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Research Paper

On the Exact and Approximate Solutions of Coupled Systems of Nonlinear Partial Differential Equations using NDM.

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Abstract*: In this research article, we are concerned about finding exact and approximate solutions to coupled systems of nonlinear PDEs using the Natural Decomposition Method (NDM). We applied this method to obtain exact solution of (1+1)-dimensional nonlinear Boussinesq Equation and approximate numerical solution of the Broer-Kaup (BK) system of equations.*
The approximate numerical solutions are calculated in power series form with easily computable components and obtained result are in good agreement with the exact solutions.
The application further demonstrates the efficiency and accuracy of the method as it reduces significantly the numerical computations as compared to some other known techniques. The method is effective and applicable for many other NPDEs in mathematical physics.
keywords*: Adomian Polynomials, Boussinesq Equation, Broer-Kaup Equation, Systems of nonlinear Partial Differential Equations, NDM.*

I. INTRODUCTION

Nonlinear Evolution Equations have attracted the attention of researchers in recent years due to their wide applicability in various fields such as Fluid Mechanics, Bio Mathematics, Physics and other areas of Science and Engineering. As a result, considerable interests have been shown by researchers in the investigation of exact solutions of nonlinear evolution equations.

Among methods proposed for solving nonlinear differential equations are exp-function method [1], F-expansion method [2], Homotopy perturbation method (HPM) [3], Reduced differential transform method (RDTM) [4,5,6,7] and so on.

The Natural decomposition method (NDM) is a combination of Natural transform method and Adomian decomposition method and is useful in obtaining solutions of nonlinear differential equations.

For nonlinear models, NDM has shown dependable results. It has been employed to obtain the exact solutions of linear and nonlinear Schrodinger Equations [8], Gas Dynamic Equation [9], linear and nonlinear Klein-Gordon Equations [10] and nonlinear 2-dimensional Brusselator Equations [11].

In this paper, we apply the Natural Decomposition Method (NDM) to coupled system of nonlinear partial differential equations.

First we find the exact solution of the (1+1)-dimensional nonlinear Boussinesq equation:

$$\begin{aligned} v_t + w_x + vv_x &= 0 \\ w_t + (wv)_x + v_{xxx} &= 0 \end{aligned} \qquad (1.1)$$

Subject to the initial conditions:

$$\begin{aligned} v(x,0) &= 2x \\ w(x,0) &= x^2 \end{aligned} \qquad (1.2)$$

Secondly, we obtain the approximate solution of the Broer-Kaup (BK) system of equations:

$$\begin{aligned} v_t + vv_x + w_x &= 0 \\ w_t + v_x + (vw)_x + v_{xxx} &= 0 \end{aligned} \qquad (1.3)$$

Subject to the initial conditions:

$$v(x,0) = 1 + 2\tanh x;$$
$$w(x,0) = 1 - 2\tanh x \qquad (1.4)$$

The rest of the paper is organized as follows: In section 2, the NDM is introduced and in section 3, the definitions and properties of the N-Transform are discussed. In section 4, the NDM is applied to (1+1)-dimensional nonlinear Boussinesq equation and Broer-Kaup (BK) system of equations. Section 5 is the conclusion.

II. BASIC IDEA OF THE NATURAL TRANSFORM METHOD:

Here we discuss some preliminaries about the nature of the Natural Transform Method (NTM).
Consider a function $f(t),\ t \in (-\infty,\infty)$, then the general Integral transform is defined as follows [10,12]:

$$\Im[f(t)](s) = \int_{-\infty}^{\infty} k(s,t) f(t) dt \qquad (2.1)$$

where $k(s,t)$ represent the kernel of the transform, s is the real (complex) number which is independent of t.

