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Research Paper

Asymptotic Solutions of Fifth Order Critically Damped Non-linear Systems with Pair Wise Equal Eigenvalues and another is Distinct

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ABSTRACT : *The Krylov-Bogoliubov-Mitropolskii (KBM) method is one of the most used techniques to investigate transient behavior of vibrating systems. Initially the method was developed for obtaining the periodic solutions of second order nonlinear differential systems with small nonlinearities. Afterward many researchers have studied and modified the method for achieving solutions of higher order nonlinear systems. In this article, we have modified the KBM method to examinethe asymptotic solutions of fifth order critically damped nonlinear systems.*

Keywords: -*KBM, periodic solution, nonlinearity, asymptotic solution,critically damped system.*

I. INTRODUCTION

In nonlinear differential equations, one of most used techniques to examine weakly nonlinear oscillatory and non-oscillatory differential systems is the Krylov-Bogoliubov-Mitropolskii (KBM) ([1], [2]) method, which was first developed by Krylov and Bogoliubov [3] to find the periodic solutions of second order nonlinear differential systems with small nonlinearities. Subsequently,Bogoliubov and Mitropolskii [4] improved and justified this method in mathematical terms. It was further extended by Popov [5] to damped oscillatory nonlinear systems. Later on over-damped nonlinear systems were examined byMurty and Deekshatulu [6]using Bogoliubov's method. An asymptotic solution of a second order critically damped nonlinear systemwas pointed out by Sattar [7]. Moreover, a new technique was introduced by Shamsul [8] to obtain approximate solutions of second order both over-damped and critically damped nonlinear systems. Further, Osiniskii [9] investigated the third order nonlinear systems using Bogoliuvob's method and imposed some restrictions upon the parameters, which made the solutions over-simplified and revealed incorrect results. However, Mulholland [10] removed these restrictions to obtain desired results. Consequently, thesolutions of nonlinear systems were transformed by Bojadziv [11] to a three dimensional differential system. Sattar [12]also assessedthe solutions of third order over-damped nonlinear systems. Shamsul [13]analyzed the solutions of third order over-damped systems whose eigenvalues were integral multiple. Shamsul and Sattar [14] introduced a unified KBM method to obtain approximate solutions of third order damped and over-damped nonlinear systems. Kawser and Akbar [15] suggested an asymptotic solution for the third order critically damped nonlinear system with pair wise equal eigenvalues. Kawser and Sattar [16] proposed an asymptotic solution of a fourth order critically damped nonlinear system with pair wise equal eigenvalues. The KBM method was further extended by Akber and Tanzer [17] to obtain solutions for the fifth order over-damped nonlinear systems with cubic nonlinearity. Rahaman and Rahman [18] expounded the analytical approximate solutions of fifth order more critically damped systems in the case of smaller triply repeated roots.
The aim of this article is to obtain the asymptotic solutions of fifth order critically damped non-linear systems by extending the KBM method. The results obtained by the perturbation method were compared with those obtained by the fourth order Runge-Kutta method, whichdemonstrated perfect coincident with the numerical solutions.

II. THE METHOD

We are going to propose a perturbation technique to solve fifth order non-linear differential systems of the form

$$x^{v}+k_1x^{iv}+k_2\dddot{x}+k_3\ddot{x}+k_4\dot{x}+k_5x=-\in f(x,\dot{x},\ddot{x},\dddot{x},x^{(iv)}) \quad (1)$$

where x^{v} and x^{iv} stand for the fifth and fourth derivatives respectively and over dots are used for the first, second and third derivatives of *x* with respect to *t*. k_1, k_2, k_3, k_4, k_5 are constants, $\in$ is a sufficiently small parameter and f is the given nonlinear function. As the unperturbed equation (1) is of order five so it has five real negative eigenvalues, where four eigenvalues are pair wise equal and other one is distinct. Suppose the eigenvalues are $-\lambda, -\lambda, -\mu, -\mu, -\nu$.

When $\in=0$ the equation (1) becomes linear and the solution of the corresponding linear equation is

$$x(t,0)=(a_0+b_0t)e^{-\lambda t}+(c_0+d_0t)e^{-\mu t}+h_0e^{-\nu t} \quad (2)$$

where a_0, b_0, c_0, d_0, h_0 are constants of integration

When $\in\neq 0$ following Shamsul [19] an asymptotic solution of the equation (1) is sought in the form

$$x(t,\in)=(a+bt)e^{-\lambda t}+(c+dt)e^{-\mu t}+he^{-\nu t}+\in u_1(a,b,c,d,h,t)+\ldots \quad (3)$$

where a, b, c, d, h are the functions of *t* and they satisfy the differential equations

$$\begin{aligned}
\dot{a}&=\in A_1(a,b,c,d,h,t)+\ldots\\
\dot{b}&=\in B_1(a,b,c,d,h,t)+\ldots\\
\dot{c}&=\in C_1(a,b,c,d,h,t)+\ldots\\
\dot{d}&=\in D_1(a,b,c,d,h,t)+\ldots\\
\dot{h}&=\in H_1(a,b,c,d,h,t)+\ldots
\end{aligned} \quad (4)$$

Now differentiating (3) five times with respect to *t*, substituting the value of *x* and the derivatives $x^{v}, x^{iv}, \dddot{x}, \ddot{x}, \dot{x}$ in the original equation (1) utilizing the relations presented in (4) and finally bringing out the coefficients of $\in$ we obtain

$$\begin{aligned}
&e^{-\lambda t}(D+\mu-\lambda)^2(D+\nu-\lambda)\left(\frac{\partial A_1}{\partial t}+2B_1+t\frac{\partial B_1}{\partial t}\right)+e^{-\mu t}(D+\lambda-\mu)^2\\
&\quad(D+\nu-\mu)\left(\frac{\partial C_1}{\partial t}+2D_1+t\frac{\partial D_1}{\partial t}\right)+e^{-\nu t}(D+\lambda-\nu)^2(D+\mu-\nu)^2H_1 \quad (5)\\
&\quad+(D+\lambda)^2(D+\mu)^2(D+\nu)u_1=-f^{(0)}(a,b,c,d,h,t)
\end{aligned}$$

where $f^{(0)}(a,b,c,d,h,t)=f(x,\dot{x},\ddot{x},\dddot{x},x^{iv})$

and $x(t,0)=(a_0+b_0t)e^{-\lambda t}+(c_0+d_0t)e^{-\mu t}+h_0e^{-\nu t}$

We have extended the function $f^{(0)}$ in the Taylor's series (Sattar [20], Shamsul and Sattar [21]) about the origin in power of *t*. Therefore, we obtain

$$f^{(0)}=\sum_{q=0}^{\infty}\left\{t^q\sum_{i,j,k,l=0}^{\infty}F_{q,l}(a,b,c,d,h)e^{-(i\lambda+j\mu+k\nu)t}\right\} \quad (6)$$

Thus, using (6), the equation (5) becomes

$$\begin{aligned}
&e^{-\lambda t}(D+\mu-\lambda)^2(D+\nu-\lambda)\left(\frac{\partial A_1}{\partial t}+2B_1+t\frac{\partial B_1}{\partial t}\right)+e^{-\mu t}(D+\lambda-\mu)^2\\
&(D+\nu-\mu)\left(\frac{\partial C_1}{\partial t}+2D_1+t\frac{\partial D_1}{\partial t}\right)+e^{-\nu t}(D+\lambda-\nu)^2(D+\mu-\nu)^2H_1+ \quad (7)\\
&(D+\lambda)^2(D+\mu)^2(D+\nu)u_1=-\sum_{q=0}^{\infty}\left\{t^q\sum_{i,j,k,l=0}^{\infty}F_{q,l}(a,b,c,d,h)e^{-(i\lambda+j\mu+k\nu)t}\right\}
\end{aligned}$$

