Quest Journals
Journal of Research in Applied Mathematics
Volume 2~ Issue 9 (2016) pp: 01-06
ISSN(Online) : 2394-0743 ISSN (Print):2394-0735
www.questjournals.org

Research Paper

Minimum Absolute Deviation Method of Estimation of Cobb-Douglas Frontier Production Function

K. Muthyalappa[1], K. Pushpanjali[2], Dr. B.R. Sreedhar[3]
[1]*Research Scholar, Department Of Statistics, Sri Krishnadevaraya University, Ananthapuramu, Andhrapradesh, India.*
[2]*professor, Department Of Statistics, Sri Krishnadevaraya University, Ananthapuramu, Andhrapradesh, India.*
[3]*Assit. Professor, CBIT, Gandipet, Hyderabad, Telangana State, India.*

Received 25 May, 2016; Accepted 13 June, 2016 © The author(s) 2015. Published with open access at**www.questjournals.org**

ABSTRACT:- *This paper contains a number of to measure technical efficiency of Decision Making Units (DMU's)***.** *This approach engages the linear programming technique (L.P.P) with parametric and non-parametric production frontiers in easy way. The parametric estimates cannot be subjected to significance tests due to the non-obtainability of standard errors (S.E's). In this Paper we proposed MAD (Minimum Absolute Deviation) method of estimation of Cobb-Douglas frontier production function as a linear programming problem (L.P.P).This method can be stretched in easy way to any parametric frontier production or cost function which is linear in parameters.*

Keywords:- *Decision Making Units, Minimum Absolute Deviation,*

I. INTRODUTION

Efficiency is critical for organizations that seek to be both environmentally conscious and profitable. Efficiency has implications for a "win-win" situation to arise. Studying and managing organizations from this perspective requires an evaluation of efficiency. To aid researchers and managers develop measures for efficiency we review the use of data envelopment analysis (DEA) for this purpose. DEA theory and application has increased greatly. Its use as a tool for environmental performance evaluation has been limited. In this paper we provide MAD (Minimum Absolute Deviation) method of estimation of Cobb-Douglas frontier production function as a linear programming problem (L.P.P).

II. INPUT LEVEL SETS

$\mathrm{L}(\mathrm{u}) = \{\mathrm{x} : \mathrm{x} \text{ produces } \mathrm{u}\}$, Where x, u, are input and output vectors respectively. The input level set $\mathrm{L}(\mathrm{u})$ satisfies the following properties.

1. $\mathrm{L}(0) = \mathrm{R}_+^n, \quad 0 \notin \mathrm{L}(\mathrm{u}) \text{ for } \mathrm{u} > 0$
2. $\mathrm{x} \in \mathrm{L}(\mathrm{u}), \quad \mathrm{x}' \geq \mathrm{x} \Rightarrow \mathrm{x}' \in \mathrm{L}(\mathrm{u})$
3. $\mathrm{u}_2 \geq \mathrm{u}_1 \geq 0 \Rightarrow \mathrm{L}(\mathrm{u}_2) \subseteq \mathrm{L}(\mathrm{u}_1)$

III. FRONTIER PRODUCTION FUNCTION

Let $\phi(x), x \in R_+^n$ be a frontier production function. As an optimization problem $\phi(x)$ may be expressed as,

$$\phi(\mathrm{x}) = \mathrm{Max}\{\mathrm{u} : \mathrm{x} \in \mathrm{L}(\mathrm{u})\}, \qquad 0 \leq \mathrm{u} < \infty$$

$\phi(x)$ succeed properties from $\mathrm{L}(\mathrm{u})$

(i) $\phi(0) = 0$, Maximum output produced by a null input vector is zero.

$$\phi(0) = Max\{u : o \in L(u)\} \Rightarrow 0 \in L(u) \Rightarrow u = 0 \quad \Rightarrow \phi(0) = 0$$

*Corresponding Author: K. Muthyalappa[1]
[1]*Research Scholar, Department Of Statistics, Sri Krishnadevaraya University, Ananthapuramu, Andhrapradesh, India.*

(ii) $x' > x \Rightarrow \phi(x') \geq \phi(x)$, Maximum output produced by a larger input vector is larger.

