total** investment, is to invest in the risky asset at each time t;

$$\pi_T{}^*(t) = \sigma b\left[\frac{p(t)-1}{\beta}\right] + \left[\frac{(\mu+b)-r}{\zeta\beta^2}\right]V,$$

with optimal proportion reinsured

$$p_T{}^*(t) = \left[1 + \frac{\sigma\beta\pi(t)}{b}\right] - \frac{\eta a}{b^2\zeta}V,$$

Optimal consumption

$$C_T{}^*(t) = V\left\{e^{(1-n)[q'(T-t)+\frac{s'}{2}(T^2-t^2)]}\left[(1-n)z\int_t^T e^{(1-n)[q'(T-\tau)+\frac{s'}{2}(T^2-\tau^2)]}d\tau + 1\right]\right\}^{\frac{1}{\zeta-1}}$$

and the optimal value function

$$H_T{}^*(V,t,T) = \frac{V^{1-\zeta}}{1-\zeta}\left\{e^{(1-n)[q'(T-t)+\frac{s'}{2}(T^2-t^2)]}\left[(1-n)z\int_t^T e^{(1-n)[q'(T-\tau)+\frac{s'}{2}(T^2-\tau^2)]}d\tau + 1\right]\right\}^{\frac{\zeta}{1-\zeta}}.$$

Proof:

In this the case, going through the steps of (21) through (26), obtains the H-J-B equation;

$$\frac{C^{1-\zeta}}{1-\zeta} + H_t + H_V\{[(\mu+b)-r]\pi\varepsilon + [r-(\alpha+\lambda)]V(t) - C(t) + [\theta - \eta p(t)]a\} + \frac{H_{VV}}{2}\{[\pi(t)\beta]^2 + [b(1-p(t))]^2 + 2\sigma b\beta(1-p(t))\pi(t)\} = 0. \quad (54)$$

Differentiating (54) with respect to $C(t)$, obtains the optimal consumption;

$$C_V{}^*(t) = H_V{}^{\frac{-1}{\zeta}} \qquad (55)$$

The substitution of (55) into (54) yields;

$\frac{(H_v^{-\frac{1}{\zeta}})^{1-\frac{1}{\zeta}}}{1-\zeta} + H_t + H_V\left\{[(\mu+b)-r]\pi(t) + [r-(\alpha+\lambda)]V(t) - H_V^{-\frac{1}{\zeta}} + [\theta-\eta p(t)]a\right\} +$

$\frac{H_{vv}}{2}\{[\pi(t)\beta]^2 + [b(1-p(t))]^2 + 2\sigma b\beta(1-p(t))\} = 0 .$ (56)

The first order condition applied to (56) with respect to $\pi(t)$ gives the optimal value of $\pi(t)$ as;

$\pi_T{}^*(t) = \frac{\sigma b\,[p(t)-1]}{\beta} - \frac{[(\mu+b)-r]}{\beta^2 H_{VV}} H_V$ (57)

Differentiating 562) with respect to $p(t)$, obtains;

$-\zeta a H_V - b^2 H_{VV} + b^2 p(t) H_{VV} - \gamma b\beta\pi(t) H_{VV}$ (58)

which simplifies to;

$p_T{}^*(t) = \left(1 + \frac{\sigma\beta\pi\,(t)}{b}\right) + \eta\frac{a}{b^2}\frac{H_v}{H_{vv}}.$ (59)

Applying (36) to (57), give the optional investment in the risky asset as;

$\pi_T^*(t) = \frac{\sigma b\,[p(t)-1)}{\beta} + \frac{[(\mu+b)-r]V}{\zeta\beta^2}$ (60)

which is dependent on horizon and wealth . It becomes purely horizon when $(u+b) = r$.
Comparing (60) and (38) obtains,

$$\pi_R^*(t) = \gamma b\left[\frac{p(t)}{\beta} - 1\right] + \frac{[(\mu+b)-(\alpha+\lambda+r)]V}{\varepsilon\beta^2}$$

$= \gamma b\left[\frac{p(t)}{\beta} - 1\right] + \frac{[(\mu+b)-r)V}{\zeta\beta^2} - \frac{[\alpha+\lambda]V}{\zeta\beta^2}.$ (61)

This implies,

$\pi_R^*(t) = \pi_T^*(t) - \frac{[\alpha+\lambda]}{\zeta\beta^2}V$ (62)

Equation (62) shows that charging transaction costs and taxes on the insurance company's total investment will warrant an increment in the risky investment by $[\frac{\alpha+\lambda}{\zeta\beta^2}]$of the total wealth.

Also, applying (36) to (59) gives the optimal reinsured proportion as;

$p_T{}^*(t) = [(1+\sigma b\beta\pi(t))] - \frac{\eta a V}{\zeta b}$ (63)

Comparing (63) and (40) shows that spreading transaction cost and taxes over the total investment or limiting them to the investment in the risky asset does not alter the reinsured proportion of the insurance company's investments .