Following the KBM method, Murty and Deekshatulu [22], Sattar [23], Shamsul[24], Shamsul and Sattar ([25], [26]) imposed the condition that$u_1$does not contain the fundamental terms of $f^{(0)}$. Therefore, equation (7) can be separated for unknown functions A_1,B_1,C_1,D_1, H_1 andu_1 in the following way:

$$e^{-\lambda t}(D+\mu-\lambda)^2(D+\nu-\lambda)\left(\frac{\partial A_1}{\partial t}+2B_1+t\frac{\partial B_1}{\partial t}\right)+e^{-\mu t}(D+\lambda-\mu)^2$$
$$(D+\nu-\mu)\left(\frac{\partial C_1}{\partial t}+2D_1+t\frac{\partial D_1}{\partial t}\right)+e^{-\nu t}(D+\lambda-\nu)^2(D+\mu-\nu)^2H_1= \quad (8)$$
$$-\sum_{q=0}^{1}\left\{t^q\sum_{i,j,k,l=0}^{\infty}F_{q,l}(a,b,c,d,h)e^{-(i\lambda+j\mu+k\nu)t}\right\}$$

and $(D+\lambda)^2(D+\mu)^2(D+\nu)u_1=-\sum_{q=2}^{\infty}\left\{t^q\sum_{i,j,k,l=0}^{\infty}F_{q,l}(a,b,c,d,h)e^{-(i\lambda+j\mu+k\nu)t}\right\}$(9)

Now equating the coefficients of t^0, t^1 from equation (8), we obtain

$$e^{-\lambda t}(D+\mu-\lambda)^2(D+\nu-\lambda)\left(\frac{\partial A_1}{\partial t}+2B_1\right)+e^{-\mu t}(D+\lambda-\mu)^2$$
$$(D+\nu-\mu)\left(\frac{\partial C_1}{\partial t}+2D_1\right)+e^{-\nu t}(D+\lambda-\nu)^2(D+\mu-\nu)^2H_1 \quad (10)$$
$$=-\sum_{i,j,k,l=0}^{\infty}F_{0,l}(a,b,c,d,h)e^{-(i\lambda+j\mu+k\nu)t}$$

$$e^{-\lambda t}(D+\mu-\lambda)^2(D+\nu-\lambda)\frac{\partial B_1}{\partial t}+e^{-\mu t}(D+\lambda-\mu)^2(D+\nu-\mu)\frac{\partial D_1}{\partial t}=$$
$$-\sum_{i,j,k,l=0}^{\infty}F_{1,l}(a,b,c,d,h)e^{-(i\lambda+j\mu+k\nu)t} \quad (11)$$

Here, from equations (10) and (11) determining the unknown functions $A_1,B_1,C_1,D_1$and H_1. Thus, to obtain the unknown functions $A_1,B_1,C_1,D_1$and H_1 we need to impose some conditions (Shamsul [27], [28], [29]) between the eigenvalues.

In this article, we have examined solutions for the cases $\lambda >> \mu >> \nu$. As a result, we will be able to separate the equation (11) for unknown functions B_1 and D_1; and solving them for B_1 and D_1 substituting the values of B_1 and D_1 into the equation (11) and applying the conditions $\lambda >> \mu >> \nu$, we can separate the equation (10) for three unknown functions B_1, D_1 and H_1; and solving them forH_1. As $\dot{a}$, $\dot{b}$, $\dot{c}$, $\dot{d}$, $\dot{h}$ are proportional to small parameter, they are slowly varying functions of time t and for first approximate solutions; we may consider them as constants in the right side. Murty and Deekshatulu [30] first made this assumption. Therefore, the solutions of the equation (4) become

$$a=a_0+\in\int_0^t A_1(a,b,c,d,h,t)dt$$
$$b=b_0+\in\int_0^t B_1(a,b,c,d,h,t)dt$$
$$c=c_0+\in\int_0^t C_1(a,b,c,d,h,t)dt \quad (12)$$
$$d=d_0+\in\int_0^t D_1(a,b,c,d,h,t)dt$$
$$h=h_0+\in\int_0^t H_1(a,b,c,d,h,t)dt$$

Equation (9) is a non-homogeneous linear ordinary differential equation; therefore, it can be solved by the well-known operator method. Substituting the values of a, b, c, d, h and u_1 in the equations (3), we shall get the comprehensive solution of (1). Thus, the determination of the first estimated solution is far-reaching.

III. EXAMPLE

In this article, we have taken into account the Duffing type equation of fifth order nonlinear differential system as an example of the above method:

$$x^{v} + k_1 x^{iv} + k_2 \dddot{x} + k_3 \ddot{x} + k_4 \dot{x} + k_5 x = -\in x^3 \qquad (13)$$

Comparing (13) and (1), we obtain $f(x, \dot{x}, \ddot{x}, \dddot{x}, x^{iv}) = x^3$

Therefore,

$$\begin{aligned}
f^{(0)} &= a^3 e^{-3\lambda t} + 3a^2 c e^{-(2\lambda+\mu)t} + 3a^2 h e^{-(2\lambda+\nu)t} + 3ac^2 e^{-(\lambda+2\mu)t} + 6ache^{-(\lambda+\mu+\nu)t} + 3ah^2 e^{-(\lambda+2\nu)t} \\
&+ c^3 e^{-3\mu t} + 3c^2 h e^{-(2\mu+\nu)t} + 3ch^2 e^{-(\mu+2\nu)t} + h^3 e^{-3\nu t} + 3t\Big\{a^2 b e^{-3\lambda t} + 2abce^{-(2\lambda+\mu)t} + a^2 d e^{-(2\lambda+\mu)t} \\
&+ 2abhe^{-(2\lambda+\nu)t} + 2acde^{-(\lambda+2\mu)t} + bc^2 e^{-(\lambda+2\mu)t} + 2adhe^{-(\lambda+\mu+\nu)t} + 2bche^{-(\lambda+\mu+\nu)t} + bh^2 e^{-(\lambda+2\nu)t} \\
&+ c^2 d e^{-3\mu t} + 2cdhe^{-(2\mu+\nu)t} + dh^2 e^{-(\mu+2\nu)t}\Big\} + 3t^2\Big\{ab^2 e^{-3\lambda t} + b^2 c e^{-(2\lambda+\mu)t} + 2abde^{-(2\lambda+\mu)t} + b^2 h e^{-(2\lambda+\nu)t} \\
&+ ad^2 e^{-(\lambda+2\mu)t} + 2bcde^{-(\lambda+2\mu)t} + 2bdhe^{-(\lambda+\mu+\nu)t} + cd^2 e^{-3\mu t} + d^2 h e^{-(2\mu+\nu)t}\Big\} + \\
& t^3\Big\{b^3 e^{-3\lambda t} + b^2 d e^{-(2\lambda+\mu)t} + bd^2 e^{-(\lambda+2\mu)t} + d^3 e^{-3\mu t}\Big\}
\end{aligned} \qquad (14)$$