(iii) $\phi(x)$ is concave function of x

IV. INPUT LEVEL SETS INDUCED BY A PRODUCTION FRONTIER $\phi(x)$:

$$\mathrm{L}_{\phi}(\mathrm{u}) = \{\mathrm{x}:\mathrm{D}(\mathrm{u},\mathrm{x}) \geq 1\} \text{ Where } D(u,x) = \left[Min\{\lambda : \lambda x \in L(u)\}\right]^{-1} = \frac{\phi(\mathrm{x})}{\mathrm{u}}$$

It can be written $L_{\phi}(u) = L(u) = \left\{x : \frac{\phi(\mathrm{x})}{\mathrm{u}} \geq 1\right\} = \{x : \phi(\mathrm{x}) \geq u\}$

V. THE COBB-DOUGLAS PRODUCTION FRONTIER:

The Cobb-Douglas production frontier is given by,

$\hat{y}_i = A\prod_{j=1}^{n} x_{ij}^{\alpha j}$,It is the i[th] decision making unit

Taking logarithms on both sides,

$$\Rightarrow \ln \hat{y}_i = \ln A + \sum_{j=1}^{n} \alpha_j X_{ij} \quad \Rightarrow \hat{Y}_i = a + \sum_{j=1}^{n} X_{ij}\alpha_j$$

If $\hat{Y}_i \geq Y_i$ then $a + \sum_{j=1}^{n} X_{ij}\alpha_i \geq Y_i$ ….(1.1)

If there are k decision making units(DMUs), then i=1,2,……….,k, Introducing slack variables s_i , The inequation is converted equation.

$$\Rightarrow a + \sum_{j=1}^{n} X_{ij}\alpha_j - s_i = Y_i \Rightarrow \left[a + \sum_{j=1}^{n} X_{ij}\alpha_j - Y_i\right] = s_i$$

Taking summation on both sides

$$\Rightarrow \sum_{i=1}^{k} s_i = ka + \sum_{i=1}^{k}\sum_{j=1}^{n} X_{ij}\alpha_j - \sum_{i=1}^{k} Y_i \quad \text{….(1.2)}$$

By dividing this equation by k

$$\Rightarrow \bar{s} = a + \sum_{j=1}^{n} \bar{X}_{.j}\alpha_j - \bar{Y} \quad \text{….(1.3)}$$

Minimization of (1.2) is same as minimization of (1.3), $\bar{Y}$ being a constant, minimization of (1.2) is same as minimization of ,

$$\Rightarrow a + \sum_{j=1}^{n} \bar{X}_{.j}\alpha_j \quad \text{....(1.4)}$$

Combining (1.1) and (1.4) we obtain a linear programming problem (L.P.P) for which decision variables are a and α_j .

$$\begin{aligned} &Min \quad a + \sum_{j=1}^{n} \bar{X}_{.j}\alpha_j \\ &subject\ to \qquad \text{... (1.5)} \\ &\quad a + \sum_{j=1}^{n} X_{ij}\alpha_j \geq Y_i \\ &\quad \alpha_j \geq 0, is\ conditional\ for\ sign. \end{aligned}$$

Let $a = a^{+} - a^{-}$, the L.P.P can be expressed as follows:

$$\text{Minimize} \quad Z = a^{+} - a^{-} + \sum_{j=1}^{n} \overline{X}.j\alpha_j$$

subject to $a^{+} - a^{-} + \sum_{j=1}^{n} X_{ij}\alpha_j \geq Y_i$ ...(1.6)

$$a^{+}, a^{-}, \alpha_j \geq 0$$
$$i = 1,2,3,.........k$$

VI. MAD METHOD OF ESTIMATION OF COBB-DOUGLAS PRODUCTION FRONTIER:

With two errors u and v, one sided and the two sided disturbance terms, then the model is given by

$$a^{+} - a^{-} + \sum_{j=1}^{n} X_{ij}\alpha_j = Y_i + u_i + v_i, \ \ where \ \ 0 \leq u_i < \infty, \quad -\infty < v_i < \infty$$