Now, simplify (56) gives;

$H_V^{1-\frac{1}{\zeta}} + H_t + H_V\left\{[(\mu+b)-r]\pi(t) + [r-(\alpha+\lambda)]V(t) - H_V^{-\frac{1}{\zeta}} + [\theta-\eta p(t)]a\right\} +$

$\frac{H_{VV}}{2}\{[\pi(t)\beta]^2 + [b(1-p(t)]^2 + 2\sigma b\beta(1-p(t))\pi(t)\} = 0.$ (64)

Applying (36) to (64) and simplifying results to ordinary differential equation;

$Q^{1-\frac{1}{\zeta}}(t,T) + \frac{1}{2(1-\zeta)}Q'(t,T) + Q(t,T)\{\frac{V^{-1}}{2}\{[(\mu+b)-r]\pi(t) + [r-(\alpha+\lambda)]V(t) + [\theta-\eta p(t)]a\} -$

$\zeta\frac{v^{-2}}{4}\left\{[\pi(t)\beta]^2 + [b(1-p(t))]^2 + 2\sigma b\beta(1-p(t))\pi(t)\right\} = 0.$ (65)

Equation (65) reduces to;

$Q^n(t;T) + \frac{1}{2(1-\zeta)}Q'(t;T) + Q(t;T)\{[(\mu+b)-r]\pi(t) + [r-(\alpha+\lambda)]V(t) + [\theta - \eta p(t)]a\} -$

$\frac{\zeta v^{-2}}{4}\left\{[\pi(t)\beta]^2 + [b(1-p(t))]^2 + 2\sigma b\beta(1-p(t))\pi(t)\right\} = 0,$ (66)

where $n = 1 - \frac{1}{\zeta}$.

This becomes

$Q^n(t;T) + kQ'(t,T) + (q+st)Q(t,T) = 0$ (67a)

$k = \frac{1}{2(1-\zeta)}$ *and* $q + st =$

$\{[(\mu+b)-r]\pi(t) + [r-(\alpha+\lambda)]V(t) + [\theta - \eta p(t)]a\} - \frac{\zeta v^{-2}}{4}\{[\pi(t)\beta]^2 +$

b1−pt2+2σbβ1−ptπt. (67b)

The division of (67a) by Q^n results to;

$$Q^{-n}(t,T) + (q^{'} + s^{'}t)Q^{1-n}(t;T) = \epsilon \qquad (68a)$$

where

$$q^{'} = \frac{q}{k}, s^{'} = \frac{s}{k} \ and\ \epsilon = \frac{-1}{k} \qquad (68b)$$

Let

$$Q^{1-n}(t,T) = A(t,T), \qquad (69)$$

so that

$$\frac{dA(t;T)}{dt} = (1-n)\frac{dA(t;T)}{dQ(t;T)}\frac{dQ(t;T)}{dt} = (1-n)\frac{dQ(t;T)}{dt}, \qquad (70)$$

and reduces to;

$$\frac{dA}{dt}(t,T) + (1-n)(q^1 + s^1 t)\ A(t,T) = (1-n)\epsilon . \qquad (71)$$

The solution of (71) using **theorem 2**, is;

$$A(t,T) = e^{(1-n)[q^{'}(T-t)+\frac{s^{'}}{2}(T^2-t^2)}\left[(1-n)\epsilon\int_t^T e^{(1-n)[q^{'}(T-\tau)+\frac{s^{'}}{2}(T^2-\tau^2)}d\tau + z\right]. \qquad (72)$$

Applying the boundary condition, gives;

$$A(T,T) = Z = 1, \qquad (73)$$

such that

$$A(t,T) = e^{(1-n)[q^{'}(T-t)+\frac{s^{'}}{2}(T^2-t^2)}\left[(1-n)\epsilon\int_t^T e^{(1-n)[q^{'}(T-\tau)+\frac{s^{'}}{2}(T^2-\tau^2)}d\tau + 1\right] \qquad (74)$$

Therefore,

$$Q(t,T) = \left\{e^{(1-n)[q^{'}(T-t)+\frac{s^{'}}{2}(T^2-t^2)}\left[(1-n)\epsilon\int_t^T e^{(1-n)[q^{'}(T-\tau)+\frac{s^{'}}{2}(T^2-\tau^2)}d\tau + 1\right]\right\}^{\frac{1}{1-n}}, \qquad (75)$$

and the optimal value function ,

$$H(V,t;T) = \frac{V^{1-\zeta}}{1-\zeta}\left\{e^{(1-n)[q^{'}(T-t)+\frac{s^{'}}{2}(T^2-t^2)}\left[(1-n)\epsilon\int_t^T e^{(1-n)[q^{'}(T-\tau)+\frac{s^{'}}{2}(T^2-\tau^2)}d\tau + 1\right]\right\}^{\frac{1}{1-n}} \qquad (76)$$

and at the terminal time T,

$$H(V,T;T) = \frac{V^{1-\zeta}}{1-\zeta},$$

as expected.

IV. CONCLUSION

In this paper, the effect of taxation and transaction costs on stochastic power utility maximization of an insurance company's wealth with consumption and dividends was considered where the company had the liberty to reinsure a proportion of his investment. The model used has the basic claim process assumed to follow a Brownian motion with drift. The insurance company traded in two assets; a risky asset and a risk free asset. The trading was done under dividends yields, transaction costs, tax, and consumption where, transaction costs and tax were considered in two perspectives; when they were charged on the risky asset only and when they were charged on the total investment of the company.

Explicit optimal strategies were obtained solving the resulting H-J-B equations. It was found that charging, transaction costs and tax on the company's total investment increased the company's investment in the risky asset as compared to when transaction cost and tax were charged on the risky asset only by a fraction of the company's total wealth.

These conditions did not alter the optimal reinsured proportion of the company's investment and consumption.

The optimal strategies were found to be both horizon and wealth dependent and the condition for only horizon dependency obtained.

Finally the optimal value functions for the company's expected power utility maximization for both cases considered were obtained.

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