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Research Paper

Distributive and Standard Ideals

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ABSTRACT: *This paper investigates the concepts of distributive ideal, dually distributive ideal and standard ideal in a join semilattice.*
It concerns with the property of ideals in a distributive semilattice. We obtain a characterization theorem for distributive (dually distributive) and standard ideal in a join semi lattice. We establish the necessary and sufficient condition for a distributive ideal to be standard ideal.
KEYWORDS: *Distributive ideal, Distributive semi lattice, Dually Distributive ideal, Standard ideal, Join Semi Lattice.*

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I. INTRODUCTION:

The concept of distributive ideal, Standard ideal and neutral ideal in lattice L have been introduced and studied by Hashimoto,J[9]., Gratzer,G.,and Schmidt,E.T[4]. In this chapter we study the concept of distributive and dually distributive ideal in a semilattice. We observed that every ideal need not be a distributive ideal(dually distributive). The properties of ideals in a Distributive semilattice were also studied. We study that for distributive (dually distributive) ideal of join semilattice a binary relation θ defined on set of all ideals of a semilattice I(S) is a congruence relation. We give notion of Standard and Dual standard ideal in a join semilattice and established characterization theorem for standard ideal in a join semilattice. And given necessary and sufficient condition for a distributive ideal to be standard ideal in a semilattice,also given the Fundamental theorem of homomarphism and isomorphism theorem for Standard ideals in a semilattice.

Distributive Ideals:

1.1 Definition : A semilattice is a partially ordered set (S, ≤) in which any two elements in S have the least upper bound in S.

1.2 Definition: A semilattice is a non empty set S with binary operation ∨ defined on it and satisfy the following: Idempotent law : a ∨ a = a for all a in S, Commutative law : a ∨ b = b∨a for all a, b in S, Associative law : a∨ (b∨ c) = (a ∨ b)∨ c for all a,b,c in S.

1.3 Theorem : In a semilattice S, define a ≤ b if and only if a ∨ b = b for all a, b in S. Then (S, ≤) is an ordered set in which every pair of elements has a least upper bound, conversely, given an ordered set P with that property, define a ∨ b = l.u.b.(a, b). then (P, ≤) is a semilattice.

Proof : Let (S, ≤) be a semilattice and define ≤ as a ≤ b if and only if a ∨ b = b. First, we check that ≤ is a partial order.

Reflexive : Clearly a ≤ a ⇔ a ∨ a = a

Anti symmetry : Suppose a ≤ b and b ≤ a Thus a ∨ b = b and b ∨ a = a ⇔ a = b ∨ a = a ∨ b = b ⇔ a = b

Transitive : Suppose a ≤ b, b ≤ c, then a ∨ b = b and b ∨ c = c

Now a ∨ c = a ∨ (b ∨ c) =(a ∨ b) ∨ c = b ∨ c = c

Therefore a ∨ c = c ⇔ a ≤ c 25

Now a ∨ (a ∨ b) = (a ∨ a) ∨ b = a ∨ b

Therefore a ≤ a ∨ b

Therefore a ∨ b is an upper bound of a Similarly, we can prove a ∨ b is an upper bound of b.

Therefore a ∨ b = upper bound of {a, b}. Suppose c is any upper bound of {a, b}, Then a ≤ c and b ≤ c ⇒ a ∨ c = c and b ∨ c = c.

Now consider c ∨ (a ∨ b) = (c ∨ a) ∨ b = c ∨ b = c, then a ∨ b ≤ c.