For i^{th} Decision making unit(DMU)

$$\Rightarrow a^{+} - a^{-} + \sum_{j=1}^{n} X_{ij}\alpha_j - Y_i - u_i = v_i$$

Taking Modulus on both sides

$$\Rightarrow \left| a^{+} - a^{-} + \sum_{j=1}^{n} X_{ij}\alpha_j - Y_i - u_i \right| = |v_i| \ \Rightarrow a^{+} - a^{-} + \sum_{j=1}^{n} X_{ij}\alpha_j - Y_i - u_i = v_i^{+} + v_i^{-}$$

where $v_i = v_i^{+} + v_i^{-}$, $\mathrm{v}_i^{+} = \mathrm{Max}\{0, \mathrm{v}_i\}$, $v_i^{-} = -Min\{0, v_i\}$

The optimization problem is equal to MAD estimation model and it is given by

$$\text{Min} \quad \sum_{i=1}^{k} \left(v_i^{+} + v_i^{-} \right)$$

subject to

$$a^{+} - a^{-} + \sum_{j=1}^{n} X_{ij}\alpha_j - u_i - v_i^{+} - v_i^{-} = Y_i \qquad \ldots(1.7)$$

$$a^{+}, a^{-}, \alpha_j, u_i, v_i^{+}, and \quad v_i^{-} \geq 0$$

i=1,2,3,……….m, j=1,2,3,……….n

The decision variables of the above Linear Programming are $A, \alpha_j, u_i, v_i^{+} and \quad v_i^{-}$ The optimal solution of L.P.P (1.7) tells DMU specific technical efficiency.

VII. EFFICIENCY ESTIMATION IN COBB-DOUGLAS PRODUCTION FUNCTION USING MODIFIED LEAST SQUARES

Consider the Cobb-Douglas production function,

$$y = A\prod_{i=1}^{m} x_i^{\alpha_i} u \quad \text{where} \quad 0 \leq u \leq 1 \qquad ...(1.8)$$

Define $\mathrm{u} = \mathrm{e}^{-z}; \qquad 0 < \mathrm{z} < \infty$

Let the random variable Z follows Gamma distribution, so that,

$$f(z,\lambda) = \frac{1}{\Gamma(\lambda)} z^{\lambda-1} \exp(-z), \text{ where } \Gamma(\lambda) = \int_0^{\infty} z^{\lambda-1} e^{-z} dz$$

$$\ln u = -z \quad \Rightarrow -\ln u = z \quad \Rightarrow dz = -\frac{du}{u} \qquad \Rightarrow z = \ln\left(\frac{1}{u}\right)$$

$$z = 0 \Rightarrow u = 1, z = \infty \Rightarrow u = 0$$

$$= \frac{1}{\Gamma(\lambda)}\left(\ln\frac{1}{u}\right)^{\lambda-1} du$$

The probability density function of u is given by,

$$g(u,\lambda) = \frac{1}{\Gamma(\lambda)}\left(\ln\frac{1}{u}\right)^{\lambda-1} \qquad \ldots (1.9)$$

Here

1. λ is shape parameter of the distribution, $g(u.\lambda)$
2. $\lambda < 1$ implies that a greater proportion of DMUs are efficient
3. $\lambda = 1$ implies uniform efficiency
4. $\lambda > 1$ implies that a greater proportion of DMUs are inefficient

The average level of efficiency of the industry comprised of several DMU is,

$$\bar{u} = E(u) = \int_0^\infty \exp(-z)\frac{1}{\Gamma(\lambda)} z^{\lambda-1}\exp(-z)\,dz$$

$$= \frac{1}{\Gamma(\lambda)}\int_0^\infty \exp(-2z)\,z^{\lambda-1}dz \qquad [put\; 2z = v,\; 2dz = dv\;\; dz = \frac{1}{2}dv]$$

$$= \frac{1}{\Gamma(\lambda)}\int_0^\infty \exp(-v)\left(\frac{v}{2}\right)^{\lambda-1} 2^{-1}dv$$

$$= \frac{2^{-\lambda}}{\Gamma(\lambda)}\int_0^\infty \exp(-v)(v)^{\lambda-1}\,dv$$

$$= \frac{2^{-\lambda}}{\Gamma(\lambda)}\Gamma(\lambda)$$

$$\Rightarrow \bar{u} = 2^{-\lambda}$$

VIII. THE METHOD OF MODIFIED LEAST SQUARES

Consider the Cobb-Douglas production function specification

$$y_i = A\prod_{j=1}^{m} x_{ij}^{\beta j}\, u_i, \qquad i=1,2,3,\ldots\ldots k$$

$$\ln y_i = \ln A + \sum_{j=1}^{m}\beta_j \ln x_{ij} + \ln u_i$$

$$\Rightarrow Y_i = a + \sum_{j=1}^{m}\beta_j X_{ij} - z_i \qquad \ldots. (1.10)$$

