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Research Paper

Characterization of ProjectiveEquivalence Between two Important Subclasses of(α, β)-Metrics

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ABSTRACT: *In the present article, we devoted to characterize the necessary and sufficient condition to characterize the projective relation between two subclasses of* (α, β)*-metrics* $F = \alpha + \beta + \frac{2\beta^2}{\alpha} - \frac{\beta^4}{3\alpha^3}$ *and Kropina metric on ann-dimensional manifold with dim* $n \geq 3$, *where* α *and* $\bar{\alpha}$ *are two Riemannianmetrics,* β *and* $\bar{\beta}$*are two non-zero 1-forms.*

Keywords:*Finsler space,* $(\boldsymbol{\alpha}, \boldsymbol{\beta})$*-metric, Douglas tensors, Kropina metric, Projective change.*
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I. INTRODUCTION

Projective differential geometry provides the most basic application of what has come tobe known as the Bernstein-Gelfand machinery. As such, it is completely parallel to conformaldifferential geometry. On the other hand, there are direct applications within Riemannian differential geometry.

In Finsler geometry, a change of $F \to \bar{F}$ of a Finsler metric on a same under lying manifold M is called projective change if any geodesic in (M, F) remains to be a geodesic in $(M, \bar{F})$ and vice versa. We say that a Finsler metric is projectively related to another Finsler metric if they have the same geodesic as point sets. An interesting result concerned with the theory of projective change was given by Rapscak's paper [12]. He proved the necessary and sufficient condition for projective change between Finsler spaces with (α, β)-metric.

By considering the (α, β)-metric $F = \frac{\alpha^2}{\beta}$is calledKropina metric. Kropina metric was first introduced by L.Berwald in connection with two dimensional Finsler space with rectilinear extremal and was investigated by V.K. Kropina [7]. They together with Randers metric are C-reducible [10]. However, Randers metric are regular Finsler metric but Kropina metric is non-regular Finsler metric. Kropina metric seem to be among the simplest nontrivial Finsler metric with many interesting applications in physics, electron optics with a magnetic field, dissipative mechanics and irreversible thermodynamics [5], [13]. Also, there are interesting applications in relativistic field theory, evolution and developmental biology. According to [13], if two (α, β)-metrics$F = \alpha\phi(s)$and $\bar{F} = \bar{\alpha}\phi(\bar{s})$are the same anisotropy directions (or, they have the same axis rotation to their indicatrices),then their 1-form β and $\bar{\beta}$ are collinear, there is a function $\mu \in C^\infty(M)$ such that $\beta(x, y) = \mu\bar{\beta}(x, y)$.The theory of projective change between two Finsler space have been treated by many authors ([1], [3], [4], [6], [11], [14], [15], [16]).

By [4], the projective equivalence between a general (α, β)-metric and a Kropina metric, we have the following lemma:

Lemma 1.1: Let $F = \alpha\phi(s)$ be an (α, β)-metric on n-dimensional manifold $M(n \geq 3)$satisfying that β is not parallel with respect to $\alpha, db \neq 0$ everywhere (or)$b = constant$ and F is not of Randers type. Let $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ be a Kropina metric on the manifold M, where $\bar{\alpha} = \lambda(x)\alpha$ and $\bar{\beta} = \mu(x)\beta$. Then F is projectively equivalent to $\bar{F}$ if and only if the following equations holds,

$$[1 + (k_1 + k_2 s^2)^2 + k_3 s^2]\phi'' = (k_1 + k_2 s^2)(\phi - s\phi'), \tag{1.1}$$

$$G^i_\alpha = \bar{G}^i_{\bar{\alpha}} + \theta y^i - \sigma(k_1\alpha^2 + k_2\beta^2)b^i, \tag{1.2}$$

$$b_{i|j} = 2\sigma\left[(1 + k_1 b^2)a_{ij} + (k_2 b^2 + k_3)b_j b_j\right], \tag{1.3}$$

$$\bar{s}_{ij} = \frac{1}{b^2}(b_i\bar{s}_j - \bar{b}_j\bar{s}_i), \tag{1.4}$$

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where $\sigma = \sigma(x)$ is a scalar function and k_1, k_2, k_3 are constants. In this case, both $F = \alpha\phi(s)$and $\bar{F} = \bar{\alpha}\phi(\bar{s})$ are Douglas metrics.