Now comparing equations (14) and (2), we obtain

$$\begin{aligned}
\sum_{i,j,k,l=0}^{\infty} F_{0,l}(a,b,c,d,h)\, e^{-(i\lambda+j\mu+k\nu)t} &= a^3 e^{-3\lambda t} + 3a^2 c e^{-(2\lambda+\mu)t} + 3a^2 h e^{-(2\lambda+\nu)t} + 3ac^2 e^{-(\lambda+2\mu)t} \\
&+ 6ache^{-(\lambda+\mu+\nu)t} + 3ah^2 e^{-(\lambda+2\nu)t} + c^3 e^{-3\mu t} + 3c^2 h e^{-(2\mu+\nu)t} + 3ch^2 e^{-(\mu+2\nu)t} + h^3 e^{-3\nu t}
\end{aligned}$$

$$\begin{aligned}
\sum_{i,j,k,l=0}^{\infty} F_{1,l}(a,b,c,d,h)\, e^{-(i\lambda+j\mu+k\nu)t} &= 3\Big\{a^2 b e^{-3\lambda t} + 2abce^{-(2\lambda+\mu)t} + a^2 d e^{-(2\lambda+\mu)t} \\
&+ 2abhe^{-(2\lambda+\nu)t} + 2acde^{-(\lambda+2\mu)t} + bc^2 e^{-(\lambda+2\mu)t} + 2adhe^{-(\lambda+\mu+\nu)t} + 2bche^{-(\lambda+\mu+\nu)t} \\
&+ bh^2 e^{-(\lambda+2\nu)t} + c^2 d e^{-3\mu t} + 2cdhe^{-(2\mu+\nu)t} + dh^2 e^{-(\mu+2\nu)t}\Big\}
\end{aligned}$$

$$\begin{aligned}
\sum_{i,j,k,l=0}^{\infty} F_{2,l}(a,b,c,d,h)\, e^{-(i\lambda+j\mu+k\nu)t} &= 3\Big\{ab^2 e^{-3\lambda t} + b^2 c e^{-(2\lambda+\mu)t} + \\
& 2abde^{-(2\lambda+\mu)t} + b^2 h e^{-(2\lambda+\nu)t} + ad^2 e^{-(\lambda+2\mu)t} + 2bdhe^{-(\lambda+\mu+\nu)t} \\
&+ cd^2 e^{-3\mu t} + d^2 h e^{-(2\mu+\nu)t} + 2bcd\, e^{-(\lambda+2\mu)t}\Big\}
\end{aligned} \qquad (15)$$

$$\sum_{i,j,k,l=0}^{\infty} F_{3,l}(a,b,c,d,h)\, e^{-(i\lambda+j\mu+k\nu)t} = b^3 e^{-3\lambda t} + b^2 d e^{-(2\lambda+\mu)t} + bd^2 e^{-(\lambda+2\mu)t} + d^3 e^{-3\mu t}$$

For equation (13), the equations (9) to (11) respectively become

$$(D+\lambda)^2(D+\mu)^2(D+\nu)u_1=-3t^2\left\{ab^2e^{-3\lambda t}+b^2ce^{-(2\lambda+\mu)t}+2abde^{-(2\lambda+\mu)t}\right.$$
$$+b^2he^{-(2\lambda+\nu)t}+ad^2e^{-(\lambda+2\mu)t}+2bcde^{-(\lambda+2\mu)t}+2bdhe^{-(\lambda+\mu+\nu)t}+cd^2e^{-3\mu t}$$
$$\left.+d^2he^{-(2\mu+\nu)t}\right\}-t^3\left\{b^3e^{-3\lambda t}+b^2de^{-(2\lambda+\mu)t}+bd^2e^{-(\lambda+2\mu)t}+d^3e^{-3\mu t}\right\} \quad (16)$$

$$e^{-\lambda t}(D+\mu-\lambda)^2(D+\nu-\lambda)\left(\frac{\partial A_1}{\partial t}+2B_1\right)+e^{-\mu t}(D+\lambda-\mu)^2$$
$$(D+\nu-\mu)\left(\frac{\partial C_1}{\partial t}+2D_1\right)+e^{-\nu t}(D+\lambda-\nu)^2(D+\mu-\nu)^2H_1=$$
$$-\left\{a^3e^{-3\lambda t}+3a^2ce^{-(2\lambda+\mu)t}+3a^2he^{-(2\lambda+\nu)t}+3ac^2e^{-(\lambda+2\mu)t}+6ache^{-(\lambda+\mu+\nu)t}\right.$$
$$\left.+3ah^2e^{-(\lambda+2\nu)t}+c^3e^{-3\mu t}+3c^2he^{-(2\mu+\nu)t}+3ch^2e^{-(\mu+2\nu)t}+h^3e^{-3\nu t}\right\} \quad (17)$$

$$e^{-\lambda t}(D+\mu-\lambda)^2(D+\nu-\lambda)\frac{\partial B_1}{\partial t}+e^{-\mu t}(D+\lambda-\mu)^2(D+\nu-\mu)\frac{\partial D_1}{\partial t}$$
$$=-3\left\{a^2be^{-3\lambda t}+2abce^{-(2\lambda+\mu)t}+a^2de^{-(2\lambda+\mu)t}+2abhe^{-(2\lambda+\nu)t}+2acde^{-(\lambda+2\mu)t}\right. \quad (18)$$
$$+bc^2e^{-(\lambda+2\mu)t}+2adhe^{-(\lambda+\mu+\nu)t}+2bche^{-(\lambda+\mu+\nu)t}+bh^2e^{-(\lambda+2\nu)t}+c^2de^{-3\mu t}$$
$$\left.+2cdhe^{-(2\mu+\nu)t}+dh^2e^{-(\mu+2\nu)t}\right\}$$

Since the relations $\lambda >> \mu >> \nu$ among the eigenvalues, then the equation (18) can be separated for the unknown functionsB_1 andD_1 in the following way:

$$e^{-\lambda t}(D+\mu-\lambda)^2(D+\nu-\lambda)\frac{\partial B_1}{\partial t}=-3\left\{a^2be^{-3\lambda t}+2abce^{-(2\lambda+\mu)t}\right.$$
$$+a^2de^{-(2\lambda+\mu)t}+2abhe^{-(2\lambda+\nu)t}+2acde^{-(\lambda+2\mu)t}+bc^2e^{-(\lambda+2\mu)t} \quad (19)$$
$$\left.+2adhe^{-(\lambda+\mu+\nu)t}+2bche^{-(\lambda+\mu+\nu)t}+bh^2e^{-(\lambda+2\nu)t}\right\}$$

$$e^{-\mu t}(D+\lambda-\mu)^2(D+\nu-\mu)\frac{\partial D_1}{\partial t}=-3c^2de^{-3\mu t}-$$
$$6cdhe^{-(2\mu+\nu)t}-3dh^2e^{-(\mu+2\nu)t} \quad (20)$$

Solving equations (19) and (20), we get

$$B_1=l_1a^2be^{-2\lambda t}+l_2(abc+a^2d)e^{-(\lambda+\mu)t}+l_3(2acd+bc^2)e^{-2\mu t}$$
$$+l_4abhe^{-(\lambda+\nu)t}+l_5(adh+bch)e^{-(\mu+\nu)t}+l_6bh^2e^{-2\nu t} \quad (21)$$
$$D_1=p_1c^2de^{-2\mu t}+p_2cdhe^{-(\mu+\nu)t}+p_3dh^2e^{-2\nu t} \quad (22)$$

where $l_1=\dfrac{3}{2\lambda(3\lambda-\mu)^2(\nu-3\lambda)}$, $\quad l_2=\dfrac{3}{2\lambda^2(\lambda+\mu)(\nu-\mu-2\lambda)}$,

$l_3=\dfrac{3}{2\mu(\lambda+\mu)^2(\nu-\mu-2\lambda)}$, $\quad l_4=\dfrac{-3}{\lambda(\lambda+\nu)(2\lambda+\nu-\mu)^2}$,

$l_5=\dfrac{-6}{(\lambda+\mu)(\lambda+\nu)^2(\nu+\mu)}$, $\quad l_6=\dfrac{-3}{2(\lambda+\nu)(\mu-\lambda+2\nu)^2}$,

$p_1=\dfrac{3}{2\mu(\lambda-3\mu)^2(\nu-3\mu)}$, $\quad p_2=\dfrac{-3}{\mu(\lambda-\nu-2\mu)^2(\nu+\mu)}$,

