Quest Journals
Journal of Research in Applied Mathematics
Volume 2 ~ Issue 12 (2016) pp: 20-26
ISSN(Online) : 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Research Paper

Fixed Points Theorems for Variants of Compatible Mappings of Types in Menger Space

Balbir Singh
Department of Mathematics
B.M.Institute of Engineering and Technology, Sonipat, Haryana, India.

Pawan Kumar
Department of Mathematics Maitreyi College, University of Delhi
Chankya puri, New Delhi-110021

ABSTRACT. *In this paper, we introduce the notions of compatible mappings of type(R), of type (K) and of type (E) in Menger space and prove some common fixed point theorems for these mappings.*
Mathematics Subject Classification: *47H10, 54H25.*
KEY WORDS: *Menger space, Compatible mappings, Compatible mappings of type (R), of type (K), of type (E).*

Received 14 May, 2016; Revised: 10 July, 2016; Accepted 27 September 2016, Published: 30 September 2016

I. INTRODUCTION

The notion of probabilistic metric space as a generalization of metric space was introduced by Menger [12]. In Menger theory, the notion of probabilistic metric space corresponds to situations when we do not know exactly the distance between two points, but we know probabilities of possible values of this distance. In this note he explained how to replace the numerical distance between two points x and y by a function $F_{x,y}(t)$ whose value $F_{x,y}(t)$ at the real number t is interpreted as the probability that the distance between x and y is less than t. In fact the study of such spaces received an impetus with the pioneering work of Schweizer and Sklar [17]. The theory of probabilistic metric space is of paramount importance in Probabilistic Functional Analysis especially due to its extensive applications in random differential as well as random integral equations.

Now, we give preliminaries and basic definitions in Menger space which are useful in this paper.

Definition 1.1[17] A mapping $F: \mathbb{R}^+ \to \mathbb{R}^+$ is called distribution function if it is non decreasing and left continuous with inf $\{F(t): t \in \mathbb{R}^+\} = 0$ and sup $\{F(t): t \in \mathbb{R}^+\} = 1$. We will denote the set of all distribution functions by $\mathcal{L}$.

Let $\mathcal{L}$ be the set of all distribution functions whereas H be the set of specific distribution function (Also known as Heaviside function) defined by

$$H(t) = \begin{cases} 0, if\ t \leq 0 \\ 1, if\ t > 0. \end{cases}$$

Definition 1.2[12] A probabilistic metric space is a pair (X, F), where X is a nonempty set and $F: X \times X \to \mathcal{L}$ is a mapping satisfying the following:

For all $x, y, z \in X$ and $t, s \geq 0$,

(p_1) $F_{x,y}(t) = 1$ if and only if $x = y$;

(p_2) $F_{x,y}(0) = 0$;

(p_3) $F_{x,y}(t) = F_{y,x}(t)$;

(p_4) $F_{x,y}(t) = 1$ and $F_{y,z}(s) = 1$, then $F_{x,z}(t+s) = 1$.

Every metric space (X, d) can always be realized as a Probabilistiv metric space by $F_{x,y}(t) = H(t - d(x,y))$, for all $x, y \in X$, where H be the set of specific distribution function (Also known as Heaviside function) defined by

$$H(t) = \begin{cases} 0, if\ t \le 0 \\ 1, if\ t > 0 \end{cases}; \quad \text{and } F: X \times X \to \mathcal{L}.$$

Probabilistic metric space offers a wider framework than that of the metric space and cover even wider statistical situations.

Definition 1.3[17] A mapping $\Delta : [0,1] \times [0,1] \to [0,1]$ is called a t-norm if for all $a, b, c \in [0,1]$,

(1) $\Delta(a, 1) = a, \Delta(0,0) = 0;$

(2) $\Delta(a, b) = \Delta(b, a);$

(3) $\Delta(c, d) \ge \Delta(a, b)\ for\ c \ge a, d \ge b;$

(4) $\Delta(\Delta(a, b), c) = \Delta(a, \Delta(b, c)).$

Example 1.1 The following are the four basic t-norms:

(i) The minimum t-norm: $\Delta_M(a, b) = \min\{a, b\}$.

(ii) The product t-norm: $\Delta_P(a, b) = \text{ab}$.

(iii) The Lukasiewicz t-norm: $\Delta_L(a, b) = \max\{\text{a} + \text{b} - 1{,}0\}$.

(iv) The weakest t-norm, the drastic product:

$$\Delta_D(a, b) = \begin{cases} \min\{a, b\}\ if \max\ \{a, b\} = 1, \\ 0, \quad otherwise. \end{cases}$$

We have the following ordering in the above stated norms:

$$\Delta_D < \Delta_L < \Delta_P < \Delta_M\ .$$

Definition1.4 [12] A Menger space is a triplet (X, F, Δ), where (X, F) is a probabilistic metric space and Δ is a t-norm with the following condition:

For all $x, y, z \in X$ and $t, s \ge 0$,

(p_5) $F_{\text{x},\text{z}}(\text{t} + \text{s}) \ge \Delta(F_{\text{x},\text{y}}(\text{t}), F_{\text{y},\text{z}}(\text{s}))$.