The present article is organized as follows: In the first part, we prove that both the Finsler metrics are Douglasmetrics. Further in the next part, we study the projective relation between special (α,β)-metric metrics $F = \alpha + \beta + \frac{2\beta^2}{\alpha} - \frac{\beta^4}{3\alpha^3}$ and Kropina metric. The main result of the paper is as follows:

Theorem 1.1. Let$F = \alpha + \beta + \frac{2\beta^2}{\alpha} - \frac{\beta^4}{3\alpha^3}$ and $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ be a Kropina metric on a n-dimensional manifold $M(n \geq 3)$ where α and $\bar{\alpha}$ are two Riemannian metrics, β and $\bar{\beta}$ are two nonzero collinear 1-forms. Then F is projectively equivalent to $\bar{F}$ if and only if they are Douglas metrics and the geodesic co-efficient of α and $\bar{\alpha}$ have the following relations,

$$G^i_\alpha + \tau\alpha^2 b^i = \bar{G}^i_{\bar{\alpha}} + \frac{1}{2b^2}\left(\bar{\alpha}^2\bar{s}^i + \bar{r}_{00}\bar{b}^i\right) + \theta y^i, \tag{1.5}$$

where $b^i = a^{ij}b_j$, $\bar{b}^i = \bar{a}^{ij}\bar{b}_j$, $\bar{b}^2 = \|\bar{\beta}\|^2_{\bar{\alpha}}$ and $\tau = \tau(x)$ is scaler function and $\theta = \theta y^i$ is a1-form on M.

II. PRELIMINARIES

The Finsler space $F^n = (M, F)$ is said to have an (α,β)-metric if F is positively homogeneous function of degree one in two varibles$\alpha^2 = a_{ij}(x)y^i y^j$ and $\beta = b_i(x)y^i$. In Riemannian geometry, two Riemannian metrics $\alpha$and $\bar{\alpha}$ are projectively related if and only if their spray coefficients have the relation [3],

$$G^i_\alpha = G^i_{\bar{\alpha}} + \lambda_{x^k} y^k y^i, \tag{2.1}$$

where $\lambda = \lambda(x)$ is a scalar function on the manifold and (x^i, y^j) denotes the local coordinates in the tangent bundle TM. For a given Finsler metric $F = F(x,y)$, the geodesics of F satisfy the following ODE's:

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0,$$

where $G^i = G^i(x,y)$ is called the geodesic coefficient, which is given by

$$G^i = \frac{1}{4}g^{il}\{[F^2]_{x^m y^l}y^m - [F^2]_{x^l}\}.$$

Two Finsler metrics F and $\bar{F}$ on a manifold M are said to be (point wise) projectively related if they have the same geodesics as point sets. The equivalent condition has been characterized by using spray coefficients [3],

$$G^i = \bar{G}^i + P(y)y^i, \tag{2.2}$$

where $P(y)$ is a scalar function on TM_0, homogeneous of degree one in y and G and $\bar{G}$ are the spray coefficients of F and$\bar{F}$.

The concept of (α,β)-metric $L(\alpha,\beta)$ was introduced in 1972 by M. Matsumoto. By definition, an (α,β)- metric is a Finsler metric expressed in the following form, $F = \alpha\phi(s), s = \frac{\beta}{\alpha}$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric andβ is one form with $||\beta_x|| < b_0$. The function $\phi(s)$ is a C^∞ positive function on an open interval $(-b_0, b_0)$ satisfying:

$$\phi(s) - s\phi(s) + (b^2 - s^2)\phi''(s) > 0, \qquad (|s| \leq b < b_0). \tag{2.3}$$

In this case, the fundamental form of the metric tensor induced by F is positive definite.

Let $r_{ij} = \frac{1}{2}\left(b_{i|j} + b_{j|i}\right)$, $s_{ij} = \frac{1}{2}\left(b_{i|j} - b_{j|i}\right)$, where,$b_{i|j}$means the coefficients of the covariant derivative of $\beta$with respect toα. Clearly β is closed if and only if $s_{ij} = 0$. An (α,β)-metric is said to be trivial if $r_{ij} = s_{ij} = 0$. Furthermore, we denote $s_j = b^i s_{ij}$, $r^i_j = a^{il}r_{lj}$,$s^i_j = a^{il}s_{lj}$, $s_0 = s_i y^i$, $s^i_0 = s^i_j y^j$ and$r_{00} = r_{ij}y^i y^j$.

The relation between the geodesic coefficients G^i of F and geodesic coefficients G^i_α of α is given by

$$G^i = G^i_\alpha + \alpha Q s^i_0 + \{-2Qs_0 + r_{00}\}\{\psi b^i + \theta\alpha^{-1}y^i\}, \tag{2.4}$$

Where $\theta = \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi)}{2\phi\{(\phi - s\phi') + (b^2 - s^2)\phi''\}}$, $Q = \frac{\phi'}{\phi - s\phi'}$, $\psi = \frac{1}{2}\frac{\phi''}{\{(\phi - s\phi') + (b^2 - s^2)\phi''\}}$.

