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Research Paper

Special Case of Shum's Inequality of Opial-Type

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ABSTRACT: *In this paper, we study Jensen's inequality for the case of convex functions to obtain a special case of Shum's generalization of Opial's inequality. The main objective of the work was to derive Opial's inequality using Jensen's inequality involving integral functions. The methodology adopted in this paper followed a new trend of establishing integral inequalities. The result obtained was a special case of Opial-type inequality of Shum's.*

Keywords: *Integral inequalities, Opial's inequalities, Shum's generalizations, Jensen's inequalities, convex functions.*

I. INTRODUCTION

Some decades ago a Polish Mathematician called Zdzidlaw Opial established an inequality involving integrals of a function and its derivative, which was named after him as Opial's inequality.

Since it's discovery in 1960, Opial's inequality has proved to be one of the most useful inequalities in Analysis. Opial ([5]) first established the following interesting integral inequality.

Theorem 1.1 *([5]) Let* $x(t)\in C'[0,b]$ *be such* $x(0)=x(b)=0$ *and* $x(t)>0$ *in* $(0,b)$. *Then, the following inequality holds.*

$$\int_0^b x(t)x'(t)\leq\frac{b}{4}\int_0^b(x'(t))^2\,dt. \qquad (1.1)$$

In (1.1), the constant $\frac{b}{4}$ is the best possible.

Literature abounds with papers which generalized Opial's inequality. (cf. [1, 2, 3, ?, 6, 7, ?] and the references in them).

In these generalizations or extensions, the authors have used different methods to obtain their results.

Among the different investigations is Shum's generalizations [6, 7, 8]. Shum in ([7]) gave a general and shapened form of Opial's inequality.

Our objective in this paper is to use the Jensen's inequality for convex functions to obtain an extension of a special form of Shum's generalization in [7].

II. SHUM'S INEQUALITIES RELATED TO OPIAL'S

In 1974, Shum T. D added his own contribution to literature. He used differential inequality for a result that looked more complex in the expression than other generalizations.

Shum ([7])obtained the following inequality.

Theorem 2.1 *Let* $x(t)$ *be absolutely continuous on* $[0,b]$ *and* $x(0)=0$. *If* $l>0$ *and* $\int_0^b |(x'(t))^{l+1}|\,dt<\infty$, *then*

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$$\int_0^b |x(t)|^l |x'(t)|\,dt + \frac{lb^l}{l+1}\int_0^b \frac{g(t)}{t^{l+1}}\,dt \le \frac{b^l}{l+1}\int_0^b |(x'(t))|^{l+1}\,dt \tag{2.1}$$

where

$$g(t) = (l+1)\int_0^b |x(s)|^l |x'(s)|\,ds - |x(t)|^{l+1} \ge 0. \tag{2.2}$$

If either $l < -1$ and both $\int_0^b |x(t)|^l |x'(t)|\,dt < \infty,$ and $\int_0^b |(x'(t))|^{l+1}\,dt < \infty;$ or $-1 < l < 0$ and $\int_0^b |x(t)|^l |x'(t)|\,dt < \infty,$ the reverse inequality holds. Further, for $l > 0$ or $-1 < l < 0$, equality holds in (1.2) if and only if $x(t) = ct,$ whereas for $l < -1,$ equality never holds.

Remark 2.1 *In (1.1), equality holds if and only if* $x'(t)$ *does not change sign on* $[0,b].$ In this case, the inequality (1.2) reduces to

$$\int_0^b |x(t)|^l |x'(t)|\,dt \le \frac{b^l}{l+1}\int_0^b |(x'(t))|^{l+1}\,dt \tag{2.3}$$

which is a special case of Shum's generalization.

