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Research Paper

Fixed Point and Common Fixed Point Theorems under Various Expansive Conditions in Partial b-Metric Spaces

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ABSTRACT: *In 2013, Shukla [24] generalized both the concept of b-metric and partial metric spaces by introducing the partial b-metric spaces. In the present paper, we prove some fixed point theorems for self-mappings satisfying various expansive type conditions in the setting of a partial b-metric space. The presented theorems extend, generalize and improve many existing results in the literature.*

Keywords: *partial b-metric spaces, surjection, expansive mapping, fixed point.*

I. INTRODUCTION

Fixed point theory is one of the most popular tool in nonlinear analysis. Most of the generalizations for metric fixed point theorems usually start from Banach contraction principle [4]. It is not easy to point out all the generalizations of this principle. In 1989, Bakhtin [3] introduced the concept of a b-metric space as a generalization of metric spaces. In 1993, Czerwik [7-8] extended many results related to the b-metric spaces. In 1994, Matthews [19] introduced the concept of partial metric space in which the self distance of any point of space may not be zero. In 1996, O'Neill generalized the concept of partial metric space by admitting negative distances. In 2013, Shukla [24] generalized both the concept of b-metric and partial metric spaces by introducing the partial b-metric spaces. In 1984, Wang et.al [25] introduced the concept of expanding mappings and proved some fixed point theorems in complete metric spaces. In 1992, Daffer and Kaneko [9] defined an expanding condition for a pair of mappings and proved some common fixed point theorems for two mappings in complete metric spaces. Aage and Salunke [1] introduced several meaningful fixed point theorems for one expanding mapping. For more details on expanding mapping and related results we refer the reader to [10, 12-13, 22, 26-27].

In this paper, we prove some fixed point theorems for surjective mappings satisfying various expansive type conditions in the setting of a partial *b*-metric space. The presented theorems extend, generalize and improve many existing results in the literature.

I. PRELIMINARIES

Throughout this paper $\mathbb{R}$ and $\mathbb{R}^+$ will represents the set of real numbers and nonnegative real numbers, respectively.

The following definitions are required in the sequel.

Definition 2.1 (see [3]) Let X be a nonempty set, $s \geq 1$ be a given real number and $d : X \times X \to \mathbb{R}^+$ be a function. We say d is a b-metric on X if and only if for all $x, y, z \in X$ the following conditions are satisfied:

1. $d(x,y) = 0$ if and only if $x = y$;
2. $d(x,y) = d(y,x)$;

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3. $d(x,y) \leq s[d(x,z)+d(z,y)]$.

A triplet (X,d,s) is called a b-metric space. Obviously, for $s = 1$, b-metric reduces to metric.

Definition 2.2 (see [19]) Let X be a nonempty set, and $p : X \times X \to \mathbb{R}^+$ be a function. We say p is a partial metric on X if and only if for all $x,y,z \in X$ the following conditions are satisfied:

1. $x = y$ if and only if $p(x,x) = p(x,y) = p(y,y)$;
2. $p(x,x) \leq p(x,y)$;
3. $p(x,y) = p(y,x)$;
4. $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$

The pair (X,p) is called a partial metric space.

Remark 2.3 It is clear that the partial metric space need not be a b-metric spaces, since in a partial metric space if $p(x,y) = 0$ implies $p(x,x) = p(x,y) = p(y,y) = 0$ then $x = y$. But in a partial metric space if $x = y$ then $p(x,x) = p(x,y) = p(y,y)$ may not be equal zero. Therefore the partial metric space may not be a b-metric space.

On the other hand, Shukla [24] introduced the notion of a partial b-metric space as follows:
Definition 2.4 (see [24]) Let X be a nonempty set, $s \geq 1$ be a given real number and $p_b : X \times X \to \mathbb{R}^+$ be a function. We say p is a partial b-metric on X if and only if for all $x,y,z \in X$ the following conditions are satisfied:

1. $x = y$ if and only if $p_b(x,x) = p_b(x,y) = p_b(y,y)$;
2. $p_b(x,x) \leq p_b(x,y)$;
3. $p_b(x,y) = p_b(y,x)$;
4. $p_b(x,y) \leq s[p_b(x,z) + p_b(z,y) - p_b(z,z)]$

A triplet (X,p_b,s) is called a partial b-metric space. Obviously, for $s = 1$, partial b-metric reduces to partial metric.

Remark 2.5 The class of partial b-metric space (X,p_b) is effectively larger than the class of partial metric space, since a partial metric space is a special case of a partial b-metric space (X,p_b) when $s = 1$. Also, the class of partial b-metric space (X,p_b) is effectively larger than the class of b-metric space, since a b-metric space is a special case of a partial b-metric space (X,p_b) when the self distance $p(x,x) = 0$.
The following examples shows that a partial b-metric on X need not be a partial metric, nor a b-metric on X see also [21], [24].

Example 2.6 [21] Let $X = \mathbb{R}^+$. Define a function $p_b : X \times X \to \mathbb{R}^+$ such that $p_b(x,y) = [\max\{x,y\}]^2 + |x-y|^2, \forall\, x,y \in X$. Then (X,p_b) is a partial b-metric space on X with the coefficient $s = 2 > 1$. But, p_b is neither a b-metric nor a partial metric on X.

Proposition 2.7 [24] Let X be a nonempty set, and let p be a partial metric and d be a b-metric with the coefficient $s \geq 1$ on X. Then the function $p_b : X \times X \to \mathbb{R}^+$ defined by $p_b(x,y) = p(x,y) + d(x,y), \forall x.y \in X$, is a partial b-metric on X with the coefficient s.

Proposition 2.8 [24] Let (X,p) be a partial metric space and $q \geq 1$. Then (X,p_b) is a partial b-metric space with coefficient $s = 2^{q-1}$, where p_b is defined by $p_b(x,y) = [p(x,y)]^q$.
On the other hand, Mustafa [21] modify the Definition 2.4 in order that each partial b-metric p_b generates a b-metric d_{p_b} as follows:

Definition 2.9 (see [21]) Let X be a nonempty set, $s \geq 1$ be a given real number and $p_b : X \times X \to \mathbb{R}^+$ be a function. We say p is a partial b-metric on X if and only if for all $x,y,z \in X$ the following conditions are satisfied:

1. $x = y$ if and only if $p_b(x,x) = p_b(x,y) = p_b(y,y)$;
2. $p_b(x,x) \leq p_b(x,y)$;
3. $p_b(x,y) = p_b(y,x)$;
4. $p_b(x,y) \leq s[p_b(x,z) + p_b(z,y) - p_b(z,z)] + \left(\frac{1-s}{2}\right)\left(p_b(x,x) + p_b(y,y)\right)$

The pair (X,p_b) is called a partial b-metric space. The number $s \geq 1$ is called the coefficient of (X,p_b).