In [8], the authors characterized the (α,β)-metrics of Douglas type.

Lemma 2.2. Let $F = \alpha\phi\left(\frac{\beta}{\alpha}\right)$ be a regular (α,β)- metric on an n-dimensional manifold $M(n \geq 3)$. Assume that $\beta$is not parallel with respect toα and $db \neq 0$ everywhere or $b = constant$, and F is not of Randers type. Then F is a Douglas metric if and only if the function $\phi = \phi(s)$ with $\phi(0) = 1$ satisfies the following ODE,

$$[1 + (k_1 + k_2 s^2)^2 + k_3 s^2]\phi'' = (k_1 + k_2 s^2)(\phi - s\phi'), \tag{2.5}$$

And β satisfies

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$b_{i|j} = 2\sigma[(1 + k_1 b^2)a_{ij} + (k_2 b^2 + k_3)b_j b_j]$, (2.6)
Where $b^2 = \|\beta\|_\alpha^2$ and $\sigma = \sigma(x)$ is scalar function and $k_1, k_2 and k_3$ are constants with $(k_2, k_3) \neq (0,0)$.
For Kropina metric we have the following,

Lemma 2.3. [9]: Let $F = \frac{\alpha^2}{\beta}$ be a Kropina metric on an n-dimensional manifold M. Then
(i) $(n \geq 3)$, Kropina metric F with $b^2 \neq 0$ is Douglas metric if and only if

$$s_{ik} = \frac{1}{b^2}(b_i s_k - b_j s_i). \tag{2.7}$$

(ii) $(n = 2)$ Kropina metric F is a Douglas metric.

Definition 2.1: The tensor $D = D^i_{jkl}\,\partial i \otimes dx^j \otimes dx^k \otimes dx^l$, where

$$D^i_{jkl} = \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m} y^i\right), \tag{2.8}$$

is called the Douglas tensor. A Finsler metric is called Douglas metric if the Douglas tensor vanishes.

The Douglas tensor D is a non-Riemannian projective invariant, namely, if two Finslermetrics$F$ and $\bar{F}$ are projectively equivalent, that is $G^i = \bar{G}^i + Pyi$, where $P = P(x, y)$ is positively y-homogeneous of degree 1, then the Douglas tensor of F is the same as that of $\bar{F}$. Finsler metrics with vanishing Douglas tensor are called Douglas metrics. In [3], the Douglas tensor of a general (α, β) −metric is determine by

$$D^i_{jkl} = \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(T^i - \frac{1}{n+1}\frac{\partial T^m}{\partial y^m} y^i\right), \tag{2.9}$$

Where$T^i = \alpha Q s_0^i + \psi\{-2Q\alpha s_0 + r_{00}\}b^i$. (2.10)
And$T^m_{y^m} = Q' s_0 + \psi' \alpha^{-1}(b^2 - s^2)[r_{00} - 2Q\alpha s_0] + 2\psi[r_0 - Q'(b^2 - s^2)s_0 - Qss_0]$. (2.11)

In the sequel, we use quantities with a bar to denote the corresponding quantities of the metric$\bar{F}$.
Now, let$F$ and $\bar{F}$ be the two (α, β)-metrics which have the same Douglas tensor, i.e.,$D^i_{jkl} = \bar{D}^i_{jkl}$. From (2.8) and (2.9) we have,

$$\frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(T^i - \bar{T}^i - \frac{1}{n+1}\left(T^m_{y^m} - \bar{T}^m_{y^m}\right)y^i\right) = 0. \tag{2.12}$$

Then there exists a class of scalar function $H^i_{jk} = H^i_{jk}(x)$ such that

$$T^i - \bar{T}^i - \frac{1}{n+1}\left(T^m_{y^m} - \bar{T}^m_{y^m}\right)y^i = H^i_{00}. \tag{2.13}$$

Where$H^i_{00} = H^i_{jk} y^j y^k$, T^i and $T^m_{y^m}$ are given by (2.10) and (2.11) respectively. In this paper we assume that $\lambda = \frac{1}{n+1}$.

III. CHARACTERIZATION OF PROJECTIVE RELATION BETWEEN SPECIAL (α, β)- METRIC AND KROPINA METRIC

In this section, we devoted to characterize the necessary and sufficient condition for (α, β)-metric$F = \alpha + \beta + \frac{2\beta^2}{\alpha} - \frac{\beta^4}{3\alpha^3}$ and Kropina metric is of Douglas metrics on a same underlying manifold M of dimension$n \geq 3$.