$p_3=\dfrac{-3}{2\nu(\lambda-\mu-2\nu)^2(\nu+\mu)}$

Using the values ofB_1 andD_1in equation (17), we obtain

$$e^{-\lambda t}(D+\mu-\lambda)^2(D+\nu-\lambda)\frac{\partial A_1}{\partial t}+e^{-\mu t}(D+\lambda-\mu)^2(D+\nu-\mu)\frac{\partial C_1}{\partial t}$$
$$+e^{-\nu t}(D+\lambda-\nu)^2(D+\mu-\nu)^2H_1=-a^3e^{-3\lambda t}-3a^2ce^{-(2\lambda+\mu)t}-$$
$$3a^2he^{-(2\lambda+\nu)t}\;-3ac^2e^{-(\lambda+2\mu)t}-6ache^{-(\lambda+\mu+\nu)t}-3ah^2e^{-(\lambda+2\nu)t}-c^3e^{-3\mu t}$$
$$-3c^2he^{-(2\mu+\nu)t}-3ch^2e^{-(\mu+2\nu)t}-h^3e^{-3\nu t}-2(3\lambda-\mu)^2(\nu-3\lambda)l_1a^2be^{-3\lambda t} \quad (23)$$
$$-8\lambda^2(\nu-\mu-2\lambda)l_2abce^{-(2\lambda+\mu)t}-8\lambda^2(\nu-\mu-2\lambda)l_3a^2de^{-(2\lambda+\mu)t}$$
$$+4\lambda(\mu-\nu-2\lambda)^2l_4abhe^{-(2\lambda+\nu)t}+2(\lambda+\mu)^2(\nu-2\mu-\lambda)l_5(adh+bch)$$
$$e^{-(\lambda+\mu+\nu)t}+2(\mu-\lambda-2\nu)^2(\lambda+\nu)l_6bh^2e^{-(\lambda+2\nu)t}-2(\lambda-3\mu)^2(\nu-3\mu)p_1c^2d$$
$$e^{-3\mu t}+4\mu(\lambda-\nu-2\mu)^2p_2cdhe^{-(2\mu+\nu)t}+2(\mu+\nu)(\lambda-\mu-2\nu)^2p_3dh^2e^{-(\mu+2\nu)t}$$

Again applying the conditions $\lambda >> \mu >> \nu$ in equation (23), then we obtain the following equations for unknown functionsA_1,C_1,and H_1

$$e^{-\lambda t}(D+\mu-\lambda)^2(D+\nu-\lambda)\frac{\partial A_1}{\partial t}=-a^3e^{-3\lambda t}-3a^2ce^{-(2\lambda+\mu)t}$$
$$-3ac^2e^{-(\lambda+2\mu)t}-6ache^{-(\lambda+\mu+\nu)t}-3a^2he^{(\lambda+2\eta)t}-2(3\lambda-\mu)^2$$
$$(\nu-3\lambda)l_1a^2be^{-3\lambda t}-8\lambda^2(\nu-\mu-2\lambda)l_2(2abc+bc^2)e^{-(2\lambda+\mu)t} \quad (24)$$
$$-8(\lambda+\mu)^2(\nu-\mu-2\lambda)l_3(2acd+bc^2)e^{-(\lambda+2\mu)t}$$
$$+4\lambda(\mu-\nu-2\lambda)^2l_4abhe^{-(2\lambda+\nu)t}+2(\lambda+\nu)^2(\nu+\mu)$$
$$l_5(adh+bch)e^{-(\lambda+\mu+\nu)t}+2(\lambda+\nu)(\mu-\lambda-2\nu)^2l_6bh^2e^{-(\lambda+2\nu)t}$$

$$e^{-\mu t}(D+\lambda-\mu)^2(D+\nu-\mu)\frac{\partial C_1}{\partial t}=-c^3e^{-3\mu t}-2(\lambda-3\mu)^2(\nu-3\mu)$$
$$p_1c^2de^{-3\mu t}-3c^2he^{-(2\mu+\nu)t}-3ch^2e^{-(\mu+2\nu)t}+4\mu(\lambda-\nu-2\mu)^2 \quad (25)$$
$$p_2cdhe^{-(2\mu+\nu)t}+2(\mu+\nu)(\lambda-\mu-2\nu)^2p_3dh^2e^{-(\mu+2\nu)t}$$

$$e^{-\nu t}(D+\lambda-\nu)^2(D+\mu-\nu)^2H_1=-h^3e^{-3\nu t} \quad (26)$$

Solving equations (24), (25) and (26), we obtain

$$A_1=(n_1a^3+n_7a^2b)e^{-2\lambda t}+(n_3ac^2+n_9acd)e^{-2\mu t}+\{n_2a^2c+$$
$$n_6a^2h+n_8(2abc+a^2d)\}e^{-(\lambda+\mu)t}+\{n_4ach+ \quad (27)$$
$$n_{11}(adh+bch)\}e^{-(\nu+\mu)t}+(n_5ah^2+n_{12}bh^2)e^{-2\nu t}+n_{10}abhe^{-(\lambda+\nu)t}$$

$$C_1=(r_1c^3+r_2c^2d)e^{-2\mu t}+(r_3c^2h+r_5cdh)e^{-(\mu+\nu)t}+(r_4ch^2+r_6dh^2)e^{-2\nu t} \quad (28)$$

$$H_1=mh^3e^{-2\nu t} \quad (29)$$

where $n_1=\dfrac{1}{2\lambda(\mu-3\lambda)^2(\nu-3\lambda)}$, $n_2=\dfrac{3}{4\lambda^2(\mu+\lambda)(\nu-\mu-2\lambda)}$,

$n_3=\dfrac{3}{2\mu(\mu+\lambda)^2(\nu-2\mu-\lambda)}$, $n_4=\dfrac{-6}{(\lambda+\mu)(\mu+\nu)(\nu+\lambda)^2}$, $n_5=\dfrac{-3}{2\nu(\nu+\lambda)(\mu-\lambda-2\nu)^2}$,

$n_6=\dfrac{3}{4\lambda^2(\mu+\lambda)(\nu-\mu-2\lambda)}$,

$$n_7=\frac{3}{2\lambda^2(\nu-3\lambda)(\mu-3\lambda)^2},\ n_8=\frac{3}{2\lambda^2(\mu+\lambda)^2(\nu-\mu-2\lambda)},$$
$$n_9=\frac{3}{2\mu^2(\mu+\lambda)^2(\nu-2\mu-\lambda)},\ n_{10}=\frac{-6}{\lambda(\nu+\lambda)^2(\mu-\nu-2\lambda)^2},$$
$$n_{11}=\frac{12}{(\lambda+\mu)(\mu+\nu)(\nu+\lambda)^2},\ n_{12}=\frac{-3}{2\nu^2(\lambda+\nu)(\nu-\lambda-2\nu)^2},$$
$$r_1=\frac{1}{2\mu(\lambda-3\mu)^2(\nu-3\mu)},\ r_2=\frac{-3}{2\mu^2(\lambda-3\mu)^2(\nu-3\mu)},$$
$$r_3=\frac{-3}{2\mu(\lambda+\nu)(\lambda-\nu-2\mu)^2},\ r_4=\frac{-3}{2\nu(\mu+\nu)(\lambda-\mu-2\nu)^2},\ r_5=\frac{6}{\mu(\lambda+\nu)(\lambda-\nu-2\mu)^2},$$
$$r_6=\frac{-3}{2\nu^2(\lambda-\mu-2\nu)^2(\mu+\nu)},$$
$$m=\frac{1}{(\lambda-3\nu)^2\left(\mu-3\nu\right)^2}$$