Example 2.10 (see also [21]) Let $X = \mathbb{R}$ is the set of real numbers. Consider the metric space (X,d) where d is the Euclidean distance metric $\mathrm{d}(x,y) = |x-y|, \forall\, x,y \in X$. Define $p_b(x,y) = (x-y)^2 + 5, \forall\, x,y \in X$. Then

p_b is a partial b-metric on X with $s = 2$, but it is not a partial metric on X. To see this, Let $x = 1, y = 4$ and $z = 2$. Then

$$p_b(1,4) = (1-4)^2 + 5 = 14 \nleq p_b(1,2) + p_b(2,4) - p_b(2,2) = 6 + 9 - 5 = 10$$

So, p_b is not a b-metric since $p_b(x,x) \neq 0, \forall\, x \in X$.

Proposition 2.11 (see [21]) Every partial b-metric p_b defines a b-metric d_{p_b}, where

$$d_{p_b}(x,y) = 2p_b(x,y) - p_b(x,x) - p_b(y,y), x, y \in X.$$

Definition 2.12 (see [21]) A sequence $\{x_n\}_{n=1}^{\infty}$ in a partial b-metric space (X, p_b) is said to be:

1. p_b-convergent to a point $x \in X$, written as $\lim_{n\to\infty} p_b(x, x_n) = p_b(x,x)$;.
2. a p_b-Cauchy sequence if $lim_{n,m\to+\infty} p_b(x_n, x_m)$ exists (and is finite);

Definition 2.13 (see [21]) A partial b-metric space (X, p_b) is said to be p_b-complete if every p_b-Cauchy sequence in X p_b-converges to a point $x \in X$ such that

$$p_b(x,x) = lim_{n\to\infty} p_b(x_n, x) = lim_{n\to\infty} p_b(x_n, x_m)$$

Lemma 2.14 (see [21]) A sequence $\{x_n\}_{n=1}^{\infty}$ is a p_b-Cauchy sequence in a partial b-metric space (X, p_b) if and only if it is a b-Cauchy sequence in the b-metric space (X, d_{p_b}).

Lemma 2.15 (see [21]) A partial b-metric space (X, p_b) is p_b-complete if and only if the b-metric space (X, d_{p_b}) is b-complete. Moreover, $lim_{n\to\infty} d_{p_b}(x_n, x_m) = 0$ if and only if

$$p_b(x,x) = lim_{n\to\infty} p_b(x_n, x) = lim_{n\to\infty} p_b(x_n, x_m)$$

Definition 2.16 Let (X, p_b) be a p_b-complete partial b-metric space with the coefficient $s \geq 1$. A mapping $T: X \to X$ is said to be expansive mapping if $p_b(Tx, Ty) \geq \lambda p_b(x,y)\ \ \forall\, x, y \in X$, where $\lambda > s$.

II. FIXED POINT THEOREMS

In this section, we prove some fixed point theorems satisfying expansive condition by considering surjective self-mappings in the context of partial b-metric space.

We begin with a simple but a useful Lemma.

Lemma 3.1 Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in a partial b-metric space (X, p_b) with the coefficient $s \geq 1$ such that

(3.1) $$p_b(x_n, x_{n+1}) \leq \lambda p_b(x_{n-1}, x_n)$$

where $\lambda \in \left[0, \frac{1}{s}\right)$ and $n = 1,2, \dots \dots ....$ Then $\{x_n\}_{n=1}^{\infty}$ is a p_b-Cauchy sequence in X.

Proof By the simple induction with the condition (3.1), we have

(3.2) $$p_b(x_n, x_{n+1}) \leq \lambda p_b(x_{n-1}, x_n) \leq \lambda^2 p_b(x_{n-2}, x_{n-1}) \leq \cdots \leq \lambda^n p_b(x_0, x_1)$$

On the other hand, since

$$\max\{\, p_b(x_n, x_n), p_b(x_{n+1}, x_{n+1})\} \leq p_b(x_n, x_{n+1})$$

then from (3.2), we have

(3.3) $$\max\{\, p_b(x_n, x_n), p_b(x_{n+1}, x_{n+1})\} \leq \lambda^n p_b(x_0, x_1)$$

Therefore

(3.4) $$\begin{aligned} d_{p_b}(x_n, x_{n+1}) &= 2p_b(x_n, x_{n+1}) - p_b(x_n, x_n) - p_b(x_{n+1}, x_{n+1}) \\ &\leq 2p_b(x_n, x_{n+1}) + p_b(x_n, x_n) + p_b(x_{n+1}, x_{n+1}) \\ &\leq 4\lambda^n p_b(x_0, x_1) \end{aligned}$$

This show that $\lim_{n\to+\infty} d_{p_b}(x_n, x_{n+1}) = 0$. Now we have

(3.5) $$\begin{aligned} d_{p_b}(x_n, x_{n+m}) &\leq s\left[d_{p_b}(x_n, x_{n+1}) + d_{p_b}(x_{n+1}, x_{n+m})\right] \\ &\leq s d_{p_b}(x_n, x_{n+1}) + s^2\left[d_{p_b}(x_{n+1}, x_{n+2}) + d_{p_b}(x_{n+2}, x_{n+m})\right] \\ &\leq s d_{p_b}(x_n, x_{n+1}) + s^2 d_{p_b}(x_{n+1}, x_{n+2}) + \cdots \\ &+ s^{n+m-1}\left[d_{p_b}(x_{n+m-2}, x_{n+m-1}) + d_{p_b}(x_{n+m-1}, x_{n+m})\right] \\ &\leq 4s\lambda^n p_b(x_0, x_1) + 4s^2\lambda^{n+1} p_b(x_0, x_1) + \cdots \\ &+ 4s^{n+m-1}\lambda^{n+m-2} p_b(x_0, x_1) + 4s^{n+m-1}\lambda^{n+m-1} p_b(x_0, x_1) \\ &\leq 4s\lambda^n\{1 + (s\lambda) + (s\lambda)^2 + \cdots \dots \dots .\} p_b(x_0, x_1) \\ &\leq \frac{4s\lambda^n}{1 - s\lambda} p_b(x_0, x_1). \end{aligned}$$

Note that $s\lambda < 1$. This show that $\{x_n\}_{n=1}^{\infty}$ is a b-Cauchy sequence in b-metric space (X, d_{p_b}), then from Lemma 2.14, $\{x_n\}_{n=1}^{\infty}$ is a p_b-Cauchy sequence in partial b-metric space (X, p_b).