We have,

$$E(z_i) = \frac{1}{\Gamma(\lambda)}\int_0^\infty z_i z_i^{\lambda-1}\exp(-z_i)\,dz_i$$

$$= \frac{1}{\Gamma(\lambda)}\int_0^\infty z_i^{\lambda+1-1}\exp(-z_i)\,dz_i$$

$$=\frac{1}{\Gamma(\lambda)}\Gamma(\lambda+1)$$

$$=\frac{1}{\Gamma(\lambda)}\Gamma(\lambda+1)$$

$$\mathrm{E}(\mathrm{z_i})=\lambda, \qquad \forall\lambda$$

$$E\left(z_i^{\,2}\right)=\frac{1}{\Gamma(\lambda)}\int_0^\infty z_i^{\lambda+2-1}\exp(-z_i)\,dz_i$$

$$=\frac{\Gamma(\lambda+2)}{\Gamma(\lambda)}=\frac{(\lambda+1)\lambda\Gamma(\lambda)}{\Gamma(\lambda)}$$

$$=\lambda(\lambda+1)$$

$$V(z_i)=E\left(z_i^2\right)-\left[E(z_i)\right]^2=\lambda(\lambda+1)-\lambda^2$$

$$\mathrm{V}(\mathrm{z_i})=\lambda, \ \forall \mathrm{i}$$

We shall assume that Cov $(z_j, z_l)=0 \qquad j\neq l$

Define $\alpha_0=a-\lambda, \quad v_i=\lambda-z_i$ …. (1.11)

Combine (1.10) and (1.11) to obtain,

$$\hat{y}_i=\alpha_0+\sum_{j=1}^{m}\alpha_j X_{ij}+v_i$$

where $Y_i=\ln y_i, \quad X_{ij}=\ln x_{ij}$

Let v_t be an disturbance term that satisfies the following properties:

1. $\mathrm{E}(\mathrm{v_i})=0, \qquad \forall \mathrm{i}$
2. $V(v_i)=E\left(v_i^2\right)=\lambda$
3. $\mathrm{Cov}(\mathrm{v_i},\mathrm{v_j})=0, \qquad \mathrm{i}\neq \mathrm{j}$
4. $E\left(v/X\right)=0$

Under above conditions the OLS estimators are BLUEs of $\alpha_0, \alpha_1, \ldots\ldots.\alpha_{n.}$. Since $\hat{\lambda}$ is variance of v_i, the OLS estimator of λ is,

$$\hat{\lambda}=\frac{\sum_{i=1}^{k}\left(Y_i-\hat{\alpha}_0-\sum_{j=1}^{m}\hat{\alpha}_j X_{ij}\right)}{k-m-1} \qquad \text{…. (1.12)}$$

$$E(\hat{\alpha}_0)=\alpha_0=a-\lambda \qquad \text{…. (1.13)}$$

$$E(\hat{\alpha}_i)=\alpha_i, i=1,2,3\ldots\ldots$$

$$E\left(\hat{\lambda}\right)=\lambda$$

$$E(\hat{\alpha}_0)=a-E\left(\hat{\lambda}\right)$$

$$E\left(\hat{\alpha}_0+\hat{\lambda}\right)=a \qquad \text{…. (1.14)}$$

$\hat{\alpha}_0 + \hat{\lambda}$ is an unbiased estimator of a, $2^{-\hat{\lambda}}$ is a consistent but upward biased estimator of average technical efficiency.