For (α, β)-metric,$F = \alpha + \beta + \frac{2\beta^2}{\alpha} - \frac{\beta^4}{3\alpha^3}$, by (2.3) F is regular Finsler metric if and only if 1-form β satisfies the condition $\|\beta_x\|_\alpha < 1$ for any $x \epsilon M$. The geodesic coefficients are given by (2.4) with

$$\theta = \frac{9 - 54s^2 - 120s^3 + 45s^4 + 144s^5 - 24s^7}{6(3 + 3s + 6s^2 - s^4)\{(1 + 4b^2) - (6 + 4b^2)s^2 + 5s^4\}},\quad Q = \frac{(3 + 12s - 4s^3)}{(3 - 6s^2 + 3s^4)},\quad \psi = \frac{2 - 2s^2}{\{(1 + 4b^2) - (6 + 4b^2)s^2 + 5s^4\}}. \tag{3.1}$$

For Kropina-metric $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$, one can see that F is not a regular (α, β)-metric, butthe relation $\phi(s) - s\phi'(s) + (b^2 - s^2)\phi'' > 0$ is still true for $|s| > 0$. In view of equation (2.4), geodesic coefficients of the Finsler metric $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$are given by

$$\bar{G}^i = \bar{G}^i_\alpha + \bar{\alpha}\bar{Q}\bar{s}^i_0 + \{-2\,\bar{Q}\bar{\alpha}\bar{s}_0 + \bar{r}_{00}\}\{\bar{\psi}\bar{b}^i + \bar{\theta}\bar{\alpha}^{-1}\bar{y}^i\} \tag{3.2}$$

With $\bar{Q} = -\frac{1}{2s}$, $\bar{\theta} = -\frac{s}{b^2}$, $\bar{\psi} = \frac{1}{2b^2}$. (3.3)

Now let us prove the following theorem.

Theorem 3.1. Let $F = \alpha + \beta + \frac{2\beta^2}{\alpha} - \frac{\beta^4}{3\alpha^3}$ be an (α, β)-metric and $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ be an Kropina metric on an n-dimensional manifold $M(n \geq 3)$ where α and $\bar{\alpha}$ are two Riemannian metrics, β and $\bar{\beta}$ are two non-zero 1-forms. Then F and $\bar{F}$ have the same Douglas tensors if and only if they are all Douglas metrics.

Proof: First we prove the sufficient condition.

Let F and $\bar{F}$ be Douglas metrics and corresponding Douglas tensors be D^i_{jkl} and $\bar{D}^i_{jkl}$.Then by the definition of Douglas metric, we have $D^i_{jkl} = 0$ and $\bar{D}^i_{jkl} = 0$, that is both F and $\bar{F}$ have same Douglas tensor.

To prove the necessary condition,

If F and $\bar{F}$ have the same Douglas tensors, then (2.13) holds. Plugging (3.1) and (3.3) into (2.13), we have

$$H^i_{00} = \frac{A_{17}\alpha^{17}+A_{16}\alpha^{16}+A_{15}\alpha^{15}+A_{14}\alpha^{14}+A_{13}\alpha^{13}+A_{12}\alpha^{12}+A_{11}\alpha^{11}+A_{10}\alpha^{10}+A_9\alpha^9+A_8\alpha^8+A_7\alpha^7+A_6\alpha^6+A_5\alpha^5+A_4\alpha^4+A_2\alpha^2+A_1}{B_1\alpha^{16}+B_2\alpha^{14}+B_3\alpha^{12}+B_4\alpha^{10}+B_5\alpha^8+B_6\alpha^6+B_7\alpha^4+B_8\alpha^2+B_9} + \frac{\bar{A}^i\bar{\alpha}^2+\bar{B}^i}{2\,\bar{b}^2\bar{\beta}} \tag{3.4}$$

where

$A_{17} = 9(1+4b^2)\{(1+4b^2)s^i_0 - 4s_0b^i\}$,

$A_{16} = (1+4b^2)[36\beta\{(1+4b^2)s^i_0 - 4s_0b^i\} + 18r_{00}b^i - 36(r_0+s_0)\lambda y^i]$,

$A_{15} = -18\beta^2(1+4b^2)(7+8b^2)s^i_0 + 36\beta^2(9+16b^2)s_0b^i - (28+32b^2)\beta s_0\lambda y^i$,

$A_{14} = -12\beta^3(1+4b^2)(43+52b^2)s^i_0 + 144\beta^3(57+208b^2)s_0b^i - 18\beta^2(10+20b^2)r_{00}b^i + 9b^2(28+$
$32b2\beta r00\lambda yi+36\beta 211+24b2r0\lambda yi+18\beta 223-16b2-23b4s0\lambda yi$,