Theorem 3.2 Let (X, p_b) be a p_b-complete partial b-metric space with the coefficient $s \geq 1$. Assume that $T: X \to X$ is surjection and satisfies

(3.6) $$p_b(Tx, Ty) \geq \lambda p_b(x, y)$$

$\forall\, x, y \in X$, where $\lambda > s$. Then T has a unique fixed point in X.

Proof; Let $x_0 \in X$, since T is surjection, then there exists $x_1 \in X$ such that $x_0 = Tx_1$. By continuing this process, we get

(3.7) $$x_n = Tx_{n+1},\ \forall\, n \in \mathbb{N} \cup \{0\}.$$

In case $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N} \cup \{0\}$,then it is clear that x_{n_0} is a fixed point of T. Without loss of generality, we assume that $x_n \neq x_{n-1}$ for all n. Consider

(3.8) $$p_b(x_{n-1}, x_n) = p_b(Tx_n, Tx_{n+1})$$

Now by (3.7) and definition of the sequence

$$(x_{n-1}, x_n) = p_b(Tx_n, Tx_{n+1}) \geq \lambda p_b(x_n, x_{n+1})$$

and so

(3.9) $$p_b(x_n, x_{n+1}) \leq \frac{1}{\lambda} p_b(x_{n-1}, x_n) = h p_b(x_{n-1}, x_n)$$

where $h = \frac{1}{\lambda} < \frac{1}{s}$. Then by Lemma.3.1, $\{x_n\}_{n=1}^{\infty}$ is a p_b-Cauchy sequence in X. Since (X, p_b) is a p_b-complete, then from Lemma 2.15, (X, d_{p_b}) is b-complete and so the sequence $\{x_n\}_{n=1}^{\infty}$ is b-converges in the b-metric space (X, d_{p_b}), that is there exists $x^\star \in X$ such that $\lim_{n\to+\infty} d_{p_b}(x_n, x^\star) = 0$.

Again from Lemma 2.15, we have

(3.10) $$p_b(x^\star, x^\star) = lim_{n\to\infty} p_b(x_n, x^\star) = lim_{n\to\infty} p_b(x_n, x_m)$$

Moreover, since $\{x_n\}_{n=1}^{\infty}$ is a b-Cauchy sequence in the b-metric space (X, d_{p_b}), $lim_{n\to\infty} d_{p_b}(x_n, x_m) = 0$,

On the other hand, since

$$\max\{ p_b(x_n, x_n), p_b(x_{n+1}, x_{n+1})\} \leq p_b(x_n, x_{n+1})$$

then by the simple induction with (3.9), we have

(3.11) $$\max\{ p_b(x_n, x_n), p_b(x_{n+1}, x_{n+1})\} \leq h^n p_b(x_0, x_1)$$

Hence, we have $lim_{n\to\infty} p_b(x_n, x_n) = 0$. Thus from the definition of d_{p_b}, we have $lim_{n\to\infty} p_b(x_n, x_m) = 0$.

Therefore, from (3.10), we have

$$p_b(x^\star, x^\star) = lim_{n\to\infty} p_b(x_n, x^\star) = lim_{n\to\infty} p_b(x_n, x_m) = 0.$$

Since T is surjection on X, there exists $p \in X$ such that $x^\star = Tp$. From (3.6), we have

(3.12) $$p_b(x_n, x^\star) = p_b(Tx_{n+1}, Tp) \geq \lambda p_b(x_{n+1}, p)$$

Taking limit as $n \to +\infty$ in the above inequality, we get $0 = p_b(x^\star, x^\star) \geq \lambda p_b(x^\star, p)$. This implies that $p_b(x^\star, p) = 0$. Also from (3.6), we have

$$0 = p_b(x^\star, x^\star) = p_b(Tp, Tp) \geq \lambda p_b(p, p)$$

and so $p_b(p, p) = 0$. Thus $p_b(x^\star, x^\star) = p_b(x^\star, p) = p_b(p, p)$ implies that $x^\star = p = Tp$. Hence $x^\star$ is a fixed point of T. Finally, assume $x^\star \neq y^\star$ is also another fixed point of T. From (3.6), we get

(3.13) $$p_b(x^\star, y^\star) = p_b(Tx^\star, Ty^\star) \geq \lambda p_b(x^\star, y^\star)$$

This is true only when $p_b(\mathrm{x}^\star, \mathrm{y}^\star) = 0$. Also $p_b(x^\star, x^\star) = 0 = p_b(y^\star, y^\star)$. So $\mathrm{x}^\star = \mathrm{y}^\star$. Hence T has a unique fixed point in X.

Corollary 3.3 Let (X, p_b) be a p_b-complete partial b-metric space with the coefficient $s \geq 1$. and $T: X \to X$ be a surjection. Suppose that there exist a positive integer n and a constant $\lambda > s$ such that

(3.14) $$p_b(T^n x, T^n y) \geq \lambda p_b(x, y)\ \forall\, x, y \in X.$$

Then T has a unique fixed point in X.

Proof From Theorem 3.2, T^n has a unique fixed point $x^\star$. But $T^n(Tx^\star) = T(T^n x^\star) = Tx^\star$. So $Tx^\star$ is also a fixed point of T^n. Hence $Tx^\star = x^\star$, $x^\star$ is a fixed point of T. Since the fixed point of T is also fixed point of T^n, the fixed point of T is unique.

Theorem 3.4 Let (X, p_b) be a p_b-complete partial b-metric space with the coefficient $s \geq 1$ and $T: X \to X$ is surjection. Suppose that $a, b, c \geq 0$ with $a + sb + c > s$ such that

(3.15) $$p_b(Tx, Ty) \geq a p_b(x, y) + b p_b(x, Tx) + c p_b(y, Ty)\ \ \forall\, x, y \in X.$$

Then T has a fixed point.