$$\hat{\bar{u}} = 2^{-\hat{\lambda}} \qquad \text{…. (1.15)}$$

In Gamma distribution, we can estimate the proportion of DMUs with efficiency level at least equal to α

$$P[u \geq d] = P[e^{-z} \geq d] \qquad \text{…. (1.16)}$$

$$= P[z \leq -\ln d]$$

$$= \int_0^{-\ln d} \frac{1}{\Gamma(\lambda)} z^{\lambda-1} \exp(-z) dz \qquad \text{…. (1.17)}$$

IX. CONCLUSION

In this study we proposed MAD (Minimum Absolute Deviation) method of Cobb-Douglas frontier production function as a linear programming problem (L.P.P). In this research study Cobb-Douglas production frontier is proposed under certain assumptions that the two sided disturbance term follows normal distribution and one sided disturbance term follows truncated normal distribution with certain assumptions. The production frontier is estimated employing the maximum likelihood method of estimation. Finally Stochastic efficiency is estimated for all the DMUs.

REFERENCES

[1]. Andersen P, Petersen NC (1993) A procedure for ranking efficient units in data envelopment analysis. Management Science 39: 1261-1264.
[2]. Banker, R. D., Charnes, A., & Cooper, W. W. (1984). Some models for estimation of technical and scale efficiencies in data envelopment analysis. *Management Science*, 30(9), 1078-1092.
[3]. Ball VE, Lovell CAK, Nehring RF, Somwaru A. (1994) Incorporating undesirable outputs into models of production: An application to U.S. agriculture. Cahiers d'Economique et Sociologie Rurales 31: 59-73.
[4]. Brockett PL, Golany B., (1996) Using rank statistics for determining programmatic efficiency differences in data envelopment analysis. Management Science 42: 466-472.
[5]. Charnes A, Cooper WW, Rhodes E (1978) Measuring the efficiency of decision making units. European Journal of Operational Research 2: 429-444.
[6]. Charnes A, Cooper WW, Huang ZM, Sun, DB (1990) Polyhedral cone-ratio DEA models with an illustrative application to large commercial banks. Journal of Econometrics 30: 91-107.
[7]. Conover WJ (1980) Practical nonparametric statistics. John Wiley & Sons, New York.
[8]. Cook WD, Kress M, Seiford, LM (1996) Data envelopment analysis in the presence of both quantitative and qualitative factors. Journal of the Operational Research Society 47: 945-953.
[9]. Corson WH (1994) Changing course : an outline of strategies for a sustainable future. Futures 26: 206-223.
[10]. Criswell DR, Thompson RG (1996) Data envelopment analysis of space and terrestrially-based large-scale commercial power systems for earth. Solar Energy 56: 119-131.
[11]. Doyle J and Green R (1994) Efficiency and cross-efficiency in DEA: derivations, meanings and uses. Journal of the Operational Research Society 45: 567-578.
[12]. Fare R, Grosskopf S, Tyteca D. (1996) An activity analysis model of the environmental performance of firms - application to fossil fuelled-fired electric utilities. Ecological Economics and Statistics 18: 90-98.
[13]. Grosskopf S, Valdmanis V (1987) Measuring hospital performance: a non-parametric approach. Journal of Health Economics 6: 89-107.
[14]. Kao C, Chang PL, Hwang SN (1993) Data envelopment analysis in measuring the efficiency of forest management. Journal Of Environmental Management 38: 73-83.
[15]. Karkazis J, Boffey B (1997) Spatial organization of an industrial area: distribution of the environmental cost and equity policies. European Journal of Operational Research 101: 430-441.
[16]. Keeney R, Raiffa H (1976) Decisions with multiple objectives. Wiley, New York.
[17]. Khouja M (1995) The use of data envelopment analysis for technology selection. Computers and Industrial Engineering 28: 123-132.
[18]. Poit-Lepetit I, Vermersch D, Weaver RD, (1997) Agriculture's environmental externalities: DEA evidence for French agriculture. Applied Economics 29: 331-338.
[19]. Rousseau JJ, Semple JH (1995a) Radii of classification preservation in data envelopment analysis: A case study of 'program follow-through. Journal of the Operational Research Society 46: 943-957.
[20]. Rousseau JJ Semple JH (1995b) Two-person ratio efficiency games. Management Science 41: 435-441.
[21]. Sarkis J (1997) Evaluating flexible manufacturing systems alternatives using data envelopment analysis. Engineering Economist 43: 25-48.