Example 1.2 Let $X = \mathbb{R}$, $\Delta\,(a, b) = min\,(a, b)$, for all $a, b\ in\ [0, 1]$ and

$$F(x, y, t) = \begin{cases} H(t), & if\ x \ne y \\ 1, & if\ x = y \end{cases}; \quad \text{where } H(t) = \begin{cases} 0, if\ t \le 0 \\ 1, if\ t > 0. \end{cases}$$

Then (X, F, Δ) is a Menger space.

Definition 1.5 A sequence $\{x_n\}$ in Menger space (X, F, Δ) is said to be:

(i) Convergent at a point $x \in X$ if for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer $N_{\epsilon,\lambda}$ such that $F_{x_n,x}(\epsilon) > 1 - \lambda$ for all $n \ge N_{\epsilon,\lambda}$.

(ii) Cauchy sequence in X if for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer $N_{\epsilon,\lambda}$ such that $F_{x_n,x_m}(\epsilon) > 1 - \lambda$ for all $n, m \ge N_{\epsilon,\lambda}$.

(iii) Complete if every Cauchy sequence in X is convergent in X.

In 1996, Jungck [8] introduce the notion of weakly commuting mappings.

Definition 1.6[8] Two self-mapping f and g of a Menger space (X, F, Δ) are said to be weakly commuting if $F(fgx, gfx, t) \ge F(fx, gx, t)$ for each $x \in X$ and for each $t > 0$.

In 1982, Sessa [18] weakened the concept of commutativity to weakly commuting mappings. Afterwards, Jungck [9] enlarged the concept of weakly commuting mappings to compatible mappings. In 1991, Mishra [13] introduced the notion of compatible mappings in the setting of probabilistic metric space.

Definition 1.7[13] Let (X, F, Δ) be a Menger space such that the t –norm Δ is continuous and f, g be mappings from X into itself. Then f and g are said to be compatible if $lim_{n\to\infty}F(fgx_n, gfx_n, t) = 1$, whenever $\{x_n\}$ is a sequence in X such that $lim_{n\to\infty}fx_n = lim_{n\to\infty}gx_n = u$ for some $u \in X$ and for all $t > 0$.

Definition 1.8 Let f and g be self- mapping on Menger space $(X, F, \Delta\,)$. Then f and g are said to be non-compatible if either $lim_{n\to\infty}\, F(fgx_n, gfx_n\,, t)$ is non-existent or $lim_{n\to\infty}\, F(fgx_n, gfx_n\,, t) \ne 1$, whenever $\{x_n\}$ is a sequence in X such that $lim_{n\to\infty}\, fx_n = lim_{n\to\infty}\, gx_n = u$ for some $u \in X$ and for all $t > 0$.

Further, Singh and Jain [19] proved some fixed point theorems for weakly compatible maps in the setting of Menger space.

Definition 1.9[19] Two maps f and g are said to be weakly compatible if they commute at their coincidence points.

In 1999, Pant [15] introduced a new continuity condition in Menger space, known as reciprocal continuity as follows:

Definition 1.10[15] Let f and g be self- mapping of a Menger space $(X, F, \Delta\,)$. Then f and g are said to be reciprocally continuous if $lim_{n\to\infty}\, fgx_n = fz,\ lim_{n\to\infty}\, gfz_n = gz$, whenever $\{x_n\}$ is a sequence in X such that $lim_{n\to\infty}\, fx_n = lim_{n\to\infty}\, gx_n = z$ for some $z \in X$.

Remark 2.1[15] If f and g are both continuous, then they are obviously reciprocally continuous, but the converse is not true. Moreover, common fixed point theorems for compatible pair of self mappings satisfying contractive conditions, continuity of one of the mappings implies their reciprocal continuity, but not conversely.

In 2004, Rohan et al. [16] introduced the concept of compatible mappings of type (R) in a metric space as follows:

Definition 1.11[16] Let f and g be mappings from metric space (X, d) into itself. Then f and g are said to be compatible of type (R) if
$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$ and $\lim_{n\to\infty} d(ffx_n, ggx_n) = 0$,
whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in X.
In 2007, Singh and Singh et al. [20] introduced the concept of compatible mappings of type (E) in a metric space as follows:
Definition 1.12[20] Let f and g be mappings from metric space (X, d) into itself. Then f and g are said to be compatible of type (E) if $\lim_{n\to\infty} ffx_n = \lim_{n\to\infty} fgx_n = gt$ and $\lim_{n\to\infty} ggx_n = \lim_{n\to\infty} gfx_n = ft$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in X.
In 2014, Jha et al. [5] introduced the concept of compatible mappings of type (K) in a metric space as follows:
Definition 1.13[5] Let f and g be mappings from metric space (X, d) into itself. Then f and g are said to be compatible of type (K) if
$\lim_{n\to\infty} d(ffx_n, gt) = 0$ and $\lim_{n\to\infty} d(ggx_n, ft) = 0$,
whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in X.

II. PROPERTIES OF COMPATIBLE MAPPINGS OF TYPES.

Recently, Kang et al.[10] introduced the notion of compatible mappings and its variants in a multiplicative metric space.