Some Adaptations Of Jensen's Inequalities :

Let φ be continuous and convex and let $h(s,t)$ be non negative, $s \ge 0, t \ge 0$ and λ be non decreasing. Let $-\infty \le \xi(t) \le \eta(t) < \infty,$ and suppose φ has a continuous inverse φ^{-1} (which is necessarily concave).Then,

$$\varphi^{-1}\left[\frac{\int_{\xi(t)}^{\eta(t)} h(s,t)\,d\lambda(s)}{\int_{\xi(t)}^{\eta(t)} d\lambda(s)}\right] \ge \left[\frac{\int_{\xi(t)}^{\eta(t)} (\varphi)^{-1} h(s,t)\,d\lambda(s)}{\int_{\xi(t)}^{\eta(t)} d\lambda(s)}\right] \tag{3.1}$$

with the inequality reversed if φ is concave. The inequality (3.1) above is known as Jensen's inequality for convex function . Setting $\varphi(u) = u^l$, $\xi(t) = t$ and $\eta(t) = b$, then as a consequence of (3.1), we have for $l \ge 1$

$$\left[\frac{\int_t^b h(s,t)\,d\lambda(s)}{\int_t d\lambda(s)}\right]^{\frac{1}{l}} \ge \left[\frac{\int_t^b h(s,t)^{\frac{1}{l}}\,d\lambda(s)}{\int_t^b d\lambda(s)}\right],$$

which we write as

$$\left[\int_t^b h(s,t)^{\frac{1}{l}}\,d\lambda(s)\right]^l \le \left[\int_t^b d\lambda(s)\right]^{l-1}\left[\int_t^b h(s,t)\,d\lambda(s)\right] \tag{3.2}$$

and for $0 < l < 1,$ the inequality

$$\int_t^b h(s,t)\,d\lambda(s) \le \left[\int_t^b d\lambda(s)\right]^{1-l}\left[\int_t^b h(s,t)^{\frac{1}{l}}\,d\lambda(s)\right]^l \tag{3.3}$$

as the reverse of (3.3).

Furthermore, if $1 \le l \le p$, it follows from (3.3) that

$$\left[\int_t^b h(s,t)^{\frac{1}{lp}}\,d\lambda(s)\right]^l \le \left[\int_t^b d\lambda(s)\right]^{l-1}\left[\int_t^b h(s,t)^{\frac{1}{p}}\,d\lambda(s)\right]. \tag{3.4}$$

III. MAIN RESULT

In this section, we take a class of functions which does not change sign on $[a,b]$ and we use it to obtain a direct extension of inequality (3.4). Towards this end we need the following lemma.

Lemma 4.1 *Let* $f(t)$ *be continuous function and non-decreasing on [a,b], and* $0 < a \le b < \infty$ *with* $f(t) > 0$ for $t > 0$. *Suppose that* $p \ge l \ge 1, q > 0, 0 < l+q \le p$ and $\delta > 0.$ *Then,*

$$\left|\int_a^b f^q(t) t^{\delta l - 1}\left[\int_t^b f(s)ds\right]^l dt\right| \le [\delta^{-1}]^{\frac{lp-(l+q)+p}{p}}\left[\frac{p}{l+q}\right] a^{\frac{\delta(l+q)}{p}}\left[\left|\int_a^b f^p(s) s^{\frac{lp(1+\delta)}{(l+q)}-(1+\delta)} ds\right|\right]^{\frac{l+q}{p}}$$

$$+[\delta^{-1}]^{\frac{lp-(l+q)+p}{p}}\left[\frac{p}{l+q}\right]\left[\int_a^b \left| f^p(s) s^{\frac{lp(1+\delta)}{(l+q)}-1} ds\right|\right]^{\frac{l+q}{p}} \quad (4.1)$$

Proof:In the inequality (3.4), let

$$\left.\begin{aligned} h(s,t) &= f^{lp}(s) f^{pq}(t) t^{\delta(l+q)} s^{lp(1+\delta)} \\ d\lambda(s) &= s^{-(1+\delta)} ds \end{aligned}\right\} \quad (4.2)$$

Then, we have

$$f^q(t) t^{\frac{\delta(l+q)}{p}}\left[\int_t^b f(s)ds\right]^l \le \left[\left[\frac{s^{-(1+\delta)+1}}{-(1+\delta)+1}\right]_b^t\right]^{l-1} t^{\frac{\delta(l+q)}{p}} f^q(t)\int_t^b f^l(s) s^{(l-1)(1+\delta)} ds$$

$$= [t^{-\delta} - b^{-\delta}]^{l-1}[\delta^{-1}]^{l-1} t^{\frac{\delta(l+q)}{p}} f^q(t)\int_t^b f^l(s) s^{(l-1)(1+\delta)} ds \quad (4.3)$$