Note that when $k(s,t)$ is e^{-st}, $tJ_n(st)$ and $t^{s-1}(st)$, then Equation (2.1) gives, respectively, Laplace Transform, Hankel Transform and Mellin Transform.
Now, for $f(t),\ t \in (-\infty,\infty)$ consider the Integral transforms defined by

$$\Im[f(t)](u) = \int_{-\infty}^{\infty} k(t) f(ut) dt \qquad (2.2)$$

and

$$\Im[f(t)](s,u) = \int_{-\infty}^{\infty} k(s,t) f(ut) dt \qquad (2.3)$$

Note that:

(i) when $k(t) = e^{-t}$, Equation (2.2) gives the Integral Sumudu transform, where parameter s is replaced by u. Moreover, for any value of n, the generalized Laplace and Sumudu transform are respectively defined by [1,7]:

$$\ell[f(t)] = F(s) = s^n \int_0^{\infty} e^{-s^{n+1}t} f(s^n t) dt \qquad (2.4)$$

and

$$S[f(t)] = G(u) = u^n \int_0^{\infty} e^{-u^n t} f(tu^{n+1}) dt \qquad (2.5)$$

(ii) when $n = 0$, Equation (2.4) and Equation (2.5) are the Laplace and Sumudu transform respectively.

III. DEFINITIONS AND PROPERTIES OF THE N-TRANSFORM:

The natural transform of the function $f(t),\ t \in (-\infty,\infty)$ is defined by [13,14]:

$$\mathrm{N}[f(t)] = R(s,u) = \int_{-\infty}^{\infty} e^{-st} f(ut) dt \ ; \quad s,u \in (-\infty,\infty) \qquad (3.1)$$

where $\mathrm{N}[f(t)]$ is the natural transformation of the time function $f(t)$ and the variable s and u are the natural transform variables.

Note:

(i) Equation (3.1) can be written in the form [13,15]:

$$\mathrm{N}[f(t)] = \int_{-\infty}^{\infty} e^{-st} f(ut) dt \ ; \quad s,u \in (-\infty,\infty)$$
$$= \left[\int_{-\infty}^{0} e^{-st} f(ut) dt \ ; s,u \in (-\infty,0)\right] + \left[\int_{0}^{\infty} e^{-st} f(ut) dt \ ; \mathrm{s,u} \in (0,\infty)\right]$$

$$= \mathrm{N}^{-}[f(t)] + \mathrm{N}^{+}[f(t)] = \mathrm{N}[f(t)\mathrm{H}(-t)] + \mathrm{N}[f(t)\mathrm{H}(t)] = R^{-}(s,u) + R^{+}(s,u)$$

where $\mathrm{H}(\cdot)$ is the Heaviside function.

(ii) If the function $f(t)\mathrm{H}(t)$ is defined on the positive real axis, with $t \in (0,\infty)$ and in the set $A = \left\{ f(t) : \exists\, M, \tau_1, \tau_2 > 0, s.t. |f(t)| < Me^{\frac{|t|}{\tau_j}}, \text{where}\, t \in (-1)^j \times [0,\infty),\, j = 1,2 \right\}$, then we define the Natural transform (N-Transform) as [1,7]:

$$\mathrm{N}[f(t)\mathrm{H}(t)] = \mathrm{N}^{+}[f(t)] = R^{+}(s,u) = \int_0^{\infty} e^{-st} f(ut)dt \;\; ; \;\; s,u \in (0,\infty) \qquad (3.2)$$

(iii) If $u = 1$, Equation (3.2) can be reduced to the Laplace transform and if $s = 1$, then Equation (3.2) can be reduced to the Sumudu transform. Now, we give some of the N-Transforms and the conversion to Sumudu and Laplace [13,16].

Table 1 : Special N-Transforms and the conversion to Sumudu and Laplace.

$f(t)$	$\mathrm{N}[f(t)]$	$\mathrm{S}[f(t)]$	$\ell[f(t)]$
1	$\frac{1}{s}$	1	$\frac{1}{s}$
t	$\frac{u}{s^2}$	u	$\frac{1}{s^2}$
e^{at}	$\frac{1}{s-au}$	$\frac{1}{1-au}$	$\frac{1}{s-a}$
$\frac{t^{n-1}}{(n-1)!}, n = 1,2,\ldots$	$\frac{u^{n-1}}{s^n}$	u^{n-1}	$\frac{1}{s^n}$
$\cos t$	$\frac{s}{s^2+u^2}$	$\frac{1}{1+u^2}$	$\frac{s}{1+s^2}$
$\sin t$	$\frac{u}{s^2+u^2}$	$\frac{u}{1+u^2}$	$\frac{1}{1+s^2}$