$A_{13} = 9\beta^4\{35+208b^2+96b^4\}s^i_0 - 36\beta^4(26+24b^2)s_0b^i + 18\beta^3(28+56b^2+64b^4)s_0\lambda y^i$,

$A_{12} = 12\beta^{15}\{227+600b^2+352b^4\}s^i_0 + 48\beta^5\{-87-88b^2\}s_0b^i + 18\beta^4(41+44b^2)r_{00}b^i + 9\beta^3(8b^2 -$
$28r00\lambda yi+36\beta 4-45-60b2r0\lambda yi+\beta 4828+364b4-2736b2s0\lambda yi$,

$A_{11} = -18\beta^6(82+136b^2+32b^4)s^i_0 + 36\beta^6(14+32b^2)s_0b^i + 18\beta^5(32-72b^2-32b^4)s_0\lambda y^i$,

$A_{10} = 12\beta^7(572+1076b^2+432b^4)s^i_0 + 48\beta^7(134+72b^2)s_0b^i + 18\beta^6(-97-80b^2)r_{00}b^i +$
$18\beta^5(36+2b^2+32b^4)r_{00}\lambda y^i + 18\beta^6(190+160b^2)r_0\lambda y^i + 12\beta^6[(224+824b^2-292b^4)+$
$192b21+ 4b2\lambda s0yi$,

$A_9 = 9\beta^8(191+168b^2+16b^4)s^i_0 + 36\beta^8(-21-4b^2)s_0b^i + 18\beta^7(-54-400b^2-288b^4)s_0\lambda y^i$,

$A_8 = 12\beta^9(737+816b^2+112b^4)s^i_0 + 48\beta^9(-62-24b^2)s_0b^i + 18\beta^8(115+60b^2)r_{00}b^i + 18\beta^7(6+$
$8b2-64b4r00\lambda yi+36\beta 8115+60b2r0\lambda yi+\beta 8-12224-5456b2+2176b4s0\lambda yi$,

$A_7 = \beta^{10}(90+360b^2)s^i_0 + \beta^{10}180s_0b^i + \beta^9(522+72b^2)s_0\lambda y^i$,

$A_6 = 12\beta^{11}(-521-288b^2-164b^4)s^i_0 + \beta^{11}(1728+192b^2)s_0b^i + 18\beta^{10}(-81-24b^2)r_{00}b^i +$
$\beta^9(-116b^2+288b^4-1872)r_{00}\lambda y^i + 36\beta^{10}(61+8b^2)r_0\lambda y^i + \beta^{10}(10072+192b^2-704b^4)\lambda s_0y^i$,

$A_5 = 225\beta^{12}s^i_0 - 360\beta^{11}\lambda s_0y^i$,

$A_4 = \beta^{13}(2220+480b^2)s^i_0 - 90\beta^{13}s_0b^i + 18\beta^{12}(31+4b^2)r_{00}b^i + 9\beta^{11}(292+48b^2)r_{00}\lambda y^i +$
$36\beta^{12}(14-4b^2)r_0\lambda y^i + \beta^{12}(-1964+1536b^2)$,

$A_3 = 0$,

$A_2 = -90\beta^{14}r_{00}b^i + \beta^{13}(1080-180b^2)r_{00}\lambda y^i + 180\beta^{14}r_0\lambda y^i - 400\beta^{14}s_0\lambda y^i$,

$A_1 = 180\beta^{15}r_{00}\lambda y^i$,

And

$B_1 = 9(1+4b^2)^2$,

$B_2 = 18(1+4b^2)(-8-12b^2)\beta^2$,

$B_3 = 9(324+136b^2+240b^4)\beta^4$,

$B_4 = 18(-165-380b^2-160b^4)\beta^6$,

$B_5 = 9(830+920b^2+208b^4)\beta^8$,

$B_6 = 8(-328-324b^2-48b^4)\beta^{10}$,

$B_7 = 9(361+248b^2+16b^4)\beta^{14}$,

$B_8 = 18(-80-20b^2)\beta^{14}$,

$B_9 = 225\beta^{16}$,

And,

$\bar{A} = \bar{b}^2\bar{s}^i_0 - \bar{b}^i\bar{s}_0$,

$\bar{B}^i = \bar{\beta}[2\lambda y^i(\bar{r}_0 + \bar{s}_0) - \bar{b}^i\bar{r}_{00}]$.