Since $q > 0,$ we have $f^q(t) \le f^q(s) \ \forall s \in [t,b].$ Consequently,

$$f^q(t)\int_t^b f^l(s) s^{(l-1)(1+\delta)} ds \le \int_t^b f^{l+q}(s) s^{(l-1)(1+\delta)} ds \quad (4.4)$$

Combining (4.4) and (4.3), yields,

$$f^q(t) t^{\frac{\delta(l+q)}{p}}\left[\int_t^b f(s)ds\right]^l \le [t^{-\delta} - b^{-\delta}]^{l-1}[(\delta)^{-1}]^{l-1} t^{\frac{\delta(l+q)}{p}} \int_t^b f^{l+q}(s) s^{(l-1)(1+\delta)} ds$$

$$= [t^{-\delta} - b^{-\delta}]^{l-1}[\delta^{-1}]^{l-1}\int_t^b t^{\frac{\delta(l+q)}{p}} f^{l+q}(s) s^{(l-1)(1+\delta)} ds \quad (4.5)$$

let $\zeta = \dfrac{l+q}{p}$. Since $0 < \dfrac{l+q}{p} \le 1$, then ζ satisfies the condition in (3.3). Consequently, putting

$$\left.\begin{aligned} h(s,t) &= f^{l+q}(s) t^{\frac{\delta(l+q)}{p}} s^{l(1+\delta)} \\ d\lambda(s) &= s^{-(1+\delta)} ds \end{aligned}\right\} \quad (4.6)$$

We have,

$$\int_t^b f^{l+q}(s) t^{\frac{\delta(l+q)}{p}} s^{l(1+\delta)-(1+\delta)} ds \le \left[\int_t^b \left[f^{l+q}(s) t^{\frac{\delta(l+q)}{p}} s^{l(1+\delta)}\right]^{\frac{p}{l+q}} s^{-(1+\delta)} ds\right]^{\frac{l+q}{p}}$$

$$=[\delta^{-1}]^{1-\frac{l+q}{p}}[t^{-\delta}-b^{-\delta}]^{1-\frac{l+q}{p}}t^{\frac{\delta(l+q)}{p}}\int_t^b\left[f^p(s)s^{\frac{lp(1+\delta)}{l+q}-(1+\delta)}ds\right]^{\frac{l+q}{p}} \tag{4.7}$$

Combining (4.5) and (4.7), we have

$$f^q(t)t^{\frac{\delta(l+q)}{p}}\left[\int_t^b f(s)ds\right]^l \le [\delta^{-1}]^{1-\frac{l+q}{p}}[t^{-\delta}-b^{-\delta}]^{1-\frac{l+q}{p}}t^{\frac{\delta(l+q)}{p}}\int_t^b\left[f^p(s)s^{\frac{lp(1+\delta)}{l+q}-(1+\delta)}ds\right]^{\frac{l+q}{p}} \tag{4.8}$$

On simplifying and arranging the above inequality (4.8), it becomes

$$f^q(t)t^{\frac{\delta(l+q)}{p}}\left[\int_t^b f(s)ds\right]^l[t^{-\delta}-b^{-\delta}]^{\frac{(l+q)-lp}{p}} \le [\delta^{-1}]^{\frac{(l+q)-lp}{p}}t^{\frac{\delta(l+q)}{p}}\int_t^b\left[f^p(s)s^{\frac{lp(1+\delta)}{l+q}-(1+\delta)}ds\right]^{\frac{l+q}{p}} \tag{4.9}$$

Let

$$\delta > 0 \text{ and } \left(\frac{(l+q)-lp}{p}\right) \le 0.$$

Thus

$$\left[t^{-\delta}\right]^{\frac{(l+q)-lp}{p}} \le \left[t^{-\delta}-b^{-\delta}\right]^{\frac{(l+q)-lp}{p}} \quad \forall\ t\in[a,b] \tag{4.10}$$

Putting inequality (4.10) into (4.9), we have

$$t^{\delta l}f^q(t)\left[\int_t^b f(s)ds\right]^l \le [\delta^{-1}]^{\frac{lp-(l+q)}{p}}t^{\frac{\delta(l+q)}{p}}\left[f^p(s)s^{\frac{lp(1+\delta)}{l+q}-(1+\delta)}ds\right]^{\frac{l+q}{p}} \tag{4.11}$$