Thus the solution of the equation (16) for u_1 is

$$\begin{aligned}u_1&=ab^2e^{-3\lambda t}(q_1t^2+q_2t+q_3)+(b^2c+2abd)e^{-(2\lambda+\mu)t}(q_4t^2+q_5t+q_6)\\&+b^2he^{-(2\lambda+\nu)t}(q_7t^2+q_8t+q_9)+(ad^2+2bcd)e^{-(\lambda+2\mu)t}(q_{10}t^2+q_{11}t+q_{12})\\&+bdhe^{-(\lambda+\mu+\nu)t}(q_{13}t^2+q_{14}t+q_{15})+cd^2e^{-3\mu t}(q_{16}t^2+q_{17}t+q_{18})\\&+d^2he^{-(2\mu+\nu)t}(q_{19}t^2+q_{20}t+q_{21})+b^3e^{-3\lambda t}(q_{22}t^3+q_{23}t^2+q_{24}t+q_{25})\\&+b^2de^{-(2\lambda+\mu)t}(q_{26}t^3+q_{27}t^2+q_{28}t+q_{29})+bd^2e^{-(\lambda+2\mu)t}(q_{30}t^3+q_{31}t^2\\&+q_{32}t+q_{33})+d^3e^{-3\mu t}(q_{34}t^3+q_{35}t^2+q_{36}t+q_{37})\end{aligned}\qquad(30)$$

where $q_1=\dfrac{1}{4\lambda^2(3\lambda-\mu)^2(3\lambda-\nu)}$,

$$q_2=\frac{2}{4\lambda^2(3\lambda-\mu)^2(3\lambda-\nu)}\left(\frac{1}{\lambda}+\frac{2}{3\lambda-\mu}+\frac{1}{3\lambda-\nu}\right),$$
$$q_3=\frac{2}{4\lambda^2(3\lambda-\mu)^2(3\lambda-\nu)}\left\{\frac{3}{4\lambda^2}+\frac{2}{\lambda(3\lambda-\mu)}+\frac{3}{(3\lambda-\mu)^2}+\frac{1}{\lambda(3\lambda-\nu)}+\frac{1}{(3\lambda-\nu)^2}+\frac{2}{(3\lambda-\mu)(3\lambda-\nu)}\right\},$$
$$q_4=\frac{3}{4\lambda^2(\lambda+\mu)^2(2\lambda+\mu-\nu)},$$
$$q_5=\frac{6}{4\lambda^2(\lambda+\mu)^2(2\lambda+\mu-\nu)}\left(\frac{2}{\lambda+\mu}+\frac{1}{\lambda}+\frac{1}{2\lambda+\mu-\nu}\right),$$
$$q_6=\frac{3}{2\lambda^2(\lambda+\mu)^2(2\lambda+\mu-\nu)}\left\{\frac{3}{(\lambda+\mu)^2}+\frac{3}{4\lambda^2}+\frac{2}{\lambda(\lambda+\mu)}+\frac{1}{(2\lambda+\mu-\nu)^2}+\frac{1}{\lambda(2\lambda+\mu-\nu)}+\frac{2}{(\lambda+\mu)(2\lambda+\mu-\nu)}\right\},$$

$$q_7=\frac{3}{2\lambda(2\lambda+\nu-\mu)^2(\lambda+\nu)^2},$$

$$q_8=\frac{3}{\lambda(2\lambda+\nu-\mu)^2(\lambda+\nu)^2}\left(\frac{2}{\lambda+\nu}+\frac{2}{2\lambda+\nu-\mu}+\frac{1}{2\lambda}\right),$$

$$q_9=\frac{3}{\lambda(2\lambda+\nu-\mu)^2(\lambda+\nu)^2}\left\{\frac{3}{(\lambda+\nu)^2}+\frac{3}{(2\lambda+\nu-\mu)^2}\right.$$
$$\left.+\frac{4}{(\lambda+\nu)(2\lambda+\nu-\mu)}+\frac{1}{4\lambda^2}+\frac{1}{\lambda(\lambda+\nu)}+\frac{1}{\lambda(2\lambda+\nu-\mu)}\right\},$$

$$q_{10}=\frac{3}{4\mu^2(\lambda+\mu)^2(\lambda+2\mu-\nu)},$$

$$q_{11}=\frac{3}{2\mu^2(\lambda+\mu)^2(\lambda+2\mu-\nu)}\left(\frac{2}{\lambda+\mu}+\frac{1}{\mu}+\frac{1}{\lambda+2\mu-\nu}\right),$$

$$q_{12}=\frac{3}{2\mu^2(\lambda+\mu)^2(\lambda+2\mu-\nu)}\left\{\frac{3}{(\lambda+\mu)^2}+\frac{3}{4\mu^2}+\frac{2}{\mu(\lambda+\mu)}\right.$$
$$\left.+\frac{1}{(\lambda+2\mu-\nu)^2}+\frac{1}{\mu(\lambda+2\mu-\nu)}+\frac{2}{(\lambda+\mu)(\lambda+2\mu-\nu)}\right\},$$

$$q_{13}=\frac{6}{(\lambda+\mu)(\mu+\nu)^2(\lambda+\nu)^2},$$

$$q_{14}=\frac{12}{(\lambda+\mu)(\mu+\nu)^2(\lambda+\nu)^2}\left(\frac{1}{\lambda+\mu}+\frac{2}{\mu+\nu}+\frac{2}{\lambda+\nu}\right),$$

$$q_{15}=\frac{12}{(\lambda+\mu)(\mu+\nu)^2(\lambda+\nu)^2}\left\{\frac{3}{(\lambda+\nu)^2}+\frac{3}{(\mu+\nu)^2}+\frac{4}{(\mu+\nu)(\lambda+\nu)}\right.$$
$$\left.+\frac{1}{(\lambda+\mu)^2}+\frac{2}{(\lambda+\mu)(\mu+\nu)}+\frac{2}{(\lambda+\mu)(\lambda+\nu)}\right\},$$

$$q_{16}=\frac{3}{4\mu^2(\lambda-3\mu)^2(3\mu-\nu)},\quad q_{17}=\frac{3}{2\mu^2(\lambda-3\mu)^2(3\mu-\nu)}\left(\frac{1}{\mu}+\frac{2}{3\mu-\lambda}+\frac{1}{3\mu-\nu}\right),$$

$$q_{18}=\frac{3}{2\mu^2(\lambda-3\mu)^2(3\mu-\nu)}\left\{\frac{3}{4\mu^2}+\frac{2}{\mu(3\mu-\lambda)}+\frac{3}{(3\mu-\lambda)^2}\right.$$
$$\left.+\frac{1}{\mu(3\mu-\nu)}+\frac{1}{(3\mu-\nu)^2}+\frac{2}{(3\mu-\lambda)(3\mu-\nu)}\right\},$$

$$q_{19}=\frac{3}{2\mu(2\mu+\nu-\lambda)^2(\mu+\nu)^2},$$

$$q_{20}=\frac{3}{\mu(2\mu+\nu-\lambda)^2(\mu+\nu)^2}\left(\frac{2}{\mu+\nu}+\frac{2}{2\mu+\nu-\lambda}+\frac{1}{2\mu}\right),$$

$$q_{21}=\frac{3}{\mu(2\mu+\nu-\lambda)^2(\mu+\nu)^2}\left\{\frac{3}{(\mu+\nu)^2}+\frac{3}{(2\mu+\nu-\lambda)^2}\right.$$
$$\left.+\frac{4}{(\mu+\nu)(2\mu+\nu-\lambda)}+\frac{1}{4\mu^2}+\frac{1}{\mu(\mu+\nu)}+\frac{1}{\mu(2\mu+\nu-\lambda)}\right\},$$

$$q_{22}=\frac{1}{4\lambda^2(3\lambda-\mu)^2(3\lambda-\nu)},$$

$$q_{23}=\frac{3}{4\lambda^2(3\lambda-\mu)^2(3\lambda-\nu)}\left(\frac{1}{\lambda}+\frac{2}{3\lambda-\mu}+\frac{1}{3\lambda-\nu}\right),$$