Some basic properties of the N-Transform are given as follows [13,16]:

(i) If $R(s,u)$ is the natural transform and $F(s)$ is the Laplace transform of the function $f(t)$, then

$$\mathrm{N}^{+}[f(t)] = R(s,u) = \frac{1}{u}\int_0^{\infty} e^{-\frac{st}{u}} f(t)dt = \frac{1}{u}F\left(\frac{s}{u}\right).$$

(ii) If $R(s,u)$ is the natural transform and $G(u)$ is the Sumudu transform of the function $f(t)$, then

$$\mathrm{N}^{+}[f(t)] = R(s,u) = \frac{1}{s}\int_0^{\infty} e^{-t} f\left(\frac{ut}{s}\right)dt = \frac{1}{s}G\left(\frac{u}{s}\right).$$

(iii) If $\mathrm{N}^{+}[f(t)] = R(s,u)$, then $\mathrm{N}^{+}[f(at)] = \frac{1}{a}R(s,u)$.

(iv) If $\mathrm{N}^{+}[f(t)] = R(s,u)$, then $\mathrm{N}^{+}[f'(t)] = \frac{s}{u}R(s,u) - \frac{f(0)}{u}$.

(v) If $\mathrm{N}^{+}[f(t)] = R(s,u)$, then $\mathrm{N}^{+}[f''(t)] = \frac{s^2}{u^2}R(s,u) - \frac{s}{u^2}f(0) - \frac{f'(0)}{u}$.

(vi) Linearity property [16]: If a and b are non-zero constants, and $f(t)$ and $g(t)$ are functions, then $N^{+}[af(t)\pm bg(t)] = a\,N^{+}[f(t)]\pm b\,N^{+}[g(t)] = a\,F^{+}(s,u)\pm b\,G^{+}(s,u)$.

Moreover, $F^{+}(s,u)$ and $G^{+}(s,u)$ are the N-transforms of $f(t)$ and $g(t)$ respectively.

APPLICATIONS

Example 4.1: Here, we consider the following (1+1)-dimensional nonlinear Boussinesq equation:

$$\begin{aligned} v_t + w_x + vv_x &= 0 \\ w_t + (wv)_x + v_{xxx} &= 0 \end{aligned} \qquad (4.1)$$

Subject to the initial conditions

$$\begin{aligned} v(x,0) &= 2x; \\ w(x,0) &= x^2 \end{aligned} \qquad (4.2)$$

We first take the N-Transform of Equation (4.1), to obtain

$$\begin{aligned} N^{+}[v_t] + N^{+}[w_x] + N^{+}[vv_x] &= 0 \\ N^{+}[w_t] + N^{+}[(wv)_x] + N^{+}[v_{xxx}] &= 0 \end{aligned}$$

Using the properties in Table 1 and properties of the N-Transform, we have

$$\begin{aligned} \frac{s}{u}v(x,s,u) - \frac{v(x,0)}{u} &= -N^{+}[w_x + vv_x] \\ \frac{s}{u}w(x,s,u) - \frac{w(x,0)}{u} &= -N^{+}[(wv)_x + v_{xxx}] \end{aligned} \qquad (4.3)$$

Substituting Equation (4.2) into Equation (4.3), we have:

$$\begin{aligned} v(x,s,u) &= \frac{2x}{s} - \frac{u}{s}N^{+}[w_x + vv_x] \\ w(x,s,u) &= \frac{x^2}{s} - \frac{u}{s}N^{+}[(wv)_x + v_{xxx}] \end{aligned} \qquad (4.4)$$

Now , taking the inverse N-Transform of Equation (4.4), we have

$$\begin{aligned} v(x,t) &= 2x - N^{-1}\left[\frac{u}{s}N^{+}[w_x + vv_x]\right] \\ w(x,t) &= x^2 - N^{-1}\left[\frac{u}{s}N^{+}[(wv)_x + v_{xxx}]\right] \end{aligned} \qquad (4.5)$$