Multiplying both sides of (4.11) with t^{-1} then,integrate over [a, b] with respect to t, we obtain

$$\int_a^b f^q(t)t^{\delta l-1}f^q(t)\left[\int_t^b f(s)ds\right]^l dt \le [\delta^{-1}]^{\frac{lp-(l+q)}{p}}\int_a^b t^{\frac{\delta(l+q)}{p}-1}\left[\int_t^b f^p(s)s^{\frac{lp(1+\delta}{l+q}-(1+\delta)}ds\right]^{\frac{l+q}{p}}dt \tag{4.12}$$

Take the RHS of (4.12) and then integrate by parts to obtain

$$[\delta^{-1}]^{\frac{lp-(l+q)}{p}}\int_a^b t^{\frac{\delta(l+q)}{p}-1}\left[\int_t^b f^p(s)s^{\frac{lp(1+\delta)}{l+q}-(1+\delta)}ds\right]^{\frac{l+q}{p}}dt$$

$$=(-)[\delta^{-1}]^{\frac{lp-(l+q)}{p}}\left[\frac{p}{l+q}\right]a^{\delta(l+q)/p}\left[\int_a^b f^p(s)s^{\frac{lp(1+\delta)}{l+q}-(1+\delta)}ds\right]^{\frac{l+q}{p}}ds$$

$$+[\delta^{-1}]^{\frac{lp-(l+q)}{p}}\int_a^b f^p(t)t^{\frac{\delta(l+q)}{p}-1}t^{\frac{lp(1+\delta)}{l+q}-1}\times\left[\int_t^b f^p(t)s^{\frac{lp(1+\delta)}{l+q}-(1+\delta)}ds\right]^{\frac{l+q}{p}-1}dt. \tag{4.13}$$

Simplifying (4.13), we have

$$[\delta^{-1}]^{\frac{lp-(l+q)}{p}}\int_a^b t^{\frac{\delta(l+q)}{p}-1}\left[\int_t^b f^p(s)s^{\frac{lp(1+\delta)}{l+q}-(1+\delta)}ds\right]^{\frac{l+q}{p}}dt$$

$$=(-)[\delta^{-1}]^{\frac{lp-(l+q)}{p}}\left[\frac{p}{l+q}\right]a^{\delta(l+q)/p}\left[\int_a^b f^p(s)s^{\frac{lp(1+\delta)}{l+q}-(1+\delta)}ds\right]^{\frac{l+q}{p}} ds$$

$$+[\delta^{-1}]^{\frac{lp-(l+q)}{p}}\int_a^b f^p(t)t^{\frac{\delta(l+q)}{p}-1}\times\left[t^{\delta}\int_t^b f^p(t)s^{\frac{lp(1+\delta)}{l+q}-(1+\delta)}ds\right]^{\frac{l+q}{p}-1} dt. \qquad (4.14)$$

using the fact that for

$$\delta > 0, t^{\delta} \le s^{\delta}\ \forall [t,b]$$

we have

$$[\delta^{-1}]^{\frac{lp-(l+q)}{p}}\int_a^b t^{\frac{\delta(l+q)}{p}-1}\left[\int_t^b f^p(s)s^{\frac{lp(1+\delta)}{l+q}-(1+\delta)}ds\right]^{\frac{l+q}{p}} dt$$

$$=(-)[\delta^{-1}]^{\frac{lp-(l+q)}{p}}\left[\frac{p}{l+q}\right]a^{\delta(l+q)/p}\left[\int_a^b f^p(s)s^{\frac{lp(1+\delta)}{l+q}-(1+\delta)}ds\right]^{\frac{l+q}{p}}$$

$$+[\delta^{-1}]^{\frac{lp-(l+q)}{p}}\int_a^b f^p(t)t^{\frac{lp(1+\delta)}{l+q}-1}\times\left[\int_t^b f^p(s)s^{\frac{lp(1+\delta)}{l+q}-(1+\delta)}s^{\delta}ds\right]^{\frac{l+q}{p}-1} dt. \qquad (4.15)$$