$$q_{24}=\frac{3}{2\lambda^2(3\lambda-\mu)^2(3\lambda-\nu)}\left\{\frac{3}{4\lambda^2}+\frac{2}{\lambda(3\lambda-\mu)}+\frac{3}{(3\lambda-\mu)^2}\right.$$
$$\left.+\frac{1}{\lambda(3\lambda-\nu)}+\frac{1}{(3\lambda-\nu)^2}+\frac{2}{(3\lambda-\mu)(3\lambda-\nu)}\right\},$$

$$q_{25}=\frac{3}{2\lambda^2(3\lambda-\mu)^2(3\lambda-\nu)}\left\{\frac{1}{2\lambda^3}+\frac{3}{2\lambda^2(3\lambda-\mu)}+\frac{3}{\lambda(3\lambda-\mu)^2}\right.$$
$$+\frac{4}{(3\lambda-\mu)^3}+\frac{1}{\lambda(3\lambda-\nu)^2}+\frac{2}{(3\lambda-\mu)(3\lambda-\nu)^2}+\frac{1}{(3\lambda-\nu)^3}$$
$$\left.+\frac{3}{4\lambda^2(3\lambda-\nu)}+\frac{2}{\lambda(3\lambda-\mu)(3\lambda-\nu)}+\frac{3}{(3\lambda-\mu)^2(3\lambda-\nu)}\right\},$$

$$q_{26}=\frac{3}{4\lambda^2(\lambda+\mu)^2(2\lambda+\mu-\nu)},$$

$$q_{27}=\frac{9}{4\lambda^2(\lambda+\mu)^2(2\lambda+\mu-\nu)}\left(\frac{2}{\lambda+\mu}+\frac{1}{\lambda}+\frac{1}{2\lambda+\mu-\nu}\right),$$

$$q_{28}=\frac{9}{2\lambda^2(\lambda+\mu)^2(2\lambda+\mu-\nu)}\left\{\frac{3}{(\lambda+\mu)^2}+\frac{3}{4\lambda^2}+\frac{2}{\lambda(\lambda+\mu)}\right.$$
$$\left.+\frac{1}{(2\lambda+\mu-\nu)^2}+\frac{1}{\lambda(2\lambda+\mu-\nu)}+\frac{2}{(\lambda+\mu)(2\lambda+\mu-\nu)}\right\},$$

$$q_{29}=\frac{9}{2\lambda^2(\lambda+\mu)^2(2\lambda+\mu-\nu)}\left\{\frac{4}{(\lambda+\mu)^3}+\frac{3}{2\lambda^3}+\frac{3}{\lambda(\lambda+\mu)^2}+\frac{3}{2\lambda^2(\lambda+\mu)}\right.$$
$$+\frac{1}{(2\lambda+\mu-\nu)^3}+\frac{2}{(\lambda+\mu)(2\lambda+\mu-\nu)^2}+\frac{1}{\lambda(2\lambda+\mu-\nu)^2}$$
$$\left.+\frac{2}{(\lambda+\mu)^2(2\lambda+\mu-\nu)}+\frac{3}{4\lambda^2(2\lambda+\mu-\nu)}+\frac{2}{\lambda(\lambda+\mu)(2\lambda+\mu-\nu)}\right\},$$

$$q_{30}=\frac{3}{4\mu^2(\lambda+\mu)^2(\lambda+2\mu-\nu)},$$

$$q_{31}=\frac{9}{4\mu^2(\lambda+\mu)^2(\lambda+2\mu-\nu)}\left(\frac{2}{\lambda+\mu}+\frac{1}{\mu}+\frac{1}{\lambda+2\mu-\nu}\right),$$

$$q_{32}=\frac{9}{2\mu^2(\lambda+\mu)^2(\lambda+2\mu-\nu)}\left\{\frac{3}{(\lambda+\mu)^2}+\frac{3}{4\mu^2}+\frac{2}{\mu(\lambda+\mu)}\right.$$
$$\left.+\frac{1}{(\lambda+2\mu-\nu)^2}+\frac{1}{\mu(\lambda+2\mu-\nu)}+\frac{2}{(\lambda+\mu)(\lambda+2\mu-\nu)}\right\},$$

$$q_{33}=\frac{9}{2\mu^2(\lambda+\mu)^2(\lambda+2\mu-\nu)}\left\{\frac{4}{(\lambda+\mu)^3}+\frac{3}{2\mu^2(\lambda+\mu)}+\frac{1}{(\lambda+2\mu-\nu)^3}+\frac{3}{2\mu^3}\right.$$
$$+\frac{3}{\mu(\lambda+\mu)^2}+\frac{2}{(\lambda+\mu)(\lambda+2\mu-\nu)^2}+\frac{1}{\mu(\lambda+2\mu-\nu)^2}+\frac{3}{(\lambda+\mu)^2(\lambda+2\mu-\nu)}$$
$$\left.+\frac{3}{4\mu^2(\lambda+2\mu-\nu)}+\frac{2}{\mu(\lambda+\mu)(\lambda+2\mu-\nu)}\right\},$$

$$q_{34}=\frac{1}{4\mu^2(\lambda-3\mu)^2(3\mu-\nu)},$$

$$q_{35}=\frac{3}{4\mu^2(\lambda-3\mu)^2(3\mu-\nu)}\left(\frac{1}{\mu}+\frac{2}{3\mu-\lambda}+\frac{1}{3\mu-\nu}\right),$$

$$q_{36}=\frac{3}{2\mu^2(\lambda-3\mu)^2(3\mu-\nu)}\left\{\frac{3}{4\mu^2}+\frac{2}{\mu(3\mu-\lambda)}+\frac{3}{(3\mu-\lambda)^2}\right.$$
$$\left.+\frac{1}{\mu(3\mu-\nu)}+\frac{1}{(3\mu-\nu)^2}+\frac{2}{(3\mu-\lambda)(3\mu-\nu)}\right\},$$

$$q_{37}=\frac{3}{2\mu^2(\lambda-3\mu)^2(3\mu-\nu)}\left\{\frac{1}{2\mu^3}+\frac{3}{2\mu^2(3\mu-\lambda)}+\frac{3}{\mu(3\mu-\lambda)^2}\right.$$
$$+\frac{4}{(3\mu-\lambda)^3}+\frac{1}{\mu(3\mu-\nu)^2}+\frac{2}{(3\mu-\lambda)(3\mu-\nu)^2}+\frac{1}{(3\mu-\nu)^3}$$
$$\left.+\frac{3}{4\mu^2(3\mu-\nu)}+\frac{2}{\mu(3\mu-\lambda)(3\mu-\nu)}+\frac{3}{(3\mu-\lambda)^2(3\mu-\nu)}\right\}$$