Therefore a ∨ b is a least upper bound of {a, b} in S. Therefore (S, ∨) is a partial ordered set. Conversely, suppose that (S, ≤) is a partial ordered set. To show that (P, ≤) is a semilattice.
Define a relation a ≤ b if and only a ∨ b = b , that is a ∨ b = l.u.b {a, b} Let a ≤ a, then a ∨ a = a which is idempotent.
Let a ≤ b, then a ∨ b = b and b ≤ a which implies b ∨ a = a by the property of anti symmetry a = b. Hence a ∨ b = b ∨ a which is commutative.
Let a ≤ b and b ≤ c, then by the property of transitive we have a ≤ c.
Thus a ∨ b = b; b ∨ c = c and a ∨ c = c Now consider a ∨ (b ∨ c) = a ∨ c = c and (a ∨ b) ∨ c = a ∨ c = c .
Therefore a ∨ (b ∨ c) = (a ∨ b) ∨ c, which is associative. Therefore (P, ≤) is idempotent, commutative and Associate. Hence, (P, ≤) is a semi lattice.
1.4 Definition: A non empty subset D of a join semi lattice S is called an ideal if (i) for x in D, y in D ⇒ x ∨ y in D ,
(ii) for x in D, t in S and t ≤ x ⇒ t in D.
1.5 Theorem: If I(S) denote set of all ideals of a join semi lattice S, then I(S) is a lattice with respective to the following:
(i) D1 ≤ D2 if and only if D1 ⊆ D2
(ii) D1 ∨ D2 = { x in S / x = x1 ∨ x2 , where x1 is in D1, x2 is in D2}
(iii) D1∧ D2 = {x in S / x is in D1 and x is in D2}; where D1, D2 are in I(S). Proof: Let I(S) be set of all ideals of a semi lattice S. Claim : I(S) is a lattice. First we prove that I(S) is a partially order set .By using given three conditions (i), (ii) & (iii) we have Reflexive : D1 ≤ D1 ⇔ D1 ⊆ D1 Ant symmetric : Suppose D1 ≤ D2 and D2 ≤ D1 ⇒ D1 ⊆ D2 and D2 ⊆ D1 ⇒ D1 = D2 Transitive : Suppose D1 ≤ D2 and D2 ≤ D3 ⇒ D1 ⊆ D2 and D2 ≤ D3 ⇒ D1 ⊆ D3 and D1 ≤ D3 Therefore (I(S), ≤) is partially ordered set.
Now, to show that I(S) has least upper bound (lub) and greatest lower bound (glb) To show that D1 ∨ D2 is an ideal.
We give Proof of D1 ∨ D2 is an ideal by different cases.
(i) Let x, y ∈ D1 ∨ D2 ⇒ x ∈ D1 ∨ D2 and y ∈ D1 ∨ D2 27 ⇒ x ∈ D1 or x ∈ D2 and y ∈ D1 or y ∈ D2 ⇒ x ∨ y ∈ D1 as D1 is an ideal and x ∨ y ∈ D2 as D2 is an ideal. ⇒ x ∨ y ∈ D1 ∨ D2
(ii) Let x be any element in D1 ∨ D2 and t ∈ S such that t ≤ x, we examin the following cases. Since x ∈ D1 ∨ D2, x = x1 ∨ x2 where x1 ∈ D1 and x2 ∈ D2 (i) Suppose x1 ≤ t and t ≤ x2, then t ∈ D2 as D2 is an ideal and x2 ∈ D2 Now t ∨ x1 = t as t ∈ D2 and x1 ∈ D1 , thus t ∈ D1 ∨ D2 .