Now we introduce the notions of compatible mappings of types in the setting of a Menger space as follows:

Definition 2.1 Let f and g be self- mapping on Menger space (X, F, Δ). Then f and g are called
(1) *Compatible of type (R)* if $lim_{n\to\infty} F(fgx_n, gfx_n, t) = 1$, and $lim_{n\to\infty} F(ffx_n, ggx_n, t) = 1$ whenever $\{x_n\}$ is a sequence in X such that $lim_{n\to\infty} fx_n = lim_{n\to\infty} gx_n = u$ for some $u \in X$ and for all $t > 0$.
(2) *Compatible of type (K)* if $lim_{n\to\infty} F(ffx_n, gx, t) = 1$, and $lim_{n\to\infty} F(ggx_n, fx, t) = 1$ whenever $\{x_n\}$ is a sequence in X such that $lim_{n\to\infty} fx_n = lim_{n\to\infty} gx_n = x$ for some $x \in X$ and for all $t > 0$.
(3) *Compatible of type (E)* if if $\lim_{n\to\infty} ffx_n = \lim_{n\to\infty} fgx_n = gt$ and $\lim_{n\to\infty} ggx_n = \lim_{n\to\infty} gfx_n = ft$, whenever $\{xn\}$ is a sequence in X such that $\lim n\to\infty fxn = \lim n\to\infty gxn = x$ for some x in X.
Now we give some properties related to compatible mappings of type (R) and type (E).
Proposition 2.1 Let f and g be compatible mappings of type (R) of a Menger space (X, F, Δ) into itself. If $fx = gx$ for some $x \in X$, then $fgx = ffx = ggx = gfx$.
Proposition 2.2 Let f and g be compatible mappings of type (R) of a Menger space (X, F, Δ) into itself. Suppose that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x$ for some x in X. Then
(a) $\lim_{n\to\infty} gfx_n = fx$ if f is continuous at x.
(b) $\lim_{n\to\infty} fgx_n = gx$ if g is continuous at x.
(c) $fgx = gfx$ and $fx = gx$ if f and g are continuous at x.
Proposition 2.3 Let f and g be compatible mappings of type (E) of a Menger space (X, F, Δ) into itself. Let one of f and g be continuous. Suppose that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x$ for some $x \in X$. Then
(a) $fx == gx$ and $\lim_{n\to\infty} ffx_n = \lim_{n\to\infty} ggx_n = \lim_{n\to\infty} fgx_n = \lim_{n\to\infty} gfx_n$.
(b) If there exists $u \in X$ such that $fu = gu = x$, we have $fgu = gfu$.
LEMMA 2.1[19] Let $\{x_n\}$ be a sequence in a Menger space (X, F, Δ) with continuous t-norm Δ and $\Delta(t, t) \geq t$. If there exists a constant $k \in (0, 1)$ such that

$$F(x_n, x_{n+1}\ y, kt) \geq F(x_{n-1}, x_n, t)$$

for all $t > 0$ and $n = 1,2,3...$ then $\{x_n\}$ is a Cauchy sequence in X.
LEMMA 2.2[19] Let (X, F, Δ) be a Menger space. If there exists $k \in (0, 1)$ such that
$F(x, y, kt) \geq F(x, y, t)$ for all $x, y \in X$ and $t > 0$, then $x = y$.

III. MAIN RESULTS

Let $\boldsymbol{\Phi}$ be class of all the mappings $\phi: [0,1] \to [0,1]$ satisfying the following properties:
$(\boldsymbol{\phi_1})$ ϕ is continuous and non decreasing on [0,1],
$(\boldsymbol{\phi_2})$ $\phi(m) > m$ for all m in [0,1].
We note that if $\phi \in \boldsymbol{\Phi}$, then $\phi(1) = 1$ and $\phi(m) \geq m$ for all m in [0,1].

In 2015, Kang et al. [10] proved the following common fixed point theorem in a complete multiplicative metric space. We prove the same with a control functions ϕ in Menger space.
Theorem 3.1 Let A, B, S and T be mappings of a complete Menger space (X, F, Δ) into itself satisfying the following conditions:
(3.1) $T(X) \subset A(X)$, $S(X) \subset B(X)$;

(3.2) $F(Sx,Ty,kt) \geq \phi\left(min\begin{Bmatrix} F(Ax,By,t),F(Ax,Sx,t),F(By,Ty,t) \\ F(Sx,By,\alpha t),F(Ax,Ty,(2-\alpha)t) \end{Bmatrix}\right)$

hold for all x,y in X, where $\alpha \in (0,2)$, $t > 0$,

(3.3) one of the mappings A,B,S and T is continuous.

Assume that the pairs A,S and B,T are compatible of type (R), Then A, B,S and T have a unique common fixed point in X.

Proof Since $S(X) \subset B(X)$. Now consider a point $x_0 \in X$, there exists $x_1 \in X$ such that $Sx_0 = Bx_1 = y_0$ for this point x_1 there exists $x_2 \in X$ such that $Tx_1 = Ax_2 = y_1$. Continuing in this way, we can define a sequence $\{y_n\}$ in X such that

$$y_{2n} = Sx_{2n} = Bx_{2n+1}; \qquad y_{2n+1} = Tx_{2n+1} = Ax_{2n+2};$$

Now we prove that $\{y_n\}$ is Cauchy sequence in X.