The terms of (3.4) can be written as,
$\left(2\,\bar{b}^2\bar{\beta}\right)(B_1\alpha^{16}+B_2\alpha^{14}+B_3\alpha^{12}+B_4\alpha^{10}+B_5\alpha^8+B_6\alpha^6+B_7\alpha^4+B_8\alpha^2+B_9)H_{00}^i=\left(2\,\bar{b}^2\bar{\beta}\right)(A_{17}\alpha^{17}+$
A16α16+A15α15+A14α14+A13α13+A12α12+A11α11+A10α10+A9α9+A8α8+A7α7+A6α6+A5α5+A4α4+A2α2+A1+(Aiα2+Bi)(B1α16+B2α14+B3α12+B4α10+B5α8+B6α6+B7α4+B8α2+B9),
(3.5)

Replacing (y^i) and $(-y^i)$ in (3.5), which yields,
$\left(-2\,\bar{b}^2\bar{\beta}\right)(B_1\alpha^{16}+B_2\alpha^{14}+B_3\alpha^{12}+B_4\alpha^{10}+B_5\alpha^8+B_6\alpha^6+B_7\alpha^4+B_8\alpha^2+B_9)H_{00}^i=$
$\left(-2\,\bar{b}^2\bar{\beta}\right)(-A_{17}\alpha^{17}+A_{16}\alpha^{16}-A_{15}\alpha^{15}+A_{14}\alpha^{14}-A_{13}\alpha^{13}+A_{12}\alpha^{12}-A_{11}\alpha^{11}+A_{10}\alpha^{10}-A_9\alpha^9+A_8\alpha^8-$
A7α7+A6α6−A5α5+A4α4+A2α2+A1−(Aiα2+Bi)(B1α16+B2α14+B3α12+B4α10+B5α8+B6α6+B7α4+B8α2+B9), (3.6)

Add (3.5) and (3.6), which yields,
$$(A_{17}\alpha^{17}+A_{15}\alpha^{15}+A_{13}\alpha^{13}+A_{11}\mathrm{A}^{11}+A_9\alpha^9+A_7\alpha^7+A_5\alpha^5)=0 \tag{3.7}$$

From (3.4) and (3.7), we have
$$H_{00}^i=\frac{A_{16}\alpha^{16}+A_{14}\alpha^{14}+A_{12}\alpha^{14}+A_{10}\alpha^{10}+A_8\alpha^8+A_6\alpha^6+A_4\alpha^4+A_2\alpha^2+A_1}{B_1\alpha^{16}+B_2\alpha^{14}+B_3\alpha^{12}+B_4\alpha^{10}+B_5\alpha^8+B_6\alpha^6+B_7\alpha^4+B_8\alpha^2+B_9}+\frac{\bar{A}^i\bar{\alpha}^2+\bar{B}^i}{2\,\bar{b}^2\bar{\beta}}. \tag{3.8}$$

Again (3.8) can be written as
$\left(2\,\bar{b}^2\bar{\beta}\right)(B_1\alpha^{16}+B_2\alpha^{14}+B_3\alpha^{12}+B_4\alpha^{10}+B_5\alpha^8+B_6\alpha^6+B_7\alpha^4+B_8\alpha^2+B_9)H_{00}^i=\left(2\,\bar{b}^2\bar{\beta}\right)(A_{16}\alpha^{16}+$
A14α14+A12α12+A10α10+A8α8+A6α6+A4α4+A2α2+A1+(Aiα2+Bi
$)(B_1\alpha^{16}+B_2\alpha^{14}+B_3\alpha^{12}+B_4\alpha^{10}+B_5\alpha^8+B_6\alpha^6+B_7\alpha^4+B_8\alpha^2+B_9).$
(3.9)

From the above equation (3.9), we say $\bar{A}^i\bar{\alpha}^2(B_1\alpha^{16}+B_2\alpha^{14}+B_3\alpha^{12}+B_4\alpha^{10}+B_5\alpha^8+B_6\alpha^6+B_7\alpha^4+B_8\alpha^2+B_9)$can be divided by $\bar{\beta}$. Since $\beta=\mu\,\bar{\beta}$, then $\bar{A}^i\bar{\alpha}^2B_1\alpha^{16}$can be divided by $\bar{\beta}$. Because $\bar{\beta}$ is prime with respect to α and $\bar{\alpha}$. Therefore, $\bar{A}^i=\bar{b}^2\bar{s}_0^i-\bar{b}^i\bar{s}_0$ can be divided by $\bar{\beta}$. Hence there is a scalar function $\psi^i(x)$ such that
$$\bar{b}^2\bar{s}_0^i-\bar{b}^i\bar{s}_0=\bar{\beta}\psi^i \tag{3.10}$$
Transvecting (3.10) by $\bar{y}_i=\bar{a}_{ij}y^j$, we get $\psi^i(x)=-\bar{s}^i$. Thus we have
$$\bar{s}_{ij}=\frac{1}{\bar{b}^2}(\bar{b}_i\bar{s}_j-\bar{b}_j\bar{s}_i). \tag{3.11}$$
Thus by lemma 2.3, $\bar{F}=\frac{\bar{\alpha}^2}{\bar{\beta}}$ is a Douglas metrics. i.e., Both $F=\alpha+\beta+\frac{2\beta^2}{\alpha}-\frac{\beta^4}{3\alpha^3}$ and $\bar{F}=\frac{\bar{\alpha}^2}{\bar{\beta}}$ are Douglas metrics.