At this stage, we assume an infinite series solutions of the unknown functions v(x,t) and w(x,t) of the form:

$$\begin{aligned} v(x,t) &= \sum_{n=0}^{\infty} v_n(x,t) \\ w(x,t) &= \sum_{n=0}^{\infty} w_n(x,t) \end{aligned} \qquad (4.6)$$

Then Equation (4.5) can easily be written in the form

$$\begin{aligned} v(x,t) &= 2x - N^{-1}\left[\frac{u}{s}N^{+}\left[\sum_{n=0}^{\infty} w_{nx} + \sum_{n=0}^{\infty} A_n(v)\right]\right] \\ w(x,t) &= x^2 - N^{-1}\left[\frac{u}{s}N^{+}\left[\sum_{n=0}^{\infty} B_n(w,v) + \sum_{n=0}^{\infty} v_{nxxx}\right]\right] \end{aligned} \qquad (4.7)$$

where A_n and B_n are Adomian polynomials representing the nonlinear terms vv_x and $(wv)_x$ respectively.

Then by applying the natural decomposition techniques, we can generate the recursive relation as follows:

$$v_0(x,t) = 2x \qquad (4.8)$$

$$v_1(x,t) = \mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}\left[w_{0x} + A_0(v)\right]\right]$$

$$v_2(x,t) = \mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}\left[w_{1x} + A_1(v)\right]\right]$$

Thus,

$$v_{n+1}(x,t) = \mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}\left[w_{nx} + A_n(v)\right]\right] \quad n \geq 0 \qquad (4.9)$$

Similarly,

$$w_0(x,t) = x^2 \qquad (4.10)$$

$$w_1(x,t) = \mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}\left[B_0(w,v) + v_{0xxx}\right]\right]$$

$$w_2(x,t) = \mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}\left[B_1(w,v) + v_{1xxx}\right]\right]$$

Therefore

Eventually,

$$w_{n+1}(x,t) = \mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}\left[B_n(w,v) + v_{nxxx}\right]\right] \quad n \geq 0 \qquad (4.11)$$

from the recursive relation derived in Equation (4.9) and Equation (4.11) we can compute the remaining components of the solution as follows:

$$\begin{aligned} v_1(x,t) &= \mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}\left[w_{0x} + A_0(v)\right]\right] \qquad (4.12) \\ &= \mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}\left[w_{0x} + v_0 v_{0x}\right]\right] \\ &= \mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}\left[(6x)\right]\right] \\ &= 6x\,\mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}[1]\right] = 6xt. \end{aligned}$$

$$\begin{aligned} w_1(x,t) &= \mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}\left[(B_0(w,v))_x + v_{0xxx}\right]\right] \qquad (4.13) \\ &= \mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}\left[(w_0 v_0)_x + v_{0xxx}\right]\right] \\ &= \mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}\left[6x^2\right]\right] = 6x^2 t. \end{aligned}$$

and

$$v_2(x,y,t) = \mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}\left[w_{1x} + A_1(v)\right]\right] \qquad (4.14)$$
$$= \mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}\left[w_{1x} + v_1 v_{0x} + v_0 v_{1x}\right]\right]$$
$$= \mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}\left[36xt\right]\right]$$
$$= 36x\,\mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}\left[t\right]\right] = 36x\,\mathrm{N}^{-1}\left[\frac{u^2}{s^3}\right]$$
$$= 36x\frac{t^2}{2!}.$$

$$w_2(x,t) = \mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}\left[B_1(v,w) + v_{1xxx}\right]\right] \qquad (4.15)$$
$$= \mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}\left[v_1 w_{0x} + v_0 w_{1x} + w_1 v_{0x} + w_0 v_{1x} + v_{1xxx}\right]\right]$$
$$= \mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}\left[54x^2 t\right]\right]$$
$$= 54x^2\,\mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}\left[t\right]\right] = 54x^2\,\mathrm{N}^{-1}\left[\frac{u^2}{s^3}\right]$$
$$= 54x^2\frac{t^2}{2!} = 27x^2t^2.$$