Proof: Let $x_0 \in X$. Similar to the proof of Theorem 3.2, we can obtain a sequence $\{x_n\}_{n=1}^{\infty}$ such that

(3.16) $$x_n = Tx_{n+1},\ \forall\, n \in \mathbb{N} \cup \{0\}.$$

In case $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N} \cup \{0\}$,then it is clear that x_{n_0} is a fixed point of T. Without loss of generality, we assume that $x_n \neq x_{n-1}$ for all n. Consider, (3.17) $p_b(x_{n-1}, x_n) = p_b(Tx_n, Tx_{n+1})$

Now by (3.15) and definition of the sequence

$$\begin{aligned} p_b(x_{n-1}, x_n) &= p_b(Tx_n, Tx_{n+1}) \\ &\geq ap_b(x_n, x_{n+1}) + bp_b(x_n, Tx_n) + cp_b(x_{n+1}, Tx_{n+1}) \\ &= ap_b(x_n, x_{n+1}) + bp_b(x_n, x_{n-1}) + cp_b(x_{n+1}, x_n) \end{aligned}$$

and so

$$(1-b)p_b(x_{n-1}, x_n) \geq (a+c)p_b(x_n, x_{n+1})$$

If $a + c = 0$, then $b > 1$. The above inequality implies that a negative number is greater than or equal to zero. That is impossible. So, $a + c \neq 0$ and $1 - b > 0$. Therefore,

(3.18) $p_b(x_n, x_{n+1}) \leq hp_b(x_{n-1}, x_n)$

where $h = \frac{1-b}{a+c} < \frac{1}{s}$. Then by Lemma 3.1, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Since (X, p_b) is a p_b-complete, then from Lemma 2.14, (X, d_{p_b}) is b-complete and so the sequence $\{x_n\}_{n=1}^{\infty}$ is b-converges in the b-metric space (X, d_{p_b}), that is there exists $x^\star \in X$ such that $\lim_{n\to+\infty} d_{p_b}(x_n, x^\star) = 0$. Again from Lemma 2.15, we have

(3.19) $p_b(x^\star, x^\star) = lim_{n\to\infty} p_b(x_n, x^\star) = lim_{n\to\infty} p_b(x_n, x_m)$

Moreover, since $\{x_n\}_{n=1}^{\infty}$ is a b-Cauchy sequence in the b-metric space (X, d_{p_b}), $lim_{n\to\infty} d_{p_b}(x_n, x_m) = 0$,

On the other hand, since

$$\max\{ p_b(x_n, x_n), p_b(x_{n+1}, x_{n+1})\} \leq p_b(x_n, x_{n+1})$$

then by the simple induction with (3.18), we have

(3.20) $\max\{ p_b(x_n, x_n), p_b(x_{n+1}, x_{n+1})\} \leq h^n p_b(x_0, x_1)$

Hence, we have $lim_{n\to\infty} p_b(x_n, x_n) = 0$. Thus from the definition of d_{p_b}, we have $lim_{n\to\infty} p_b(x_n, x_m) = 0$.
Therefore, from (3.19), we have

$$p_b(x^\star, x^\star) = lim_{n\to\infty} p_b(x_n, x^\star) = lim_{n\to\infty} p_b(x_n, x_m) = 0.$$

Since T is surjection on X, there exists $p \in X$ such that $x^\star = Tp$. From (3.15), we have

(3.21)
$$\begin{aligned} p_b(x_n, x^\star) &= p_b(Tx_{n+1}, Tp) \\ &\geq ap_b(x_{n+1}, p) + bp_b(x_{n+1}, Tx_{n+1}) + cp_b(p, Tp) \\ &= ap_b(x_{n+1}, p) + bp_b(x_{n+1}, x_n) + cp_b(p, x^\star) \end{aligned}$$

Taking limit as $n \to +\infty$ in the above inequality, we get $0 = p_b(x^\star, x^\star) \geq (a+c)p_b(x^\star, p)$.This implies that $p_b(x^\star, p) = 0$. Also from (3.15), we have

$$\begin{aligned} 0 = p_b(x^\star, x^\star) = p_b(Tp, Tp) &\geq ap_b(p,p) + \mathrm{b}p_b(p, Tp) + \mathrm{c}p_b(p, Tp) \\ &= ap_b(p,p) + \mathrm{b}p_b(p, x^\star) + \mathrm{c}p_b(p, x^\star) = ap_b(p,p) \end{aligned}$$

and so $p_b(p,p) = 0$. Thus $p_b(x^\star, x^\star) = p_b(x^\star, p) = p_b(p,p)$ implies that $x^\star = p = Tp$. Hence $x^\star$ is a fixed point of T. Finally, assume $x^\star \neq y^\star$ is also another fixed point of T. Then $p_b(x^\star, x^\star) = 0 = p_b(y^\star, y^\star)$. From (3.15), we get

(3.22)
$$\begin{aligned} p_b(x^\star, y^\star) &= p_b(Tx^\star, Ty^\star) \\ &\geq ap_b(x^\star, y^\star) + bp_b(x^\star, Tx^\star) + cp_b(y^\star, Ty^\star) \\ &= ap_b(x^\star, y^\star) + bp_b(x^\star, x^\star) + cp_b(y^\star, y^\star) = ap_b(x^\star, y^\star) \end{aligned}$$

This is true only when $p_b(\mathrm{x}^\star, \mathrm{y}^\star) = 0$. Also $p_b(x^\star, x^\star) = 0 = p_b(y^\star, y^\star)$. So $\mathrm{x}^\star = \mathrm{y}^\star$. Hence T has a unique fixed point in X.

Remark 3.5 Setting $b = c = 0$ and $a = \lambda$ in Theorem 3.4, we can obtain the Theorem 3.2.

Theorem 3.6 Let (X, p_b) be a p_b-complete partial b-metric space with the coefficient $s \geq 1$ and $T: X \to X$ is a continuous surjection. Suppose that there exists a constant $\lambda > s$ such that

(3.23) $p_b(Tx, Ty) \geq \lambda u$, for some $u \in \{p_b(x,y), p_b(x, Tx), p_b(y, Ty)\}\ \forall\, x, y \in X$.

Then T has a fixed point.

Proof: Similar to the proof of Theorem 3.1, we can obtain a sequence $\{x_n\}_{n=1}^{\infty}$ such that

(3.24) $x_n = Tx_{n+1},\ \forall\, n \in \mathbb{N} \cup \{0\}$.

In case $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N} \cup \{0\}$, then it is clear that x_{n_0} is a fixed point of T. Without loss of generality, we assume that $x_n \neq x_{n-1}$ for all n. Now by (3.24) and definition of the sequence

(3.25) $p_b(x_{n-1}, x_n) = p_b(Tx_n, Tx_{n+1}) \geq \lambda u_n$

where $u_n \in \{p_b(x_{n-1}, x_n), p_b(x_n, x_{n+1})\}$.

Now we have to consider the following two cases.