If $n=2$, $\bar{F}=\frac{\bar{\alpha}^2}{\bar{\beta}}$ is a Douglas metric by lemma 2.3. Thus F and $\bar{F}$ have the same Douglas tensors means that they are Douglas metrics.

This completes the proof of Theorem 3.1.

IV. PROOF OF THEOREM

In this section, we characterize the projective relation between a special (α,β)-metric and Kropina metric.

Proof: First we prove the necessary condition.

Since Douglas tensor is an invariant under projective changes between two Finsler metrics. If F is projectively related to $\bar{F}$, then they have the same Douglas tensor. From Theorem 3.1, we obtain that both F and $\bar{F}$are Douglas metrics.

By [4], It is well knowing that Kropina metric $\bar{F}=\frac{\bar{\alpha}^2}{\bar{\beta}}$with $b^2\neq 0$ is a Douglas metric if and only if $s_{ik}=\frac{1}{b^2}(b_is_k-b_ks_i)$ and According to [2], the (α,β)-metric, $F=\alpha+\beta+\frac{2\beta^2}{\alpha}-\frac{\beta^4}{3\alpha^3}$ is a Douglas metric if and only if
$$b_{i|j}=\frac{\tau}{2}[(1+4b^2)a_{ij}-5b_ib_j], \tag{4.1}$$
Where $\tau=\tau(x)$ is a scalar function on M. Here in this case, β is closed.

Plugging (4.1) and (3.1) into (2.4), we have
$$G^i=G_\alpha^i+\left(\frac{\{(1+4b^2)\alpha^2-5\beta^2\}\{9\alpha^7-54\alpha^5\beta^2-120\alpha^4\beta^3+45\alpha^3\beta^4+144\alpha^2\beta^5-24\beta^7\}}{12\{3\alpha^4+3\alpha^3\beta+6\alpha^2\beta^2-\beta^4\}\{\alpha^4(1+4b^2)-(6+4b^2)\alpha^2\beta^2+5\beta^4\}}\right)\tau y^i+\tau\alpha^2b^i. \tag{4.2}$$

It has been proved in [9] that, $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ is a Douglas metric if and only if

$$\bar{s}_{ij} = \frac{1}{\bar{b}^2}(\bar{b}_i\bar{s}_j - \bar{b}_j\bar{s}_i). \quad (4.3)$$

Again plugging (4.3) and (3.3) into (3.2), we have

$$\bar{G} = \bar{G}^i_{\bar{\alpha}} - \frac{1}{2\,\bar{b}^2}\left[-\bar{\alpha}^2\bar{s}^i + \left(2\,\bar{s}_0 y^i - \bar{r}_{00}\bar{b}^i\right) + 2\left(\frac{\bar{r}_{00}\bar{\beta}\,y^i}{\bar{\alpha}^2}\right)\right]. \quad (4.4)$$

Since F is Projectively equivalent to $\bar{F}$, then their exit a scalar function $P = P(x,y)$ on TM_0 such that

$$G^i = \bar{G}^i + Py^i, \quad (4.5)$$

By (4.2), (4.4) and (4.5), we have

$$\left[P - \left(\frac{\{(1+4b^2)\alpha^2-5\beta^2\}\{9\alpha^7-54\alpha^5\beta^2-120\alpha^4\beta^3+45\alpha^3\beta^4+144\alpha^2\beta^5-24\beta^7\}}{12\{3\alpha^4+3\alpha^3\beta+6\alpha^2\beta^2-\beta^4\}\{\alpha^4(1+4b^2)-(6+4b^2)\alpha^2\beta^2+5\beta^4\}}\right)\tau - \frac{1}{\bar{b}^2}\left(\bar{s}_0 + \frac{\bar{r}_{00}\,\bar{\beta}}{\bar{\alpha}^2}\right)\right]y^i = G^i_\alpha - \bar{G}^i_{\bar{\alpha}} + \alpha^2\tau b^i - \frac{1}{2\,\bar{b}^2}(\bar{\alpha}^2\bar{s}^i + \bar{r}_{00}\bar{b}^i). \quad (4.6)$$

Note that RHS of above equation (4.6) is in quadratic form.