Eventually the approximate solution of the unknown functions v(x,t) and w(x,t) are given by:

$$v(x,t) = \sum_{n=0}^{\infty} v_n(x,t) \qquad (4.16)$$
$$= v_0(x,t) + v_1(x,t) + v_2(x,t) + \ldots$$
$$= 2x - 6xt - 36x\frac{t^2}{2!} - \ldots$$
$$= \frac{2x}{1+3t}.$$

And

$$w(x,t) = \sum_{n=0}^{\infty} w_n(x,t) \qquad (4.17)$$
$$= w_0(x,t) + w_1(x,t) + w_2(x,t) + \ldots$$
$$= x^2 - 6x^2t - 27x^2t^2 - \ldots$$
$$= \frac{x^2}{(1+3t)^2}.$$

Hence, the exact solutions of the given (1+1)-dimensional nonlinear Boussinesq equation are given by:

$$v(x,t)=\frac{2x}{1+3t}$$
$$w(x,t)=\frac{x^2}{(1+3t)^2} \qquad (4.18)$$

Example 4.2: we consider the following Broer-Kaup (BK) system of equations:

$$v_t+vv_x+w_x=0$$
$$w_t+v_x+(vw)_x+v_{xxx}=0 \qquad (4.19)$$

Subject to the initial conditions

$$v(x,0)=1+2\tanh x;$$
$$w(x,0)=1-2\tanh^2 x \qquad (4.20)$$

We first take the N-Transform of Equation (4.19), to obtain

$$\mathrm{N}^+[v_t]+\mathrm{N}^+[vv_x]+\mathrm{N}^+[w_x]=0$$
$$\mathrm{N}^+[w_t]+\mathrm{N}^+[v_x]+\mathrm{N}^+[(vw)_x]+\mathrm{N}^+[v_{xxx}]=0$$

Using the properties in Table 1 and properties of the N-Transform, we have

$$\frac{s}{u}v(x,s,u)-\frac{v(x,0)}{u}=-\mathrm{N}^+[vv_x+w_x]$$
$$\frac{s}{u}w(x,s,u)-\frac{w(x,0)}{u}=-\mathrm{N}^+[v_x+(vw)_x+v_{xxx}] \qquad (4.21)$$

Substituting Equation (4.20) into Equation (4.21), we have:

$$v(x,s,u)=\frac{1+2\tanh x}{s}-\frac{u}{s}\mathrm{N}^+[vv_x+w_x]$$
$$w(x,s,u)=\frac{1-2\tanh^2 x}{s}-\frac{u}{s}\mathrm{N}^+[v_x+(vw)_x+v_{xxx}] \qquad (4.22)$$

Now , taking the inverse N-Transform of Equation (4.4), we have

$$v(x,t)=1+2\tanh x-\mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^+[vv_x+w_x]\right]$$
$$w(x,t)=1-2\tanh^2 x-\mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^+[v_x+(vw)_x+v_{xxx}]\right] \qquad (4.23)$$

At this stage, we assume an infinite series solutions of the unknown functions v(x,t) and w(x,t) of the form:

$$v(x,t)=\sum_{n=0}^{\infty}v_n(x,t)$$
$$w(x,t)=\sum_{n=0}^{\infty}w_n(x,t) \qquad (4.24)$$

Then Equation (4.23) can easily be written in the form

$$v(x,t)=1+2\tanh x-\mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^+\left[\sum_{n=0}^{\infty}A_n(v)+\sum_{n=0}^{\infty}w_{nx}\right]\right]$$
$$w(x,t)=1-2\tanh^2 x-\mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^+\left[\sum_{n=0}^{\infty}v_{nx}+\sum_{n=0}^{\infty}B_n(v,w)+\sum_{n=0}^{\infty}v_{nxxx}\right]\right] \qquad (4.25)$$

where A_n and B_n are Adomian polynomials representing the nonlinear terms vv_x and $(vw)_x$ respectively.