Simplifying (4.15) further, we obtain,

$$[\delta^{-1}]^{\frac{lp-(l+q)}{p}}\int_a^b t^{\frac{\delta(l+q)}{p}-1}\left[\int_t^b f^p(s)s^{\frac{lp(1+\delta)}{l+q}-(1+\delta)}ds\right]^{\frac{l+q}{p}} dt$$

$$=(-)[\delta^{-1}]^{\frac{lp-(l+q)}{p}}\left[\frac{p}{l+q}\right]a^{\frac{\delta(l+q)}{p}}\left[\int_a^b f^p(s)s^{\frac{lp(1+\delta)}{l+q}-(1+\delta)}ds\right]^{\frac{l+q}{p}} ds$$

$$+[\delta^{-1}]^{\frac{lp-(l+q)}{p}}\int_a^b f^p(t)t^{\frac{lp(1+\delta)}{l+q}-1}\times\left[\int_t^b f^p(s)s^{\frac{lp(1+\delta)}{l+q}-1}ds\right]^{\frac{l+q}{p}-1} dt. \qquad (4.16)$$

Evaluating the last integral on the RHS of the (4.16),we have

$$[\delta^{-1}]^{\frac{lp-(l+q)}{p}}\int_a^b t^{\frac{\delta(l+q)}{p}-1}\left[\int_t^b f^p(s)s^{\frac{lp(1+\delta)}{l+q}-(1+\delta)}ds\right]^{\frac{l+q}{p}} dt$$

$$=(-)[\delta^{-1}]^{\frac{lp-(l+q)}{p}}\left[\frac{p}{l+q}\right]a^{\delta(l+q)/p}\left[\int_a^b f^p(s)s^{\frac{lp(1+\delta)}{l+q}-(1+\delta)}ds\right]^{\frac{l+q}{p}} ds$$

$$+[\delta^{-1}]^{\frac{lp-(l+q)}{p}}\left[\frac{p}{l+q}\right]\left[\int_a^b f^p(s)s^{\frac{lp(1+\delta)}{l+q}-1}ds\right]^{\frac{l+q}{p}} dt. \qquad (4.17)$$

We combine (4.12) and (4.17) to obtain,

$$\int_a^b f^q(t)t^{\delta l-1}\left[\int_t^b f(s)ds\right]^l dt \le [\delta^{-1}]^{\frac{lp-(l+q)+p}{p}}\left[\frac{p}{l+q}\right]a^{\frac{\delta(l+q)}{p}}\left[\int_a^b f^p(s)s^{\frac{lp(1+\delta)}{(l+q)}-(1+\delta)}ds\right]^{\frac{l+q}{p}}$$

$$+[\delta^{-1}]^{\frac{lp-(l+q)+p}{p}}\left[\frac{p}{l+q}\right]\left[\int_a^b f^p(s)s^{\frac{lp(1+\delta)}{(l+q)}-1}ds\right]^{\frac{l+q}{p}}. \quad (4.18)$$

Since f is non-decreasing and $f(t)>0$ whenever $t>0$, on $[a,b], 0\le a\le b<\infty$, , we can take modulus of both sides without changing the inequality sign, to obtain.

$$\left|\int_a^b f^q(t)t^{\delta l-1}\left[\int_t^b f(s)ds\right]^l dt\right| \le [\delta^{-1}]^{\frac{lp-(l+q)+p}{p}}\left[\frac{p}{l+q}\right]a^{\frac{\delta(l+q)}{p}}\left[\left|\int_a^b f^p(s)s^{\frac{lp(1+\delta)}{(l+q)}-(1+\delta)}ds\right|\right]^{\frac{l+q}{p}}$$

$$+[\delta^{-1}]^{\frac{lp-(l+q)+p}{p}}\left[\frac{p}{l+q}\right]\left[\int_a^b\left|f^p(s)s^{\frac{lp(1+\delta)}{(l+q)}-1}ds\right|\right]^{\frac{l+q}{p}}. \quad (4.19)$$

This completes the proof of the Lemma.