Putting $x = x_{2n}$, $y=x_{2n+1}$, $\alpha = 1-\beta$ with $\beta \in (0,1)$ in inequality (3.2), we have

$$F(Sx_{2n},Tx_{2n+1},kt) \geq \phi\left(min\begin{Bmatrix} (F(Ax_{2n},Bx_{2n+1},t),)F(Ax_{2n},Sx_{2n},t), \\ F(Bx_{2n+1},Tx_{2n+1},t),F(Sx_{2n},Bx_{2n+1},t) \\ F(Ax_{2n},Tx_{2n+1},(1+\beta)t) \end{Bmatrix}\right)$$

$$F(y_{2n},y_{2n+1},kt) \geq \phi\left(min\begin{Bmatrix} F(y_{2n-1},y_{2n},t),F(y_{2n-1},y_{2n},t), \\ F(y_{2n},y_{2n+1},t),F(y_{2n},y_{2n},(1-\beta)t), \\ F(y_{2n-1},y_{2n+1},(1+\beta)t) \end{Bmatrix}\right)$$

$$F(y_{2n},y_{2n+1},kt) \geq \phi\left(min\begin{Bmatrix} F(y_{2n-1},y_{2n},t),F(y_{2n-1},y_{2n},t), \\ F(y_{2n},y_{2n+1},t), \\ F(y_{2n-1},y_{2n+1},(1+\beta)t) \end{Bmatrix}\right)$$

$$F(y_{2n},y_{2n+1},kt) \geq \phi\left(min\begin{Bmatrix} F(y_{2n-1},y_{2n},t)F(y_{2n},y_{2n+1},t) \\ F(y_{2n-1},y_{2n+1},(1+\beta)t) \end{Bmatrix}\right)$$

$$\geq \phi\left(min\begin{Bmatrix} F(y_{2n-1},y_{2n},t)F(y_{2n},y_{2n+1},t) \\ F(y_{2n-1},y_{2n},(t),F(y_{2n},y_{2n+1},(\beta t) \end{Bmatrix}\right)$$

$$F(y_{2n},y_{2n+1},kt) \geq \phi\left(min\begin{Bmatrix} F(y_{2n-1},y_{2n},t)F(y_{2n},y_{2n+1},t) \\ ,F(y_{2n},y_{2n+1},(\beta t) \end{Bmatrix}\right)$$

As Δ is continuous, letting $\beta \to 1$ we get

$$F(y_{2n},y_{2n+1},kt) \geq \phi\left(min\begin{Bmatrix} F(y_{2n-1},y_{2n},t)F(y_{2n},y_{2n+1},t) \\ ,F(y_{2n},y_{2n+1},(t) \end{Bmatrix}\right)$$

$$F(y_{2n},y_{2n+1},kt) \geq \phi\left(min\begin{Bmatrix} F(y_{2n-1},y_{2n},t) \\ ,F(y_{2n},y_{2n+1},(t) \end{Bmatrix}\right)$$

Hence $F(y_{2n},y_{2n+1},kt) \geq \phi\left(min\begin{Bmatrix} F(y_{2n-1},y_{2n},t) \\ ,F(y_{2n},y_{2n+1},(t) \end{Bmatrix}\right)$

Similarly, $F(y_{2n+1},y_{2n+2},kt) \geq \phi\left(min\begin{Bmatrix} F(y_{2n},y_{2n+1},t), \\ ,F(y_{2n+1},y_{2n+2},t \end{Bmatrix}\right)$

Therefore, for all n even or odd we have

$$F(y_n,y_{n+1},kt) \geq \phi(min\{F(y_{n-1},y_n,t),F(y_n,y_{n+1},t)\})$$

Consequently,

$$F(y_n,y_{n+1},t) \geq \phi\left(min\{F(y_{n-1},y_n,\tfrac{t}{k}),F(y_n,y_{n+1},\tfrac{t}{k})\}\right)$$

By repeated application of above inequality, we get

$$F(y_n,y_{n+1},t) \geq \phi\left(min\{F(y_{n-1},y_n,\frac{t}{k}),F(y_n,y_{n+1},\frac{t}{k^m})\}\right)$$

Since $F(y_n,y_{n+1},\frac{t}{k^m}) \to 1$ as $m \to \infty$, it follows that

$F(x_n,x_{n+1}\ y,kt) \geq \phi(F(x_{n-1},x_n,t))$for all $n \in N$ for all $t > 0$, by property ϕ, we have $F(x_n,x_{n+1}\ y,kt) \geq F(x_{n-1},x_n,t)$. Therefore, by Lemma 2.1, $\{y_n\}$ is a Cauchy sequence in X and hence it converges to some point z $\in X$. Consequently, the subsequence $\{Sx_{2n}\}$, $\{Bx_{2n+1}\}$,$\{Tx_{2n+1}\}$ and $\{Ax_{2n}\}$ of $\{y_n\}$ also converges to z.

Now, suppose that A is continuous. Since A and S are compatible of type (R), by Proposition2.1, AAx_{2n} and SAx_{2n} converges to Az as $n \to \infty$.