Then by applying the natural decomposition techniques, we can generate the recursive relation as follows:

$$v_0(x,t) = 1 + 2\tanh x \qquad (4.26)$$

$$v_1(x,t) = \mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}\left[A_0(v) + w_{0x}\right]\right]$$

$$v_2(x,t) = \mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}\left[A_1(v) + w_{1x}\right]\right]$$

Thus,

$$v_{n+1}(x,t) = \mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}\left[A_n(v) + w_{nx}\right]\right] \quad n \geq 0 \qquad (4.27)$$

Similarly,

$$w_0(x,t) = 1 - 2\tanh^2 x \qquad (4.28)$$

$$w_1(x,t) = \mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}\left[v_{0x} + B_0(w,v) + v_{0xxx}\right]\right]$$

$$w_2(x,t) = \mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}\left[v_{1x} + B_1(w,v) + v_{1xxx}\right]\right]$$

Eventually,

$$w_{n+1}(x,t) = \mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}\left[v_{nx} + B_n(w,v) + v_{nxxx}\right]\right] \quad n \geq 0 \qquad (4.29)$$

Therefore from the recursive relation derived in Equation (4.27) and Equation (4.29) we can compute the remaining components of the solution as follows:

$$v_1(x,t) = \mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}\left[A_0(v) + w_{0x}\right]\right] \qquad (4.30)$$

$$= \mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}\left[v_0 v_{0x} + w_{0x}\right]\right]$$

$$= \mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}\left[\left(2\sec h^2 x\right)\right]\right]$$

$$= 2\sec h^2 x\, \mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}[1]\right] = 2t\sec h^2 x.$$

$$w_1(x,t) = \mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}\left[v_{0x} + B_0(v,w) + v_{0xxx}\right]\right] \qquad (4.31)$$

$$= \mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}\left[v_{0x} + (v_0 w_0)_x + v_{0xxx}\right]\right]$$

$$= \mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}\left[2\sec h^2 x + \left((1+2\tanh x)(1-2\tanh^2 x)\right)_x - 4\sec h^4 x + 8\tanh^2 x\sec h^2 x\right]\right]$$

$$= -4\tanh x\sec h^2 xt.$$

$$v_2(x,y,t) = \mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}\left[A_1(v) + w_{1x}\right]\right] \qquad (4.32)$$

$$= \mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}\left[v_1 v_{0x} + v_0 v_{1x} + w_{1x}\right]\right]$$

$$= \mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}\left[4\sec h^2 x\tanh xt\right]\right]$$

and

$$= 4\sec h^2 x\tanh x\,\mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}[t]\right] = 4\sec h^2 x\tanh x\,\mathrm{N}^{-1}\left[\frac{u^2}{s^3}\right]$$

$$= 4\sec h^2 x\tanh x\frac{t^2}{2!} = 2t^2\sec h^2 x\tanh x.$$

$$w_2(x,t) = \mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}\left[v_{1x} + B_1(v,w) + v_{1xxx}\right]\right] \qquad (4.33)$$

$$= \mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}\left[v_{1x} + v_1 w_{0x} + v_0 w_{1x} + w_1 v_{0x} + w_0 v_{1x} + v_{1xxx}\right]\right]$$

$$= \mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}\left[12t\sec h^4 x - 8t\sec h^2 x\right]\right]$$

$$= (12\sec h^4 x - 8\sec h^2 x)\,\mathrm{N}^{-1}\left[\frac{u}{s}\mathrm{N}^{+}[t]\right] = (12\sec h^4 x - 8\sec h^2 x)\,\mathrm{N}^{-1}\left[\frac{u^2}{s^3}\right]$$

$$= (12\sec h^4 x - 8\sec h^2 x)\frac{t^2}{2!} = \left(6\sec h^4 x - 4\sec h^2 x\right)t^2.$$

Eventually the approximate solution of the unknown functions v(x,t) and w(x,t) are given by:

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$$v(x,t)=\sum_{n=0}^{\infty} v_n(x,t) \qquad (4.34)$$
$$=v_0(x,t)+v_1(x,t)+v_2(x,t)+\ldots$$
$$=1+2\tanh x-2t\sec h^2 x-2t^2\sec h^2 x\tanh x-\ldots$$

And

$$w(x,t)=\sum_{n=0}^{\infty} w_n(x,t) \qquad (4.35)$$
$$=w_0(x,t)+w_1(x,t)+w_2(x,t)+\ldots$$
$$=1-2\tanh^2 x+4t\sec h^2 x\tanh x+4t^2\sec h^2 x-6t^2\sec h^4 x+\ldots$$

The approximate solution converges rapidly to the exact solution of the Broer-Kaup (BK) system of equations given by :

$$v(x,t)=1-2\tanh(t-x)$$
$$w(x,t)=1-2\tanh^2(t-x).$$

1.3 Table of Calculations.

In this section, we shall illustrate the accuracy and efficiency of the NDM by comparing the approximate and exact solutions.