Then there must be a one form $\theta = \theta_i y^i$ on M, such that

$$P - \left(\frac{\{(1+4b^2)\alpha^2-5\beta^2\}\{9\alpha^7-54\alpha^5\beta^2-120\alpha^4\beta^3+45\alpha^3\beta^4+144\alpha^2\beta^5-24\beta^7\}}{12\{3\alpha^4+3\alpha^3\beta+6\alpha^2\beta^2-\beta^4\}\{\alpha^4-(6+4b^2)\alpha^2\beta^2+5\beta^4\}}\right)\tau - \frac{1}{\bar{b}^2}\left(\bar{s}_0 + \frac{\bar{r}_{00}\,\bar{\beta}}{\bar{\alpha}^2}\right) = \theta.$$

Thus (4.6) becomes

$$G^i_\alpha + \alpha^2\tau b^i = \bar{G}^i_{\bar{\alpha}} + \frac{1}{2\,\bar{b}^2}\left(\bar{\alpha}^2\bar{s}^i + \bar{r}_{00}\bar{b}^i\right) + \theta y^i. \quad (4.7)$$

From (4.1), (4.3) and (4.8) completes the proof of necessity.

Conversely, Substituting (4.1) and (3.1) into (2.4) which yields (4.2). Again Substituting (4.3) and (3.3) into (2.4) which yields (4.4). Thus from (4.2), (4.4) and (4.7), we have,

$$G^i = \bar{G}^i + \left[\theta + \left(\frac{\{(1+4b^2)\alpha^2-5\beta^2\}\{9\alpha^7-54\alpha^5\beta^2-120\alpha^4\beta^3+45\alpha^3\beta^4+144\alpha^2\beta^5-24\beta^7\}}{12\{3\alpha^4+3\alpha^3\beta+6\alpha^2\beta^2-\beta^4\}\{\alpha^4-(6+4b^2)\alpha^2\beta^2+5\beta^4\}}\right)\tau + \frac{1}{\bar{b}^2}\left(\bar{s}_0 + \frac{\bar{r}_{00}\,\bar{\beta}}{\bar{\alpha}^2}\right)\right]y^i. \quad (4.8)$$

Thus F is projectively equivalent to $\bar{F}$.

Hence the proof.

From the above theorems (3.1) and (1.1), immediately we get the following corollary:

Corollary 4.1: Let $F = \alpha + \beta + \frac{2\beta^2}{\alpha} - \frac{\beta^4}{3\alpha^3}$be a special (α,β)-metric and $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ be a Kropina metric be two (α,β)-metrics on a n-dimensional manifold M with dimension $n \geq 3$, where α and $\bar{\alpha}$ are two Riemannian metrics, β and $\bar{\beta}$ are two non-zero collinear 1-forms. Then F is projectively related to $\bar{F}$ if and only if they are Douglas metrics and the spray coefficients of α and $\bar{\alpha}$ have the following relations,

$$G^i + \alpha^2\tau b^i = \bar{G}^i_{\bar{\alpha}} + \frac{1}{2\,\bar{b}^2}\left(\bar{\alpha}^2\bar{s}^i + \bar{r}_{00}\bar{b}^i\right) + \theta y^i,$$

$$s_{ij} = 0,$$

$$\bar{s}_{ij} = \frac{1}{b^2}(\bar{b}\bar{s}_j - \bar{b}_j\bar{s}_i),$$

$$b_{i|j} = \frac{\tau}{2}[(1+4b^2)a_{ij} - 5b_ib_j],$$

where $b_{i|j}$ denotes the coefficients of the covariant derivative of β with respect to α.

V. CONCLUSION

Projective differential geometry was initiated in the 1920s, especially by ElieCartan and Tracey Thomas. Projective differential geometry also provides the simplest setting in which over determined systems of partial differential equations naturally arise. In projective differential geometry, we have a remarkable theorem called Rapcsak Theorem, which plays an important role in Projective geometry of Finsler spaces. This theroem gives the necessary and sufficient conditions that a Finsler space is projective to another Finsler space. The problem of projectively related Finsler metrics id formulated in Hilbert's Fourth Problem i.e., to determine the metrics on an open subset in R^n whose geodesics are straight lines. Projective flat Finsler metrics on a convex domain in Rn are regular solutions to Hilbert's Fourth problem.

So it is an important problem in Finsler geometry to study and characterize the Projective related Finsler metrics. In this articles we are trying to characterize the projective relation betweena special (α,β)-metric and Kropina metric..

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