We claim that $z = Az$. Putting $x = Ax_{2n}$ and $y = x_{2n+1}$, $\alpha = 1$ in inequality (3.2) we have

$$F(SAx_{2n},Tx_{2n+1},kt) \geq \phi\left(min\begin{Bmatrix} F(AAx_{2n},Bx_{2n+1},t),F(AAx_{2n},SAx_{2n},t), \\ F(Bx_{2n+1},Tx_{2n+1},t),F(SAx_{2n},Bx_{2n+1},t), \\ F(AAx_{2n},Tx_{2n+1},t) \end{Bmatrix}\right)$$

Letting $n \to \infty$ we have

$$F(Az,z,kt) \geq \phi\left(min\begin{Bmatrix} F(Az,z,t), F(Az,Az,t), F(z,z,t), F(Az,z,t) \\ F(Az,z,t) \end{Bmatrix}\right)$$

$F(Az,z,kt) \geq \phi(F(Az,z,t))$, then by property ϕ, we have
$F(Az,z,kt) \geq F(Az,z,t)$. By Lemma 2.2, we $Az = z$.
Next we claim that $z = Sz$. Putting $x = z$ and $y = x_{2n+1}$, $\alpha = 1$ in inequality (3.2) we have

$$F(Sz,Tx_{2n+1},kt) \geq \phi(min\{F(Az,Bx_{2n+1},t), F(Az,Sz,t), F(Bx_{2n+1},Tx_{2n+1},t), , F(Sz,Bx_{2n+1},t), F(Az,Tx_{2n+1},t)\}).$$

Letting $n \to \infty$ we have

$$F(Sz,z,kt) \geq \phi(min\{F(z,z,t), F(z,Sz,t), F(z,z,t), F(z,z,t), F(z,z,t)\}).$$

we get $F(Sz,z,kt) \geq \phi(F(Sz,z,t))$,then by property ϕ, we have
$F(Sz,z,kt) \geq F(Sz,z,t)$. By Lemma 2.2, we $Sz = z$. Since $S(X) \subset B(X)$ and hence exists a point $u \in X$ such that $z = Sz = Bu$.

We claim that $z = Tu$. Putting $x = z$ and $y = u$, $\alpha = 1$ in inequality (3.2) we have $F(z,Tu,kt) = F(Sz,Tu,kt) \geq \phi(min\{F(Az,Bu,t), F(Az,Sz,t), F(Bu,Tu,t), F(Sz,Bu,t), F(Az,Tu,t)\})$.

$$F(z,Tu,kt) \geq \phi(min\{F(z,z,t), F(z,z,t), F(z,Tu,t), F(z,z,t), F(z,Tu,t)\}).,$$

then by property ϕ.
we get $F(z,Tu,kt) \geq F(z,Tu,t)$. By Lemma 2.2, we get $z = Tu$. Since B and T are compatible of type (R) and $Bu = Tu = z$, by Proposition2.1, $BTu = TBu$ and hence $Bz = BTu = TBu = Tz$. Also, we have

$$F(z,Bz,kt) = F(Sz,Tz,kt) \geq \phi(min\{F(Az,Bz,t), F(Az,Sz,t), F(Bz,Tz,t), F(Sz,Bz,t), F(Az,Tz,t)\}).$$

$$= \phi(min\{F(z,Bz,t), F(z,z,t), F(Bz,Bz,t), F(z,Bz,t), F(z,Bz,t)\}).$$

we get $F(z,Bz,kt) \geq F(z,Bz,t)$. By Lemma 2.2, we get $z = Bz$. Hence $z = Bz = Tz = Az = Sz$. Therefore, z is a common fixed point of A, S, B and T.
Similarly, we can complete the proof when B is continuous.
Next, suppose that S is continuous. Since A and S are compatible of type (R), by Proposition2.1, SSx_{2n} and Sx_{2n} converges to Sz as $n \to \infty$.
We claim that $z = Sz$. Putting $x = Sx_{2n}$ and $y = x_{2n+1}$, $\alpha = 1$ in inequality (3.2) we have

$$F(SSx_{2n},Tx_{2n+1},kt) \geq \phi(min\{F(ASx_{2n},Bx_{2n+1},t), F(ASx_{2n},SSx_{2n},t), F(Bx_{2n+1},Tx_{2n+1},t), F(SSx_{2n},Bx_{2n+1},t), F(ASx_{2n},Tx_{2n+1},t)\}).$$

Letting $n \to \infty$ we have

$$F(Sz,z,kt) \geq \phi(min\{F(Sz,z,t), F(Sz,Sz,t), F(z,z,t), F(Sz,z,t), F(Sz,z,t)\}).$$

we get $F(Sz,z,kt) \geq F(Sz,z,t)$. By Lemma 2.2, we $Sz = z$. Since $S(X) \subset B(X)$ and hence exists a point $v \in X$ such that $z = Sz = Bv$.
We claim that $z = Tv$. Putting $x = Sx_{2n}$ and $y = v$, $\alpha = 1$ in inequality (3.2) ,we have

$$F(SSx_{2n},Tv,kt) \geq \phi(min\{F(ASx_{2n},Bv,t), F(ASx_{2n},Sx_{2n},t), F(Bv,Tv,t) ,F(SSx_{2n},Bv,t), F(Az,Tu,t)\}).$$

the property by ϕ. *we have*
Letting $n \to \infty$ we have

$$F(z,Tv,kt) \geq \phi(min\{F(z,z,t), F(z,z,t), F(z,Tv,t), F(z,z,t), F(z,Tv,t)\}).$$

we get$F(z,Tu,kt) \geq F(z,Tu,t)$. By Lemma 2.2, we get $z = Tv$. Since B and T are compatible of type (R) and $Bv = Tv = z$, by Proposition2.1, $BTv = TBv$ and hence $Bz = BTv = TBv = Tz$.
We claim that $z = Tz$. Putting $x = x_{2n}$ and $y = z$, $\alpha = 1$ in inequality (3.2) we have

$$F(Sx_{2n},Tz,kt) \geq \phi(min\{F(Ax_{2n},Bz,t), F(Ax_{2n},Sx_{2n},t), F(Bz,Tz,t), , F(Sx_{2n},Bz,t), F(Ax_{2n},Tz,t)\}).$$