If $u_n = p_b(x_{n-1}, x_n)$, then

$$p_b(x_{n-1}, x_n) \geq \lambda p_b(x_{n-1}, x_n)$$

which implies that $p_b(x_{n-1}, x_n) = 0$, that is, $x_{n-1} = x_n$. This is a contradiction.

If $u_n = p_b(x_n, x_{n+1})$, then

$$p_b(x_{n-1}, x_n) \geq \lambda p_b(x_n, x_{n+1})$$

and so

$$p_b(x_n, x_{n+1}) \leq \frac{1}{\lambda} p_b(x_{n-1}, x_n)$$

where $\frac{1}{\lambda} < \frac{1}{s}$. Then by Lemma 3.1, $\{x_n\}_{n=1}^{\infty}$ is a p_b-Cauchy sequence in X. Since (X, p_b) is a p_b-complete, the sequence $\{x_n\}_{n=1}^{\infty}$ is p_b-converges to a point $x^\star \in X$. Since T is p_b-continuous, it is clear that $x^\star$ is a fixed point of T. This completes the proof.

Example 3.7 Let $X = [0, +\infty)$ and let $p_b(x, y) = [\max\{x, y\}]^2, \forall x, y \in X$. It is obvious that p_b is a partial b-metric on X with $s = 2 > 1$ and (X, p_b) is complete. Also, p_b is not a partial metric on X. Define $T: X \to X$ be

$$Tx = \begin{cases} 6x & \text{if } x \in [0,1), \\ 5x + 1 & \text{if } x \in [1,2), \\ 4x + 3 & \text{if } x \in [2, \infty). \end{cases}$$

Also, clearly T is surjection on X. Now we consider following cases.

- Let $x, y \in [0,1)$, then
$$p_b(Tx, Ty) = [\max\{6x, 6y\}]^2 = 36([\max\{x, y\}]^2) \geq 15[\max\{x, y\}]^2 = 15p_b(x, y)$$
- Let $x, y \in [1,2)$, then
$$p_b(Tx, Ty) = [\max\{(5x + 1), (5y + 1)\}]^2 > [\max\{5x. 5y\}]^2 = 25[\max\{x, y\}]^2 \geq 15[\max\{x, y\}]^2 = 15p_b(x, y)$$
- Let $x, y \in [2, \infty)$, then
$$p_b(Tx, Ty) = [\max\{(4x + 3), (4y + 3)\}]^2 > [\max\{4x. 4y\}]^2 = 16[\max\{x, y\}]^2 \geq 15[\max\{x, y\}]^2 = 15p_b(x, y)$$
- Let $x \in [0,1)$ and $y \in [1,2)$, then
$$p_b(Tx, Ty) = [\max\{6x, (5y + 1)\}]^2 > [\max\{6x. 5y\}]^2 > 25[\max\{x, y\}]^2 \geq 15[\max\{x, y\}]^2 = 15p_b(x, y)$$
- Let $x \in [0,1)$ and $y \in [2, \infty)$, then
$$p_b(Tx, Ty) = [\max\{6x, (4y + 3)\}]^2 > [\max\{6x. 4y\}]^2 > 16[\max\{x, y\}]^2 \geq 15[\max\{x, y\}]^2 = 15p_b(x, y)$$
- Let $x \in [1,2)$ and $y \in [2, \infty)$, then
$$p_b(Tx, Ty) = [\max\{(5x + 1), (4y + 3)\}]^2 > [\max\{5x. 4y\}]^2 > 16[\max\{x, y\}]^2 \geq 15[\max\{x, y\}]^2 = 15p_b(x, y)$$

That is $p_b(\mathrm{Tx}, \mathrm{Ty}) \geq \lambda p_b(\mathrm{x}, \mathrm{y}), \forall \mathrm{x}, \mathrm{y} \in \mathrm{X}$ where $\lambda = 15 > 2 = s$. The conditions of Theorem 3.2 are satisfied and T has a unique fixed point $x^\star = 0 \in X$.

III. COMMON FIXED POINT THEOREMS

In this section, we give a common fixed point theorem of two weakly compatible mappings in partial b-metric spaces. In [14] Jungck introduced the concept of commuting maps. In [15] Jungck introduced the concept of compatible mappings which generalize the concept of commuting maps. Jungck in [16] further generalized the concept of weakly compatible maps as follows.

Let S and T be two self-mappings on a nonempty set X. Then S and T are said to be weakly compatible if they commute at all of their coincidence points; that is, $Sx = Tx$ for some $x \in X$ and then $STx = TSx$.

Theorem 4.1 Let (X, p_b) be a p_b-complete partial b-metric space with the coefficient $s \geq 1$. Let S and T be two self-mappings of X and $T(X) \subseteq S(X)$. Suppose that there exists a constant $\lambda > s$ such that

(4.1) $$p_b(Sx, Sy) \geq \lambda p_b(Tx, Ty)$$

$\forall\, x, y \in X$. If one of the subspaces $T(X)$ or $S(X)$ is complete, then S and T have a unique point of coincidence in X. Moreover, if S and T are weakly compatible, then S and T have a unique common fixed point in X.Then S and T have a unique common fixed point in X.

Proof: Let $x_0 \in X$. Since $T(X) \subseteq S(X)$, choose $x_1 \in X$ such that $y_1 = Sx_1 = Tx_0$. In general, choose $x_{n+1} \in X$ such that $y_{n+1} = Sx_{n+1} = Tx_n$. Now by (4.1), we have

$$p_b(y_n, y_{n+1}) = p_b(Sx_n, Sx_{n+1}) \geq \lambda p_b(Tx_n, Tx_{n+1}) = \lambda p_b(y_{n+1}, y_{n+2})$$

and so

(4.2) $$p_b(y_{n+1}, y_{n+2}) \leq \frac{1}{\lambda} p_b(y_n, y_{n+1}) = hp_b(y_n, y_{n+1})$$

where $h = \frac{1}{\lambda} < \frac{1}{s}$. Then by Lemma 3.1, $\{x_n\}_{n=1}^{\infty}$ is a p_b-Cauchy sequence. Since $T(X) \subseteq S(X)$ and $T(X)$ or $S(X)$ is a complete subspace of X. Then from Lemma 2.15, $(S(X), d_{p_b})$ is b-complete and so the sequence $\{y_n\} = \{Tx_{n-1}\} \subseteq S(X)$ is b-converges in the b-metric space $(S(X), d_{p_b})$, that is, there exists $z^\star \in X$ such that $\lim_{n\to+\infty} d_{p_b}(y_n, z^\star) = 0$.