(ii) Let x1 ≤ t and x2 ≤ t , thus x1 ∨ x2 ≤ t Since, we have t ≤ x1 ∨ x2 Therefore t = x1 ∨ x2 ∈ D1 ∨ D2
(iii) Let t ≤ x1 and x2 ≤ t, thus t ∈ D1 as D1 is an ideal and x2 ∈ D2. Now x2 ∨ t = t as x2 ∈ D2 and t ∈ D1. Therefore t ∈ D1 ∨ D2 (iv) Let t ≤ x1 and t ≤ x2, thus t ∈ D1 and t ∈D2 as D1 and D2 are ideals. Now t ∨ t = t ∈ D1 ∨ D2. Therefore, by above different cases, we can conclude that D1 ∨ D2 is an ideal. To prove that D1 ⊆ D1 ∨ D2 and D2 ⊆ D1 ∨ D2 Let t ∈ D1 and x2 be any element in D2 ⇒ t ∨ x2 ∈ D1 ∨ D2 Since t ≤ t ∨ x2 and t ∨ x2 ∈ D1 ∨ D2 we have t ∈ D1 ∨ D2.
Therefore D1⊂ D1 ∨ D2. 28 Similarly D2 ⊂ D1 ∨ D2 Now to show that D1 ∨ D2 is the smallest ideal containing D1 and D2. Let D be an ideal such that D1⊂ D and D2 ⊂ D To show that D1 ∨ D2 ⊂ D Let t ∈ D1 ∨ D2 ⇒ t = t1 ∨ t2 where t1 ∈ D1; t2 ∈ D2 t1 ∈ D1 ⇒ t1 ∈ D (since D1 ⊂ D) and t2 ∈ D2 ⇒ t2 ∈ D (since D2 ⊂ D) Now t1 ∈ D and t2 ∈ D ⇒ t1 ∨ t2 ∈ D (since D is an ideal) ⇒ t ∈ D Therefore D1 ∨ D2 ⊂ D
Hence, D1 ∨ D2 is the smallest ideal containing both D1 and D2.
Define a relation as follows If D1 ≤ D2 ⇔ D1 ∨ D2 = D2 and D1 ∧ D2 = D1 Now D1 ∨ (D1 ∨ D2) = (D1 ∨ D2) ∨ D2 = D1 ∨ D2 then D1 ≤ D1 ∨ D2 . Therefore D1 ∨ D2 is an upper bound of D1 Similarly D2 ∨ (D1 ∨ D2) = D2 ∨ D1 ∨ D2 = D1 ∨ D2 ∨ D2 = D1 ∨ D2 ⇒ D2 ≤ D1 ∨ D2 Therefore D1 ∨ D2 is an upper bound of D2,
Therefore D1 ∨ D2 is an upper bound of { D1, D2} If D is any upper bound of D1, D2 , then D1 ≤ D and D2 ≤ D 29 Thus D1 ∨ D = D and D2 ∨ D = D Now (D1 ∨ D2) ∨ D = D1 ∨ (D2 ∨ D) = D1 ∨ D = D Therefore D1 ∨ D2 is a least upper bound of { D1, D2} To show that D1 ∧ D2 is an ideal of S.
(i)Let x ∈ D1 ∧ D2, y ∈ D1 ∧ D2 .
Then x ∈D1 and x ∈ D2, y ∈ D1 and y ∈ D2 Which implies x ∨ y ∈ D1 as D1 is an ideal and x ∨ y ∈ D2 as D2 is an ideal Therefore x ∨ y ∈ D1 ∧ D2 .
(ii)Let x ∈ D1 ∧ D2 and t ∈ S such that t ≤ x Then x ∈D1 and x ∈ D2. As x ∈ D1 and t ≤ x. We have t ∈ D1. As x ∈ D2 and t ≤ x, we have t ∈ D2 .Therefore t ∈ D1 ∧ D2 Hence D1 ∧ D2 is an ideal of S.
Now D1 ∧ (D1 ∧ D2) = (D1 ∧ D2) ∧ D2 = D1 ∧ D2 Which implies D1 ∧ D2 ≤ D1 . Similarly D1 ∧ D2 ≤ D2 Therefore D1 ∧ D2 is a lower bound of {D1,D2} Suppose D is any lower bound of {D1,D2} Then D1 ∧ D = D = D ∧ D2 Now (D1 ∧ D2) ∧ D = D1 ∧ (D2 ∧ D) = D1 ∧ D = D,
Which implies D ≤ D1 ∧ D2 Therefore D1 ∧ D2 is a greatest lower bound of { D1,D2} Therefore I(S) has both lub and glb.
Hence I(S) is a lattice.