Letting $n \to \infty$ we have

$$F(z,Tz,kt) \geq \phi(min\{F(z,Tz,t), F(z,z,t), F(Tz,Tz,t), F(z,Tz,t), F(z,Tz,t)\}).$$

we get $F(z,Tz,kt) \geq F(z,Tz,t)$. By Lemma 2.2, we $Tz = z$. Since $T(X) \subset A(X)$ and hence exists a point $w \in X$ such that $z = Sz = Aw$.
We claim that $z = Sw$. Putting $x = w$ and $y = z$, $\alpha = 1$ in inequality (3.2) we have

$$F(Sw,z,kt) = F(Sw,Tz,kt) \geq \phi(min\{F(Aw,Bz,t), F(Aw,Sw,t), F(Bz,Tz,t), F(Sw,Bz,t), F(Aw,Tz,t)\}).$$

$$= \phi(min\{F(z,z,t), F(z,Sw,t), F(Tz,Tz,t), F(Sw,z,t), F(z,z,t)\}).$$

we get $F(Sw,z,kt) \geq F(z,Sw,t)$. By Lemma 2.2, we get $z = Sw$. Since A and S are compatible of type (R) and $Sw = Aw = z$, by Proposition2.1, $ASw = SAw$ and hence $Az = ASw = SAw = Sz$. Hence $z = Bz = Tz = Az = Sz$. Therefore, z is a common fixed point of A, S, B and T.

Similarly, we can complete the proof when T is continuous.
Uniqueness follows easily. This completes the proof.
Next we prove the following theorem for compatible mappings of type (K).

Theorem 3.2 Let A, B, S and T be mappings of a complete Menger space (X, F, Δ) into itself satisfying the following conditions (3.1), (3.2). Suppose that the pairs A, S and B, T are reciprocally continuous.

Assume that the pairs A, S and B, T are compatible of type (K). Then A, B, S and T have a unique common fixed point in X.

Proof Now from the proof of Theorem 3.4 we can easily prove that $\{y_n\}$ is Cauchy sequence in X and hence it converges to some point z ∈ X. Consequently, the subsequence $\{Sx_{2n}\}$, $\{Bx_{2n+1}\}$,$\{Tx_{2n+1}\}$ and $\{Ax_{2n}\}$ of $\{y_n\}$ also converges to z.

Since the pairs A, S and B, T are compatible of type (K), we have $AAx_{2n} \to Sz, SSx_{2n} \to Az$ and $BBx_{2n} \to Tz, TTx_{2n+1} \to Bz$ as $n \to \infty$.

We claim that $Bz = Az$. Putting $x = Sx_{2n}$ and $y = Tx_{2n+1}$, $\alpha = 1$ in inequality (3.2) we have

$F(SSx_{2n}, TTx_{2n+1}, kt) \geq \phi(min\{ F(ASx_{2n}, BTx_{2n+1}, t), F(ASx_{2n}, SSx_{2n}, t),$
$F(BTx_{2n+1}, TTx_{2n+1}, t), F(SSx_{2n}, BTx_{2n+1}, t), F(ASx_{2n}, Tx_{2n+1}, t)\})$.the property $by\ \phi$ we have

Letting $n \to \infty$ and using reciprocal continuity of the pairs A, S and B, T we have

$F(Az, Bz, kt) \geq \phi(min\{ F(Az, Bz, t), F(Az, Az, t), F(Bz, Bz, t), F(Az, Bz, t), F(Az, Bz, t)\})$.

we get $F(Az, Bz, kt) \geq F(Az, Bz, t)$. By Lemma 2.2, we $Az = Bz$.

Next we claim that $Bz = Sz$. Putting $x = z$ and $y = Tx_{2n+1}$, $\alpha = 1$ in inequality (3.2) we have

$F(Sz, TTx_{2n+1}, kt) \geq \phi(min\{ F(Az, BTx_{2n+1}, t), F(Az, Sz, t), F(BTx_{2n}, TTx_{2n+1}, t),$
$, F(Sz, BTx_{2n+1}, t), F(Az, TTx_{2n+1}, t)\})$.

Letting $n \to \infty$ and using reciprocal continuity of the pairs A, S and B, T, we have

$F(Sz, Bz, kt) \geq \phi(min\{ F(Bz, Bz, t), F(Bz, Sz, t), F(Bz, Bz, t), F(Sz, B\,z, t), F(Bz, Bz, t)\})$.

we get $F(Sz, Bz, kt) \geq F(Sz, Bz, t)$. By Lemma 2.2, we $Sz = Bz$.

We claim that $Sz = Tz$. Putting $x = z$ and $y = z$, $\alpha = 1$ in inequality (3.2) we have $F(Sz, Tz, kt) \geq \phi(min\{ F(Az, Bz, t), F(Az, Sz, t), F(Bz, Tz, t), F(Sz, Bz, t), F(Az, Tz, t)\})$.