Consequently, we can find $u \in X$ such that $Su = z^\star$. Again from Lemma 2.15, we have

(4.3) $$p_b(Su, z^\star) = p_b(z^\star, z^\star) = lim_{n\to\infty} p_b(y_n, z^\star) = lim_{n\to\infty} p_b(y_n, y_m)$$

Moreover, since $\{y_n\}_{n=1}^{\infty}$ is a b-Cauchy sequence in the b-metric space $(S(X), d_{p_b})$, $lim_{n\to\infty} d_{p_b}(y_n, y_m) = 0$,

On the other hand, since

$$\max\{ p_b(y_n, y_n), p_b(y_{n+1}, y_{n+1})\} \le p_b(y_n, y_{n+1})$$

then by the simple induction with (3.2), we have

(4.4) $$\max\{ p_b(y_n, y_n), p_b(y_{n+1}, y_{n+1})\} \le h^n p_b(y_0, y_1)$$

Hence, we have $lim_{n\to\infty} p_b(y_n, y_n) = 0$. Thus from the definition of d_{p_b}, we have $lim_{n\to\infty} p_b(y_n, y_m) = 0$.

Therefore, from (4.3), we have

$$p_b(Su, z^\star) = p_b(z^\star, z^\star) = lim_{n\to\infty} p_b(y_n, z^\star) = lim_{n\to\infty} p_b(y_n, y_m) = 0.$$

Now to show that $Tu = z^\star$. From (4.1), we have

(4.5) $$p_b(Tu, Tx_n) \le \frac{1}{\lambda} p_b(Su, Sx_n)$$

Taking limit as $n \to +\infty$ in the above inequality, we get

$$p_b(Tu, z^\star) \le \frac{1}{\lambda} p_b(Su, z^\star) = 0,$$

This implies that $p_b(Tu, z^\star) = 0$ and so $Tu = z^\star$. Therefore, $Su = Tu = z^\star$. Since S and T be weakly compatible, $STu = TSu$, that is, $Sz^\star = Tz^\star$.

Now we show that $z^\star$ is a common fixed point of S and T. From condition (4.1)

$$p_b(Sz^\star, Sx_n) \ge \lambda p_b(Tz^\star, Tx_n)$$

Proceeding to the limit as $n \to +\infty$, we have

$$p_b(Sz^\star, z^\star) \ge \lambda p_b(Tz^\star, z^\star) = \lambda p_b(Sz^\star, z^\star),$$

which implies that $p_b(Sz^\star, z^\star) = 0$. Also $p_b(Sz^\star, Sz^\star) = 0 = p_b(z^\star, z^\star)$. Hence $Sz^\star = z^\star$ and so $Sz^\star = Tz^\star = z^\star$.

Finally, assume $z^\star \ne w^\star$ is also another common fixed point of S and T. From (4.1), we get

(4.6) $$p_b(z^\star, w^\star) = p_b(Sz^\star, Sw^\star) \ge \lambda p_b(Tz^\star, Tw^\star) = \lambda p_b(z^\star, w^\star)$$

This is true only when $p_b(z^\star, w^\star) = 0$. Also $p_b(z^\star, z^\star) = 0 = p_b(w^\star, w^\star)$. So $z^\star = w^\star$. Hence S and T have a unique fixed point in X. This completes the proof.

Theorem 4.2 Let (X, p_b) be a p_b-complete partial b-metric space with the coefficient $s \ge 1$. Let S and T be two self-mappings of X and $T(X) \subseteq S(X)$. Suppose that $a, b, c \ge 0$ with $a + sb + c > s$ such that

(4.7) $$p_b(Sx, Sy) \ge ap_b(Tx, Ty) + bp_b(Sx, Tx) + cp_b(Sy, Ty)$$

$\forall\, x, y \in X$. If one of the subspaces $T(X)$ or $S(X)$ is p_b-complete, then S and T have a point of coincidence in X. Moreover, if $a > 1$, then point of coincidence is unique. If S and T be weakly compatible and $a > 1$, then S and T have a unique common fixed point in X.

Proof: Let $x_0 \in X$. Since $T(X) \subseteq S(X)$, choose $x_1 \in X$ such that $y_1 = Sx_1 = Tx_0$. In general, choose $x_{n+1} \in X$ such that $y_{n+1} = Sx_{n+1} = Tx_n$. Now by (4.7), we have

$$\begin{aligned} p_b(y_n, y_{n+1}) &= p_b(Sx_n, Sx_{n+1}) \\ &\ge ap_b(Tx_n, Tx_{n+1}) + bp_b(Sx_n, Tx_n) + cp_b(Sx_{n+1}, Tx_{n+1} \\ &= ap_b(y_{n+1}, y_{n+2}) + bp_b(y_n, y_{n+1}) + cp_b(y_{n+1}, y_{n+2}) \end{aligned}$$

and so

$$(1 - b)p_b(y_n, y_{n+1}) \ge (a + c)p_b(y_{n+1}, y_{n+2})$$

If $a + c = 0$, then $b < 1$. The above inequality implies that a negative number is greater than or equal to zero. That is impossible. So, $a + c \ne 0$ and $1 - b > 0$. Therefore,

(4.8) $$p_b(y_{n+1}, y_{n+2}) \le hp_b(y_n, y_{n+1})$$

where $h = \frac{1-b}{a+c} < \frac{1}{s}$. Then by Lemma 3.1, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Since $T(X) \subseteq S(X)$ and $T(X)$ or $S(X)$ is a p_b-complete subspace of X. Then from Lemma 2.15, $(S(X), d_{p_b})$ is b-complete and so the sequence $\{y_n\} = \{Tx_{n-1}\} \subseteq S(X)$ is b-converges in the b-metric space $(S(X), d_{p_b})$, that is, there exists $z^\star \in X$ such that $lim_{n\to+\infty} d_{p_b}(y_n, z^\star) = 0$.