1. 6 Definition: The smallest ideal containing x in a join semi lattice S is denoted by (x] and is given by (x] = { s in S / s ≤ x} such ideal is called principal ideal generated by x. 30 1.7 Definition: An ideal D of a join semi lattice S is called distributive ideal if and only if D∨ (X ∧ Y) = (D ∨ X) ∧ (D ∨ Y) for all X, Y in I(S). 1.8 Definition: An ideal D of a join semi lattice S if called dually distributive ideal if and only if D ∧ (X ∨ Y) = (D ∧ X) ∨ (D ∧ Y) for all X, Y in I(S).

1.7 Theorem: A join semi lattice S is distributive if and only if
(i) S is directed below.
(ii) The lattice I(S) of all ideals of S is a distributive lattice.
Proof: Suppose a semi lattice S is distributive.
(i) To prove that S is directed below: Let a, b are in S ⇒ a ∨ b ∈ S. Since a ≤ a ∨ b ⇒ there exists x, y in S such that x ≤ a, y ≤ b and a = x ∨ y. Since y ≤ x ∨ y = a ⇒ y ≤ a also y ≤ b. Therefore for a, b in S there exists y in S such that y ≤ a, y ≤ b. Therefore S is directed below.
(ii) To prove that the lattice I(S) is distributive: To show that (a) D1 ∨ (D2 ∧ D3) = (D1 ∨ D2) ∧ (D1 ∨ D3) (b) D1 ∧ (D2 ∨D3) = (D1 ∧ D2) ∨ (D1 ∧ D3) where D1, D2 , D3 are in I(S) Define D1 ∨ D2 = { x in S/x = x1 ∨ x2, for x1 in D1 , x2 in D2 } D1 ∧ D2 = { x in S / x in D1 and x in D2 }.
Let x ∨ y ∈ D1 ∨ (D2 ∧ D3) then x ∨ y ∈ D1 or x ∨ y ∈ (D2 ∧ D3). ⇔ x ∈ D1, y ∈ (D2 ∧ D3) ⇔ x ∈ D1, y ∈ D2 and y ∈ D3 ⇔ x ∈ D1 , y ∈ D2 and x ∈ D1 , y ∈ D3 ⇔ x ∨ y ∈ D1 ∨ D2 and x ∨ y ∈ D1 ∨ D3 ⇔ x ∨ y ∈ (D1 ∨ D2) ∧ (D1 ∨ D3) Therefore D1 ∨ (D2 ∧ D3) = (D1 ∨ D2) ∧ (D1 ∨ D3). (b) Let x ∨ y ∈ (D1 ∧ D2) ∨ (D1 ∧ D3) , then x ∨ y ∈ (D1 ∧ D2) or x ∨ y ∈ (D1 ∧ D3) ⇔ x ∈ D1 ∧ D2 , y ∈ D1 ∧ D3 ⇔ x ∈ D1 and x ∈ D2 , y ∈ D1 and y ∈ D3 ⇔ x ∈ D1 , y ∈ D1 and x ∈ D2 , y ∈ D3 ⇔ x ∨ y ∈ D1 and x ∨ y ∈ D2 ∨ D3 ⇔ x ∨ y ∈ D1 ∧ (D2 ∨ D3).
Therefore D1 ∧ (D2 ∨ D3) = (D1 ∧ D2) ∨ (D1 ∧ D3). Hence I(S) is a distributive lattice. Conversely, suppose that S is directed below and I(S) is distributive lattice.
Claim: S is distributive semi lattice. Let w ≤ a ∨ b where a, b, w ∈ S. Now (w] = (w] ∧ ((a] ∨ (b])= ((w] ∧ (a]) ∨ ((w] ∧ (b]) = a0 ∨ a1, where a0 ∈ (a], a1 ∈ (b]. Hence there exists a0, a1 in S such that a0 ≤ a; a1 ≤ b and (w] = a0 ∨ a1. Therefore S is distributive semi lattice.
1.7 Definition: A binary relation θ on a lattice L is called congruence relation if (i) θ is reflexive : x ≡ x (θ) for all x in L
(ii) θ is symmetric : x ≡ y (θ) ⇒ y ≡ x (θ) for all x, y in L
(iii) θ is transitive : x ≡ y(θ) and y ≡ z(θ) ⇒ x ≡ z(θ) for all x, y, z in L (iv) Substitution Property : x ≡ x1(θ) and y ≡ y1(θ) ⇒ x ∨ y ≡ x1 ∨ y1(θ) and x ∧ y ≡ x1 ∧ y1 (θ) for all x, y, x1, y1 in L.
1.8 Theorem: Let D be an ideal of join semilattice S. Then the following conditions are equivalent.
(i) D is distributive.
(ii) The map ϕ : X → D ∨ X is a homomorphism of I(S) onto [D) = {X in I(S) / X ≥ D}.
(iii) The binary relation θD on I(S) is defined by X ≡ Y (θD) if and only if D ∨ X = D ∨ Y, where X, Y in I(S) is a congruence relation. Proof: Let D be an ideal of join semi lattice S.
To prove that (i) ⇒ (ii): Suppose (i) holds. Then D∨ (X ∧Y) = (D ∨ X) ∧ (D ∨ Y) for all X, Y in I(S) Define a map ϕ : X → D ∨ X by ϕ (X) = D ∨ X. 34 To prove that ϕ is a homomorphism: Let X, Y in I(S) be arbitrary. Consider ϕ (X ∨ Y) = D ∨ (X ∨ Y) = (D ∨ D) ∨ (X ∨ Y) = D ∨ [D ∨ (X ∨ Y)] = D ∨ [D ∨ X ∨ Y] = D ∨ (D ∨ X) ∨ Y)] = (D ∨ X) ∨ (D ∨ Y) = ϕ (X) ∨ ϕ (Y). Similarly, ϕ (X ∧ Y) = D ∨ (X ∧Y) = (D ∨ X) ∧ (D ∨ Y) = ϕ (X) ∧ ϕ (Y). Therefore ϕ is homomorphism.
To prove that ϕ is onto: Let X in [D)⇒ X in I(S) such that X ≥ D⇒ ϕ (X) = D ∨ X = X Therefore for any X in [D), there exists X in I(S) such that ϕ (X) = X Therefore ϕ is homomorphism of I(S) onto [D). To prove (ii) ⇒ (iii): Suppose the map ϕ : X → D ∨ X is a homomorphism of I(S) onto [D) = { X in I(S) / X ≥ D}.Define the binary relation θD in I(S) .
as X ≡ Y (θD) if and only if D ∨ X = D ∨ Y where X, Y in I(S). To show that the relation is congruence: (a) Let X in I(S) be arbitrary then D ∨ X = D ∨ X ⇒ X ≡ X (θD) for all X in I(S).

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