$F(Sz, Tz, kt) \geq \phi(min\{ F(Bz, Bz, t), F(Az, Az, t), F(Sz, Tz, t), F(Sz, Sz, t), F(Sz, Tz, t)\})$. then property by ϕ, we have

we get $F(Sz, Tz, kt) \geq F(Sz, Tz, t)$. By Lemma 2.2, we get $Sz = Tz$.

We claim that $z = Tz$. Putting $x = x_{2n}$ and $y = z$, $\alpha = 1$ in inequality (3.2) we have

$F(Sx_{2n}, Tz, kt) \geq \phi(min\{ F(Ax_{2n}, Bz, t), F(Ax_{2n}, Sx_{2n}, t),$
$F(Bz, Tz, t), F(Sx_{2n}, Bz, t), F(Ax_{2n}, Tz, t)\})$.

Letting $n \to \infty$

$F(z, Tz, kt) \geq \phi(min\{ F(z, Tz, t), F(z, z, t), F(z, Tz, t), F(z, Tz, t), F(z, Tz, t)\})$.

we get $F(z, Tz, kt) \geq F(z, Tz, t)$. By Lemma 2.2, we get $z = Tz$. Hence $z = Bz = Tz = Az = Sz$. Therefore, z is a common fixed point of A, S, B and T.

Uniqueness follows easily. This completes the proof.
Next we prove the following theorem for compatible mappings of type (E).

Theorem 3.3 Let A, B, S and T be mappings of a complete Menger space into itself satisfying the following conditions (3.1), (3.2). Suppose that one of A and S is continuous and one of B and T is continuous.

Assume that the pairs A, S and B, T are compatible of type (E), Then A, B, S and T have a unique common fixed point in X.

Proof Now from the proof of Theorem 3.4 we can easily prove that $\{y_n\}$ is Cauchy sequence in X and hence it converges to some point z ∈ X. Consequently, the subsequence $\{Sx_{2n}\}$, $\{Bx_{2n+1}\}$,$\{Tx_{2n+1}\}$ and $\{Ax_{2n}\}$ of $\{y_n\}$ also converges to z.

Now, suppose that one of the mappings A and S is continuous. Since A and S are compatible of type (E), by Proposition2.3, $Az = Sz$. Since $S(X) \subset B(X)$ and hence exists a point $w \in X$ such that $Sz = Bw$.

We claim that $Sz = Tw$. Putting $x = z$ and $y = w$, $\alpha = 1$ in inequality (3.2) we have

$F(Sz, Tw, kt) \geq \phi(\ min\{ F(Az, Bw, t), F(Az, Sz, t), F(Bw, Tw, t)$
$, F(Sz, Bw, t), F(Az, Tw, t)\})$.
$= \phi(\ min\{ F(Az, Sz, t), F(Sz, Sz, t), F(Sz, Tw, t), F(Sz, Bw, t), F(Sz, Tw, t)\})$.

we get $F(Sz, Tw, kt) \geq F(Sz, Tw, t)$. By Lemma 2.2, we get $Sz = Tw$. Thus we have $Az = Sz = Tw = Bw$.

We claim that $Sz = z$. Putting $x = z$ and $y = x_{2n+1}$, $\alpha = 1$ in inequality (3.2) we have

$F(Sz, Tx_{2n+1}, kt) \geq \phi(\ min\{ F(Az, Bx_{2n+1}, t), F(Az, Sz, t), F(Bx_{2n+1}, Tx_{2n+1}, t)$
$, F(Sz, Bx_{2n+1}, t), F(Az, Tx_{2n+1}, t)\})$.
$= \phi(min\{ F(Sz, z, t), F(z, z, t), F(z, z, t), F(Sz, z, t), F(Sz, z, t)\})$.then property by $\phi\ , we\ have$

we get $F(Sz,z,kt) \geq F(Sz,z,t)$. By Lemma 2.2, we get $z = Sz$. Hence $z = Bz = Tz = Az = Sz$. Therefore, z is a common fixed point of A, S, B and T.
Again, suppose B and T are compatible of type (E) and one of the mappings B and T is continuous. The we get $Bw = Tw = z$. By Proposition2.4, we have $BBw = BTw = TBw = TTw$. Hence $Bz = Tz$.
We claim that $z = Tz$. Putting $x = x_{2n}$ and $y = z$, $\alpha = 1$ in inequality (3.2) we have $F(Sx_{2n}, Tz, kt) \geq \phi(\, min\{ F(Ax_{2n}, Bz, t), F(Ax_{2n}, Sx_{2n}, t), F(Bz, Tz, t), , F(Sx_{2n}, Bz, t), F(Ax_{2n}, Tz, t)\})$.
Letting $n \to \infty$ we have
$F(Sz,z,kt) \geq \phi(min\{ F(Sz,z,t), F(Sz,Sz,t), F(z,z,t), F(Sz,z,t), F(Sz,z,t)\})$.
we get $F(Sz,z,kt) \geq F(Sz,z,t)$. By Lemma 2.2, we $Sz = z$. Since $S(X) \subset B(X)$ and hence exists a point $v \in X$ such that $z = Sz = Bv$.
We claim that $z = Tv$. Putting $x = Sx_{2n}$ and $y = v$, $\alpha = 1$ in inequality (3.2) ,we have

$$F(SSx_{2n}, Tv, kt) \geq \phi(min\{ F(ASx_{2n}, Bv, t), F(ASx_{2n}, Sx_{2n}, t), F(Bv, Tv, t), F(SSx_{2n}, Bv, t), F(Az, Tu, t)\}).$$