Consequently, we can find $u \in X$ such that $Su = z^\star$. Again from Lemma 2.15, we have

(4.9) $$p_b(Su, z^\star) = p_b(z^\star, z^\star) = lim_{n\to\infty} p_b(y_n, z^\star) = lim_{n\to\infty} p_b(y_n, y_m)$$

Moreover, since $\{y_n\}_{n=1}^{\infty}$ is a b-Cauchy sequence in the b-metric space $(S(X), d_{p_b})$, $lim_{n\to\infty} d_{p_b}(y_n, y_m) = 0$,

On the other hand, since

$$max\{ p_b(y_n, y_n), p_b(y_{n+1}, y_{n+1})\} \le p_b(y_n, y_{n+1})$$

then by the simple induction with (4.8), we have

(4.10) $$max\{ p_b(y_n, y_n), p_b(y_{n+1}, y_{n+1})\} \le h^n p_b(y_0, y_1)$$

Hence, we have

$$lim_{n\to\infty} p_b(y_n, y_n) = 0.$$

Thus from the definition of d_{p_b}, we have

$$lim_{n\to\infty} p_b(y_n, y_m) = 0.$$

Therefore, from (4.9), we have

$$p_b(Su, z^\star) = p_b(z^\star, z^\star) = lim_{n\to\infty} p_b(y_n, z^\star) = lim_{n\to\infty} p_b(y_n, y_m) = 0.$$

Now to show that $Tu = z^\star$. From (4.7), we have

(4.11) $$p_b(Su, Sx_n) \ge ap_b(Tu, Tx_n) + bp_b(Su, Tu) + cp_b(Sx_n, Tx_n)$$

Taking limit as $n \to +\infty$ in the above inequality, we get

$$0 = \mathrm{p_b}(\mathrm{Su}, \mathrm{z}^\star) \ge \mathrm{ap_b}(\mathrm{Tu}, \mathrm{z}^\star) + \mathrm{bp_b}(\mathrm{z}^\star, \mathrm{Tu})$$
$$= (a + b)p_b(Tu, z^\star)$$

This implies that $p_b(Tu, z^\star) = 0$ and so $Tu = z^\star$. Therefore, $Su = Tu = z^\star$. Therefore, $z^\star$ is a point of coincidence of S and T.

Now we suppose that $a > 1$. Let $w^\star$ be another point of coincidence of S and T. So $Sv = Tv = w^\star$ for some $v \in X$. Then from (4.7), we have

$$\mathrm{p_b}(\mathrm{z}^\star, \mathrm{w}^\star) = \mathrm{p_b}(\mathrm{Su}, \mathrm{Sv})$$
$$\ge \mathrm{ap_b}(\mathrm{Tu}, \mathrm{Tv}) + \mathrm{bp_b}(\mathrm{Su}, \mathrm{Tu}) + \mathrm{cp_b}(\mathrm{Sv}, \mathrm{Tv})$$
$$= \mathrm{ap_b}(\mathrm{z}^\star, \mathrm{w}^\star)$$

This is true only when $p_b(z^\star, w^\star) = 0$. Also $p_b(z^\star, z^\star) = 0 = p_b(w^\star, w^\star)$. So $z^\star = w^\star$.

Since S and T be weakly compatible, $STu = TSu$, that is, $Sz^\star = Tz^\star$. Now we show that $z^\star$ is a common fixed point of S and T. If $a > 1$, then from condition (4.7), we have

$$p_b(Sz^\star, Sx_n) \ge ap_b(Tz^\star, Tx_n) + bp_b(Sz^\star, Tz^\star) + cp_b(Sx_n, Tx_n)$$

Proceeding to the limit as $n \to +\infty$, we have

$$p_b(Sz^\star, z^\star) \ge ap_b(Tz^\star, z^\star) = ap_b(Sz^\star, z^\star),$$

which implies that $p_b(Sz^\star, z^\star) = 0$.

Also $p_b(Sz^\star, Sz^\star) = 0 = p_b(z^\star, z^\star)$. Hence $Sz^\star = z^\star$ and so $Sz^\star = Tz^\star = z^\star$. Hence S and T have a unique fixed point in X. This completes the proof.

Remark 4.3

1. If we take $s = 1$ in Theorem 3.2, then we get Corollary 2.1 of Huang et al. [26].
2. If we take $s = 1$ in Corollary 3.3, then we get Corollary 2.2 of Huang et al. [26].
3. If we take $s = 1$ in Theorem 3.4, then we get Theorem 2.1 of Huang et al. [26].
4. If we take $s = 1$ in Theorem 3.6, then we get Theorem 2.2 of Huang et al. [26].
5. If we take $s = 1, b = c = 0$ and $a = \lambda$ in Theorem 3.4, then we get Corollary 2.1 of Huang et al. [26].
6. If we take $s = 1,\ b = c = 0$ and $a = \lambda$ in Theorem 4.2, then we get Theorem 2.3 of Huang et al. [26].
7. If we take $b = c = 0$ and $a = \lambda$ in Theorem 4.2, then we get Theorem 4.1.
8. If we take $s = 1, S = T, T = I$ in Theorem 4.2, then we get Theorem 2.1 of Huang et al. [26].
9. If we take $S = T, T = I$ in Theorem 4.2, then we get Theorem 3.4.

Now, we prove the following common fixed point theorem, which is generalization of Theorem 2.2 of Shatanawi et al. [22] in the setting of partial b-metric space.

Theorem 4.4 Let $T, S: X \to X$ be two surjective mappings of a $\mathrm{p_b}$-complete partial b-metric space $(\mathrm{X}, \mathrm{p_b})$ with the coefficient $s \ge 1$. Suppose that T and S satisfying inequalities

(4.12) $$p_b(T(Sx), Sx) + kp_b(T(Sx), x) \ge ap_b(Sx, x)$$

(4.13) $$p_b(S(Tx), Tx) + kp_b(S(Tx), x) \ge bp_b(Tx, x)$$

for $x \in X$ and some nonnegative real numbers a, b and k with $a > s(1 + k) + s^2k$ and $b > s(1 + k) + s^2k$.. If T or S is p_b-continuous, then T and S have a common fixed point in X.

Proof Let x_0 be an arbitrary point in X. Since T is surjective, there exists $x_1 \in X$ such that $x_0 = Tx_1$. Also, since S is surjective, there exists $x_2 \in X$ such that $x_2 = Sx_1$. Continuing this process, we construct a sequence $\{x_n\}$ in X such that

(4.14) $$x_{2n} = Tx_{2n+1} \text{ and } x_{2n+1} = Sx_{2n+2}$$

for all $n \in N \cup \{0\}$, Now for $n \in \mathbb{N} \cup \{0\}$, by (4.12) we have

$$p_b(T(Sx_{2n+2}), Sx_{2n+2}) + kp_b(T(Sx_{2n+2}), x_{2n+2}) \ge ap_b(Sx_{2n+2}, x_{2n+2})$$

Thus, we have

$$p_b(x_{2n},x_{2n+1}) + kp_b(x_{2n},x_{2n+2}) \geq ap_b(x_{2n+1},x_{2n+2})$$

which implies that

$$p_b(x_{2n},x_{2n+1}) + sk[p_b(x_{2n},x_{2n+1}) + p_b(x_{2n+1},x_{2n+2}) - p_b(x_{2n+1},x_{2n+1})] \geq ap_b(x_{2n+1},x_{2n+2})$$