Substituting the values of A_1, B_1, C_1, D_1 and H_1 from equations (27), (21), (28), (22) and (29) into equation (4), we obtain

$$\dot{a}=\in\left\{(n_1a^3+n_7a^2b)e^{-2\lambda t}+(n_3ac^2+n_9acd)e^{-2\mu t}+n_2(a^2c+b^2c)e^{-(\lambda+\mu)t}\right.$$
$$+n_6a^2he^{-(\lambda+\mu)t}+n_8(2acd+a^2d)e^{-(\lambda+\mu)t}+n_4ache^{-(\mu+\nu)t}+$$
$$\left.n_{11}(adh+bch)e^{-(\mu+\nu)t}+(n_5ah^2+n_{12}bh^2)e^{-2\nu t}+n_{10}abhe^{-(\lambda+\nu)t}\right\}$$

$$\dot{b}=\in\left\{l_1a^2be^{-2\lambda t}+l_2(2abc+a^2d)e^{-(\lambda+\mu)t}+l_3(2acd+bc^2)e^{-2\mu t}\right.$$
$$\left.+l_5(adh+bch)e^{-(\mu+\nu)t}+l_4abhe^{-(\nu+\lambda)t}+l_6bh^2e^{-2\nu t}\right\} \qquad (31)$$

$$\dot{c}=\in\left\{r_1c^3e^{-2\mu t}+r_2c^2de^{-2\mu t}+(r_3c^2h+r_5cdh)\right.$$
$$\left.e^{-(\mu+\nu)t}+(r_4ch^2+r_6dh^2)e^{-2\nu t}\right\}$$

$$\dot{d}=\in\left\{p_1c^2de^{-2\mu t}+p_2cdhe^{-(\mu+\nu)t}+p_3dh^2e^{-2\nu t}\right\}$$

$$\dot{h}=\in mh^3e^{-2\nu t}$$

Here all of the equations (31) have no accurate solutions, but as $\dot{a},\ \dot{b},\ \dot{c},\ \dot{d},\ \dot{h}$ are proportional to the small parameter $\in$, they are slowly varying functions of time *t*. Accordingly, it is possible to replace a, b, c, d, h by their respective values obtained in linear case (i.e., the values of a, b, c, d, h obtained when $\in = 0.1$) in the right hand side of equations (31). Murty and Deekshatulu ([31], [32])first introduced this sort of replacement to resolve similar sort of nonlinear equations.
Thus, the solutions of equations (31) are

$$a = a_0 + \in \left\{ (n_1 a_0^3 + n_7 a_0^2 b_0)\frac{1-e^{-2\lambda t}}{2\lambda} + ((n_2 a_0^2 c_0 + b_0^2 c_0) + n_6 a_0^2 h_0 + n_8(2a_0 b_0 c_0 + a_0^2 d_0))\frac{1-e^{-(\lambda+\mu)t}}{\lambda+\mu} + (n_5 a_0 h_0^2 + n_{12} b_0 h_0^2)\frac{1-e^{-2\mu t}}{2\mu} + (n_4 a_0 c_0 h_0 + n_{11}(a_0 d_0 h_0 + b_0 c_0 h_0))\frac{1-e^{-(\mu+\nu)t}}{\mu+\nu} + n_{10} a_0 b_0 h_0 \frac{1-e^{-(\lambda+\nu)t}}{\lambda+\nu} \right\}$$

$$b = b_0 + \in \left\{ r_1 a_0^2 b_0 \frac{1-e^{-2\lambda t}}{2\lambda} + r_2(2a_0 b_0 c_0 + a_0^2 d_0)\frac{1-e^{-(\lambda+\mu)t}}{\lambda+\mu} + r_4 a_0 b_0 h_0 \frac{1-e^{-(\lambda+\nu)t}}{\lambda+\nu} + r_3(2a_0 c_0 d_0 + b_0 c_0^2)\frac{1-e^{-2\mu t}}{2\mu} + r_5(a_0 d_0 h_0 + b_0 c_0 h_0)\frac{1-e^{-(\mu+\nu)t}}{\mu+\nu} + r_6 b_0 h_0^2 \frac{1-e^{-2\nu t}}{2\nu} \right\}$$

$$c = c_0 + \in \left\{ r_1 c_0^3 \frac{1-e^{-2\mu t}}{2\mu} + r_2 c_0^2 d_0 \frac{1-e^{-2\mu t}}{2\mu} + (r_3 c_0^2 h_0 + r_5 c_0 d_0 h_0)\frac{1-e^{-(\mu+\nu)t}}{\mu+\nu} + (r_4 c_0 h_0^2 + r_6 d_0 h_0^2)\frac{1-e^{-2\nu t}}{2\nu} \right\} \qquad (32)$$

$$d = d_0 + \in \left\{ p_1 c_0^2 d_0 \frac{1-e^{-2\mu t}}{2\mu} + p_2 c_0 d_0 h_0 \frac{1-e^{-(\mu+\nu)t}}{\mu+\nu} + p_3 d_0 h_0^2 \frac{1-e^{-2\nu t}}{2\nu} \right\}$$

$$h = h_0 + \in m h_0^3 \frac{1-e^{-2\nu t}}{2\nu}$$

Hence, we obtain the first approximate solution of the equation (13) as

$$x(t,\in) = (a+bt)e^{-\lambda t} + (c+dt)e^{-\mu t} + he^{-\nu t} + \in u_1 \quad (33)$$

where a, b, c, d, h are given by the equations (32) and u_1 is given by (30).

IV. RESULTS AND DISCUSSION

It is usual to compare the perturbation solution to the numerical solution to test the accuracy of the approximate solution obtained by a certain perturbation method. We have computed $x(t,\in)$ using (33), in which a, b, c, d, h are obtained from (32) and u_1 is calculated from equation (30). The result obtained from (33) for various values of *t*, and the corresponding numerical solution obtained by a fourth order *Runge-Kutta* method is presented in the following **Fig.1, Fig.2, Fig.3,** and **Fig.4** respectively.

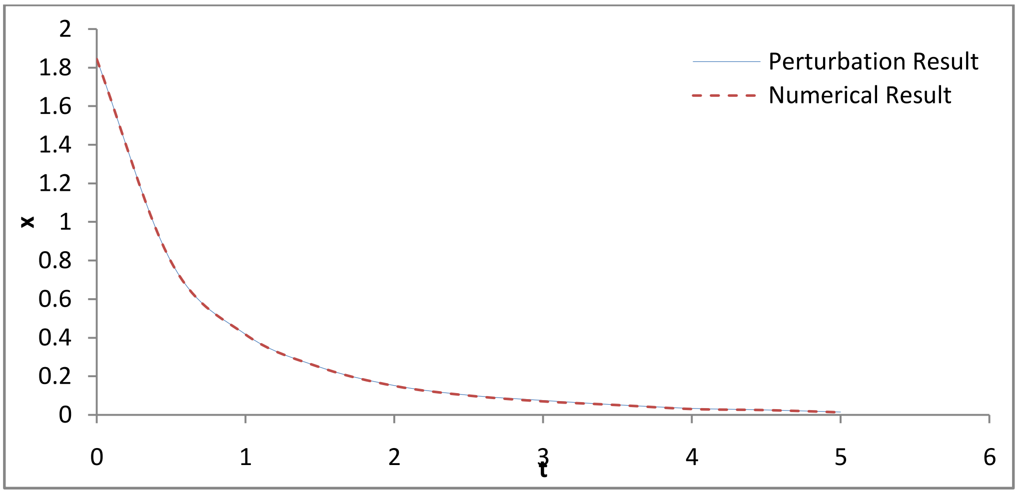

Figure 1: *Comparison between perturbation and numerical results for* $\lambda = 3.1,\ \mu = 1.5,\ \nu = 0.25$ *and* $\in = 0.1$ *with the initial conditions* $a_0 = 0.45,\ b_0 = 0.25,\ c_0 = 0.10,\ d_0 = 0.25,\ h_0 = 0.20.$