Theorem 4.1 *Let all assumptions of Lemma 4.1 hold. Then,*

$$\int_a^b |x'(t)||x^l(t)|\,dt \le \frac{b^l}{l+1}\int_a^b |(x'(t))^{l+1}|\,dt + \frac{b^l}{l+1}a^{(l+1)\frac{1}{l}}\int_a^b |(x'(t))^{l+1}|\,dt. \quad (4.20)$$

Proof : Suppose $q=1$ and $p=l+1$ in inequality (4.19) above .Then,

$$\left|\int_a^b t^{\delta l-1}f(t)\left[\int_t^b f(s)ds\right]^l dt\right| \le [\delta^{-1}]^l a^{\delta}\left[\left|\int_a^b f^{l+1}(s)s^{l(1+\delta)-(1+\delta)}ds\right|\right]$$

$$+[\delta^{-1}]^l\left[\int_a^b\left|f^{l+1}(s)s^{l(1+\delta)-1}ds\right|\right] \quad (4.21)$$

Since $f(t)>0, \forall t\in[0,b]$ we can write (4.21) as

$$\int_a^b t^{\delta l-1}\left|f(t)\left[\int_t^b f(s)ds\right]^l\right|dt \le [\delta^{-1}]^l a^{\delta}\left[\int_a^b\left|f^{l+1}(s)\right|s^{(l-1)(1+\delta)}ds\right]$$

$$+[\delta^{-1}]^l\left[\int_a^b\left|f^{l+1}(s)\right|s^{l(1+\delta)-1}ds\right] \quad (4.22)$$

Rearranging and factoring out $t^{\delta l-1}$ gives

$$0\le\int_a^b t^{\delta l-1}\left[[\delta^{-1}]^l\left|(f^{l+1}(t))\right|t^l+[\delta^{-1}]^l a^{\delta}\left|f^{l+1}(t)\right|t^{l-\delta}-\left|f(t)\left[\int_t^b f(s)ds\right]^l\right|\right]dt \quad (4.23)$$

Furthermore, since $t\ge 0$ on [a, b], it then follows from inequality (4.23) that

$$0\le\int_a^b\left[[\delta^{-1}]^l\left|(f^{l+1}(t))\right|t^l+[\delta^{-1}]^l a^{\delta}\left|f^{l+1}(t)\right|t^{l-\delta}-\left|f(t)\left[\int_t^b f(s)ds\right]^l\right|\right]dt. \quad (4.24)$$

Rearranging (4.24) gives

$$\int_a^b\left|f(t)\left[\int_t^b f(s)ds\right]^l\right|dt \le [\delta^{-1}]^l\int_a^b\left|(f^{l+1}(t))\right|t^l dt+[\delta^{-1}]^l a^{\delta}\int_a^b\left|f^{l+1}(t)\right|t^{l-\delta}dt. \quad (4.25)$$

Use the fact that $t^l\le b^l$ for $t\in[a,b]$ (and $l>0$), the inequality (4.25) above to obtain

$$\int_a^b \left| f(t)\left[\int_t^b f(s)ds\right]^l \right| dt$$

$$\leq [\delta^{-1}]^l b^l \int_a^b \left|(f^{l+1}(t))\right| dt + [\delta^{-1}]^l a^\delta b^l \int_a^b \left| f^{l+1}(t)\right| t^{-\delta} dt. \quad (4.26)$$

Note that $\left| f(t)\right| = \left|(-f(t))\right|$. Thus, from inequality (4.25)

$$\int_a^b \left|(-f(t))\left[\int_t^b f(s)ds\right]^l \right| dt$$

$$\leq [\delta^{-1}]^l b^l \int_a^b \left|(-f(t))^{l+1}\right| dt + [\delta^{-1}]^l a^\delta b^l \int_a^b \left|(-f(t))^{l+1}\right| t^{-\delta} dt. \quad (4.27)$$

Now put $\delta = (1+l)^{\frac{1}{l}}$ in the above inequality to obtain

$$\int_a^b \left|(-f(t))\left[\int_t^b f(s)ds\right]^l \right| dt \leq \frac{b^l}{l+1}\int_a^b \left|(-f(t))^{l+1}\right| dt + \frac{b^l}{l+1} a^{(l+1)^{\frac{1}{l}}} \int_a^b \left|(-f(t))^{l+1}\right| t^{-\delta} dt \quad (4.28)$$

If we put $x(t) = \int_t^b f(s)ds$ then, and noting that $|x'(t)| = |-f(t)|$, inequality (4.28) becomes inequality (4.20).

Remark 4.1 *If we let* $a \to 0^+$ *then, inequality (4.28) becomes inequality (1.4).*

We have succeeded in obtaining a special generalization of the inequality of Shum, using Jensen's inequality for convex functions.

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