That is,

$$p_b(x_{2n},x_{2n+1}) + sk[p_b(x_{2n},x_{2n+1}) + p_b(x_{2n+1},x_{2n+2})] \geq ap_b(x_{2n+1},x_{2n+2})$$

Hence

(4.15) $\quad p_b(x_{2n+1},x_{2n+2}) \leq \frac{1+sk}{a-sk}\, p_b(x_{2n},x_{2n+1})$

On other hand, we have (from (4.13))

$$p_b(S(Tx_{2n+1}),Tx_{2n+1}) + kp_b(S(Tx_{2n+1}),x_{2n+1}) \geq bp_b(Tx_{2n+1},x_{2n+1})$$

Thus we have

$$p_b(x_{2n-1},x_{2n}) + kp_b(x_{2n-1},x_{2n+1}) \geq bp_b(x_{2n},x_{2n+1})$$

which implies that

$$p_b(x_{2n-1},x_{2n}) + sk[p_b(x_{2n-1},x_{2n}) + p_b(x_{2n},x_{2n+1}) - p_b(x_{2n},x_{2n})] \geq \mathrm{bp_b}(\mathrm{x_{2n}},\mathrm{x_{2n+1}})$$

That is,

$$p_b(x_{2n-1},x_{2n}) + sk[p_b(x_{2n-1},x_{2n}) + p_b(x_{2n},x_{2n+1})] \geq bp_b(x_{2n},x_{2n+1})$$

Hence

(4.16) $\quad p_b(x_{2n-1},x_{2n}) \leq \frac{1+sk}{b-sk}\, p_b(x_{2n-1},x_{2n})$

Let $h = max\left\{\frac{1+sk}{a-sk},\frac{1+sk}{b-sk}\right\} < \frac{1}{s}$

Then by combining (4.15) and (4.16), we have

(4.17) $\quad p_b(x_n,x_{n+1}) \leq h\, p_b(x_{n-1},x_n)$

where $h \in \left[0,\frac{1}{s}\right)$, $\forall\, n \in \mathbb{N} \cup \{0\}$. Then by Lemma 3.1, $\{x_n\}_{n=1}^{\infty}$ is p_b-Cauchy sequence in the p_b-complete partial b-metric space. Then there exists $x^\star \in X$ such that $x_n \to x^\star$ as $n \to +\infty$. Therefore $x_{2n+1} \to x^\star$ and $x_{2n+2} \to v$ as $n \to +\infty$. Without loss of generality, we may assume that T is p_b-continuous, then $Tx_{2n+1} \to Tx^\star$as $n \to +\infty$. But $Tx_{2n+1} = x_{2n} \to x^\star$ as $n \to +\infty$.Thus, we have $Tx^\star = x^\star$. since S is surjective, there exists $p \in X$ such that $Sp = x^\star$.

Now

$$p_b(T(Sp),Sp) + kp_b(T(Sp),p) \geq ap_b(Sp,p)$$

implies that $kp_b(x^\star,p) \geq ap_b(x^\star,p)$

Then $p_b(x^\star,p) \leq \frac{k}{a}p_b(x^\star,p)$. Since $a > k$, we conclude that $p_b(x^\star,p) = 0$. so $x^\star = p$. Hence $Tx^\star = Sx^\star = x^\star$**.** Therefore $x^\star$ is a common fixed point of T and S.

By taking $b = a$ in theorem 4.4, we have the following result.

Corollary 4.5 Let $T,S: X \to X$ be two surjective mappings of a p_b-complete partial b-metric space (X,p_b) with the coefficient $s \geq 1$. Suppose that T and S satisfying inequalities

(4.18) $\quad p_b(T(Sx),Sx) + kp_b(T(Sx),x) \geq ap_b(Sx,x)$

(4.19) $\quad p_b(S(Tx),Tx) + kp_b(S(Tx),x) \geq ap_b(Tx,x)$

for $x \in X$ and some nonnegative real numbers a and k with $a > s(1+k) + s^2k$. If T or S is continuous, then T and S have a common fixed point in X.

By taking $S = T$ in Corollary 4.5, we have the following Corollary.

Corollary 4.6 Let $T: X \to X$ be a surjective mappings of a p_b-complete partial b-metric space (X,p_b) with the coefficient $s \geq 1$. Suppose that T satisfying inequality

(4.20) $\quad p_b(T(Tx),Tx) + kp_b(T(Tx),x) \geq ap_b(Tx,x)$

for $x \in X$ and some nonnegative real numbers a and k with $a > s(1+k) + s^2k$. If T is continuous, then T has a fixed point in X.

Now, we present an example to illustrate the usability of Corollary 4.6.

Example 4.7 Let $X = [0,+\infty)$ and define $p_b: X \times X \to \mathbb{R}^+$ by

$$d(x,y) = p_b(x,y) = [max\{x,y\}]^2 + |x-y|^2, \forall\, x,y \in X.$$

Then $(\mathrm{X},\mathrm{p_b})$ is a complete partial b-metric space with $\mathrm{s} = 2$. Define $\mathrm{T}: \mathrm{X} \to \mathrm{X}$ by $\mathrm{T(x)} = 2\mathrm{x}$. Then T has a fixed point.

Proof Note that

$$\begin{aligned} p_b(T(Tx),Tx)\ + p_b(T(Tx),x) &= p_b(4x,2x) + p_b(4x,x) \\ &= [max\{4x,2x\}]^2 + |4x-2x|^2 + [max\{4x,x\}]^2 + |4x-x|^2 \end{aligned}$$

$$= 16x^2 + 4x^2 + 16x^2 + 9x^2 = 45x^2$$
$$> 40x^2 = 8(4x^2 + x^2)$$
$$= 8\{[max\{2x, x\}]^2 + |2x - x|^2\}$$
$$= 8p_b(Tx, x)$$

for all $x \in X$. Here $k = 1$ and $a = 8$. Clearly $8 = a > s(1 + k) + sk^2 = 2(1 + 1) + 2(1)^2 = 6$. Also T is surjection on X. Thus T satisfies all the hypotheses of Corollary 4.5 and hence T has a fixed point. Here $0 \in X$ is the fixed point of T.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

AUTHOR'S CONTRIBUTIONS

All authors contributed equally and significantly to writing this paper. All authors read and approved the final manuscript.

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