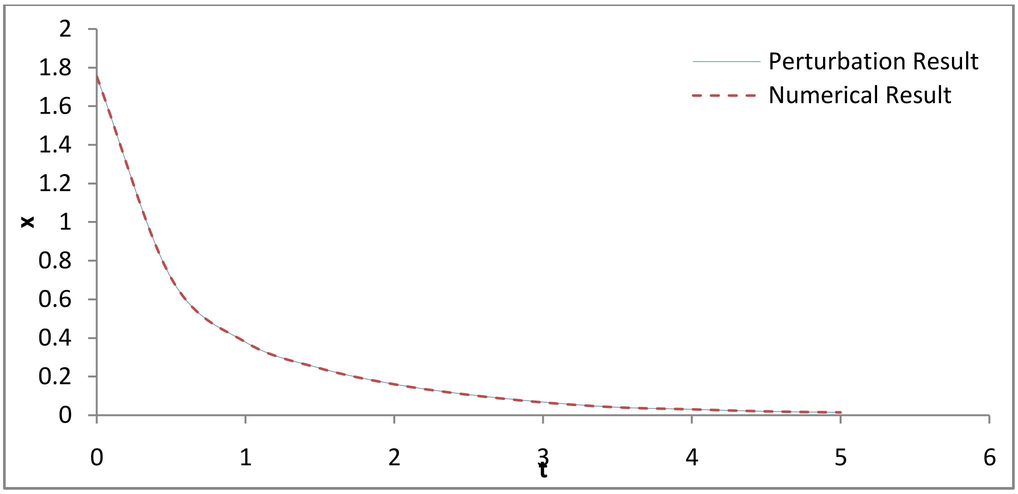

Figure 2: *Comparison between perturbation and numerical results for* $\lambda = 3,\ \mu = 1.4,\ \nu = 0.20$ *and* $\in = 0.1$ *with the initial conditions* $a_0 = 0.40,\ b_0 = 0.30,\ c_0 = 0.20,\ d_0 = 0.20,\ h_0 = 0.25.$

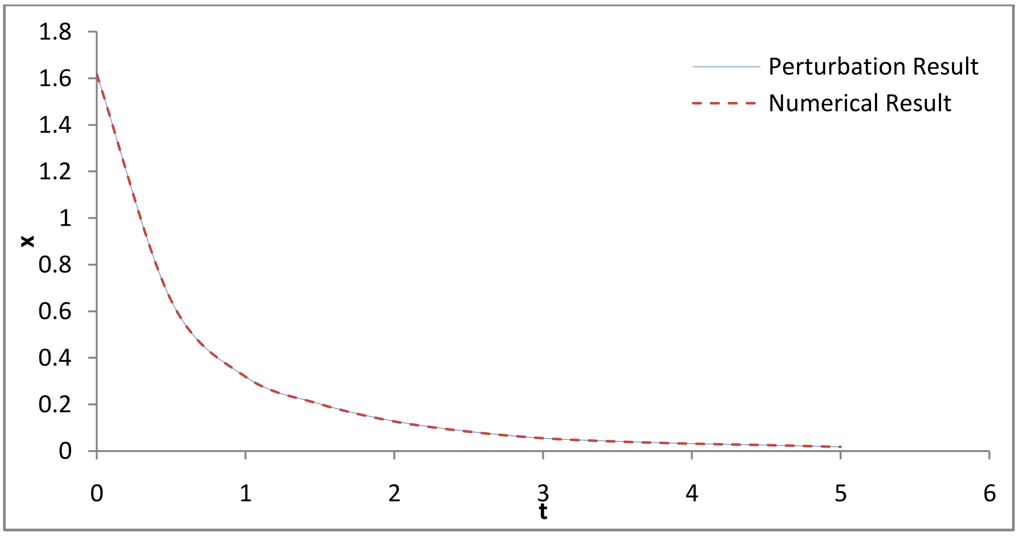

Figure 3: *Comparison between perturbation and numerical results for* $\lambda = 2.8,\ \mu = 1.3,\ \nu = 0.27$ *and* $\in = 0.1$ *with the initial conditions* $a_0 = 0.45,\ b_0 = 0.30,\ c_0 = 0.20,\ d_0 = 0.25,\ h_0 = 0.30,$

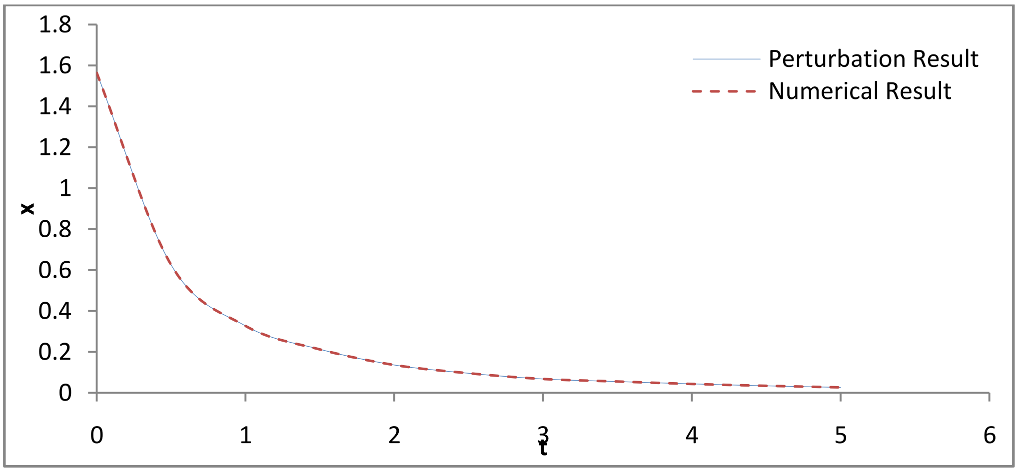

Figure 4: *Comparison between perturbation and numerical results for* $\lambda = 2.7,\ \mu = 1.25,\ \nu = 0.29$ *and* $\in = 0.1$ *with the initial conditions* $a_0 = 0.40,\ b_0 = 0.35,\ c_0 = 0.10,\ d_0 = 0.30,\ h_0 = 0.15,$

The corresponding perturbation andnumerical results for various values of *t* are demonstrated in the following **Table 1**, **Table 2**, **Table 3** and **Table 4** respectively.

Table 1: *Comparison between perturbation and Numerical results*

Time(t)	Perturbation Result(x)	Numerical Result(x)
0	1.843764	1.843764
0.5	0.793825	0.793371
1	0.417044	0.4164837
1.5	0.247136	0.246934
2	0.151797	0.150252
2.5	0.101659	0.100124
3	0.074406	0.070024
3.5	0.051458	0.051003
4	0.033197	0.030325
4.5	0.025163	0.024913
5	0.014364	0.013267

Table 2: *Comparison between perturbation and numerical results*

Time(t)	Perturbation Result(x)	Numerical Result(x)
0	1.754502	1.754502
0.5	0.710845	0.710617
1	0.378653	0.378532
1.5	0.244124	0.243042
2	0.161075	0.160257
2.5	0.106945	0.105749
3	0.068722	0.066325
3.5	0.041457	0.040952
4	0.030795	0.030563
4.5	0.020115	0.020012
5	0.015124	0.015023

Table 3: *Comparison between perturbation and Numerical results*

Time(t)	Perturbation Result(x)	Numerical Result(x)
0	1.617436	1.617436
0.5	0.646166	0.645918
1	0.318416	0.318302
1.5	0.201731	0.200983
2	0.127807	0.126352
2.5	0.083765	0.082674
3	0.054353	0.054202
3.5	0.041241	0.040123
4	0.031448	0.030823
4.5	0.025275	0.025013
5	0.017206	0.017079

Table 4: *Comparison between perturbation and numerical results*

Time(t)	Perturbation Result(x)	Numerical Result(x)
0	1.564734	1.564734
0.5	0.625894	0.625683
1	0.327245	0.327038
1.5	0.212863	0.211957
2	0.135976	0.135858
2.5	0.095642	0.095489
3	0.067589	0.067328
3.5	0.055134	0.055045
4	0.043754	0.042975
4.5	0.034087	0.034011
5	0.026572	0.026293

V. CONCLUSION

In this article, we have modified the KBM method and applied it to fifth order critically damped nonlinear systems. On the basis of the modification,transient responses of nonlinear differential systems have been examined. For fifth order critically damped nonlinear systems, the solutions are searched for such circumstances where $\lambda >> \mu >> \nu$. Thus, the results obtained for different set of initial conditions as well as different eigenvalues have shown excellent agreement with the numerical results obtained by the Runge-Kutta method.

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