Letting $n \to \infty$ we have
$F(z,Tv,kt) \geq \phi(min\{ F(z,z,t), F(z,z,t), F(z,Tv,t), F(z,z,t), F(z,Tv,t)\})$.the property by ϕ, we have
we get$F(z,Tu,kt) \geq F(z,Tu,t)$. By Lemma 2.2, we get $z = Tv$. Since B and T are compatible of type (R) and $Bv = Tv = z$, by Proposition2.2, $BTv = TBv$ and hence $Bz = BTv = TBv = Tz$.
We claim that $z = Tz$. Putting $x = x_{2n}$ and $y = z$, $\alpha = 1$ in inequality (3.2) we have

$$F(Sx_{2n}, Tz, kt) \geq \phi(min\{ F(Ax_{2n}, Bz, t), F(Ax_{2n}, Sx_{2n}, t), F(Bz, Tz, t), , F(Sx_{2n}, Bz, t), F(Ax_{2n}, Tz, t)\}).$$

Letting $n \to \infty$ we have
$F(z,Tz,kt) \geq \phi(min\{ F(z,Tz,t), F(z,z,t), F(Tz,Tz,t), F(z,Tz,t), F(z,Tz,t)\})$. then property by ϕ, we have
we get $F(z,Tz,kt) \geq F(z,Tz,t)$. By Lemma 2.2, we $Tz = z$.
Hence $z = Bz = Tz$ Therefore, z is a common fixed point of A, S, B and T.
Similarly, we can complete the proof when T is continuous.
Uniqueness follows easily. This completes the proof.

REFERENCES

[1]. M. Akkouchi, A Meir Keeler type common fixed point theorems in four mappings, Opuscula Mathematica,31(1) (2011), 5-14.
[2]. M. Aamri and D.EI Moutawakil, Some new common fixed point theorems under strict contractive conditions, J. Math. Anal. Appl., 270 (2002), 181-188.
[3]. S. Chauhan, W. Sintunavarat and P. Kumam, Common Fixed point theorems for weakly compatible mappings in fuzzy metric spaces using (JCLR) property, Applied Mathematics (in press).
[4]. Y.J. Cho, Fixed points in fuzzy metric spaces, J. Fuzzy Math., 5 (1997), 949-962.
[5]. K. Jha, V. Popa and K.B. Manandhar, Common fixed points theorem for compatible of type (K) in metric space, Int. J. Math. Sci.Eng. Appl. 89(2014), 383-391.
[6]. G. Jungck, Commuting mappings and fixed points, Amer. Math. Mon., 83(1976), 261-263.
[7]. G. Jungck, Compatible mappings and common fixed points, Internet. J. Math. Sci., 9(1986), 771-779.
[8]. G. Jungck., Common fixed points for non-continuous non-self maps on non-metric spaces, Far East Journal of Mathematical Sciences, 4(2), (1996), 199-212.
[9]. G. Jungck., Compatible mappings and common fixed points, International Journal of Mathematics and Mathematical Sciences, 9(4) (1986), 771-779.
[10]. S.N. Kang, S.Kuamr and Lee, Common fixed points for compatible mappings of types in multiplicative metric spaces, Int. J. Mathematical Anl., 9(36) (2015), 1755-1767.
[11]. A. Meir and E. Keeler, A theorem on contraction mappings, J. Math. Anal. Appl., 28(1969), 326-329.
[12]. K. Menger, Statistical metrices, Proc. Nat. Acad. Sci. (USA), 28 (1942), 535-537.
[13]. S.N. Mishra, Common fixed points of compatible mappings in probabilistic metric spaces, Math. Japon., 36, 2(1991), 283-289.
[14]. S. Manro, S. Kumar and S.S. Bhatia, Fuzzy version of Meir-Keeler type contractive condition and existence of fixed point, DE Gruyter, Doi 10.2478/tmj-2014-0008.
[15]. R.P. Pant, A common fixed point theorem under a new condition, Indian Journal of Pure and applied Mathematics, 30(1999), 147-152.
[16]. Y. Rohan, M.R. Singh and L. Shambu, Common fixed points of compatible mapping of type (C) in Banach Spaces, Proc. Math. Soc. 20(2004),77-87.
[17]. B. Schweizer and A. Sklar, Probabilistic Metric Spaces, North Holland Series in Probability and Applied Math., North-Holland Publ. Co., New York, 1983.
[18]. S. Sessa, On a weak commutativity condition of mappings in fixed point considerations, Publications Del' Institute Mathematique, Nouvelle Serie Tome, 32(1982), 149-153.
[19]. B. Singh and S. Jain, A fixed point theorem in Menger space through weak compatibility, J. Math. Anal. Appl., 301(2005), 439-448.
[20]. M.R. Singh and Y.M. Singh, Compatible mappings of type (E) and common fixed point theorems of Meir-Keeler type, Int. J. Math. Sci. Eng. Appl., 19(2007), 299-315.