Table1. Numerical Calculations for Broer-Kaup (BK) system of equations using NDM for different values of x,t.

x	t	$v(x,t)$ *Exact*	$v(x,t)$ *Approx*	$w(x,t)$ *Exact*	$w(x,t)$ *Approx*
-1	.002	-0.5248656521	-0.557896063	-0.1626076285	-0.154769675
	.004	-0.5265378847	-0.5904789092	-0.1651589567	-0.267441272
	.006	-0.5282050196	-0.622538456	-0.1677052910	-0.3223911288
	.01	-0.5315240365	-0.5404073992	-0.1727829372	-0.1865850859
-0.5	.002	0.0726228035	0.0726169870	0.5699857677	0.5758006529
	.004	0.0694857454	0.0569026089	0.5670716110	0.5561166838
	.006	0.0663545110	0.04773750416	0.5641530588	0.5484648803
	.01	0.0601096021	0.07098330791	0.5583030199	0.5685163735
0.5	.002	1.9210856177	1.92327251	0.5758006425	0.573786036
	.004	1.9179311150	1.904660749	0.5787012341	0.5916412341
	.006	1.9147708158	1.894607532	0.5815971773	0.6017515758
	.01	1.9084328654	1.919407532	0.5873748646	0.5773974894
1	.002	2.5215058541	2.51971278	-0.1574900320	-0.1547821537
	.004	2.5198182687	2.451669043	-0.1549237850	-0.0560077068
	.006	2.5181255459	2.414323657	-0.1523525866	-0.0052407811
	.01	2.5147246484	2.492622481	-0.1471953803	-0.1337266946

Table2. Comparison of absolute errors of Broer-Kaup system of equations using NDM.

x	t	$Error\,(NDM)$ $v(x,t)$	$Error\,(NDM)$ $w(x,t)$
-1	.002	$3.30304109E^{-2}$	$7.8379535E^{-3}$
	.004	$6.39410245E^{-2}$	$1.022823153E^{-1}$
	.006	$9.43334364E^{-2}$	$1.546858378E^{-1}$
	.01	$8.8833627E^{-3}$	$1.38021487E^{-2}$
-0.5	.002	$5.8165E^{-6}$	$5.8148852E^{-3}$
	.004	$1.25831365E^{-2}$	$1.09549272E^{-2}$
	.006	$1.861700684E^{-2}$	$1.56881785E^{-2}$
	.01	$1.087370581E^{-2}$	$1.02133536E^{-2}$
0.5	.002	$2.1868923E^{-3}$	$2.0146065E^{-3}$
	.004	$1.3270366E^{-2}$	$1.294E^{-2}$
	.006	$1.0974666E^{-2}$	$2.01543985E^{-2}$
	.01	$1.0974666E^{-2}$	$9.9773752E^{-3}$
1	.002	$1.7930741E^{-3}$	$2.7078783E^{-3}$
	.004	$6.81492257E^{-2}$	$9.89160782E^{-2}$
	.006	$1.038018889E^{-1}$	$1.471118055E^{-1}$
	.01	$2.21021674E^{-1}$	$1.34686857E^{-2}$

IV. CONCLUSION

In this article, the Natural Decomposition Method (NDM) was implemented for solving (1+1)-dimensional nonlinear Boussinesq Equation and the Broer-Kaup (BK) system of equations.

We successfully found an exact solution for the (1+1)- dimensional nonlinear Boussinesq Equation and approximate solution for the Broer-Kaup (BK) system of equations. The approximate results were in good agreement with the exact solution. The NDM introduces significant improvement in the field over existing techniques as demonstrated by lesser calculations and fewer iterations.

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