Quest Journals
Journal of Research in Applied Mathematics
Volume 3 ~ Issue 3 (2016) pp: 05-11
ISSN(Online) : 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Research Paper

K- Monophonic Number of A Connected Graph

P. Arul Paul Sudhahar[1], M. Mohammed Abdul Khayyoom[2]
[1]*Department of Mathematics, Rani Anna Govt. College (W), Tirunalveli-620 008, Tamilnadu, India*
arulpaulsudhahar@gmail.com
[2]*HSST Mathematics, GHSS Pandikkad, Malappuram-676521, Kerala, India* khayyoom.m@gmail.com

Received 28 Nov, 2016; **A**ccepted 20 Dec, 2016 © The author(s) 2016. **P**ublished with open access at **www.questjournals.org**

ABSTRACT: *In this paper the concept of k-monophonic set and k-monophonic number of a connected graph were introduced. For a connected graph G of order $n \geq 2$, a set $M \subseteq V$ is a k-monophonic set of G if each vertex $x \in (V(G) - M)$ lies on a u-v monophonic path of length k, for some vertices u and v in M. The minimum cardinality of a k-monophonic set in G is the k-monophonic number of G, denoted by $m_k(G)$. The 2-monophonic sets and 2-monophonic numbers of Cartesian product graphs were studied. For each pair k, n of integers with $4 \leq k \leq n$, there is a connected graph G of order n such that $m_2(G \times K_2) = k$. Also, k-monophonic numbers of certain standard graphs were identified.*

Keywords: *2- monophonic number, k- monophonic number, k-monophonic set, Monophonic number.*

I. INTRODUCTION

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. For vertices u and v in a connected graph G, the distance $d(u,v)$ is the length of a shortest $u - v$ path in G. A $u - v$ path of length $d(u,v)$ is called a $u - v$ *geodesic*. The set of all vertices u, that are adjacent with v is called the neighbourhood of v and is denoted by $N(v)$. If the sub graph induced by the neighborhood of a vertex is complete, that vertex is called *extreme vertex*. A vertex of degree one is called *end vertex*. Every end vertices are extreme vertices [5, 8].

A *chord* of a path P: $u_1, u_2 \ldots u_n$ is an edge $u_i u_j$ with $j \geq i + 2$. Any chordless path connecting u and v are called $u - v$ *monophonic path*. The monophonic closure of a subset M of $V(G)$ is $J_G[M] = \cup_{u,v \in M} J_G[u,v]$ where $J_G[u,v]$ is the set containing u and v and all vertices lying in some $u - v$ monophonic path. If $J_G[M] = V(G)$, then M is called *monophonic set* in G. The order of a minimum monophonic set is called *monophonic number* and is denoted by $m(G)$ [1, 2, 3, 7].

An integer $k \geq 1$, is a *geodesic* in a connected graph G of length k is called a *k-geodesic*. A vertex v is called a *k-extreme vertex* if v is not the internal vertex of a k-geodesic joining any pair of distinct vertices of G. Each extreme vertex of a connected graph G is a k-extreme vertex of G. Each end vertex of G is a k-extreme vertex of G. A set $S \subseteq V$ is called a *k-geodetic set* of G if each vertex in $V - S$ lies on a k-geodesic of vertices in S. The minimum cardinality of a k-geodetic set of G is its *k-geodetic number* and is denoted by $g_k(G)$. There are interesting applications of these concepts to the problem of designing the route for a shuttle and communication network design [6, 10].

For two graphs G and H, their Cartesian product is denoted by $G \times H$, and has the vertex set $V(G) \times V(H)$, where two distinct vertices (x_1, y_1) and (x_2, y_2) in $G \times H$ are adjacent if and only if either $x_1 = x_2$ and $y_1 y_2 \in E(H)$, or $y_1 = y_2$ and $x_1 x_2 \in E(G)$. The mappings $\pi(G) : V(G \times H) \rightarrow V(G)$ and $\pi(H) : V(G \times H) \rightarrow V(H)$ defined by $\pi_G(x,y) = x$ and $\pi_H(x,y) = y$, respectively for all $(x,y) \in V(G \times H)$ are called *projections.* For a set $S \subseteq V(G \times H)$, define the projection of S on G as $\pi_G(S) = \{x \in V(G) : (x,y) \in S$ for some $y \in V(H)\}$ and the projection of S on H as $\pi_H(S) = \{y \in V(H) : (x,y) \in S$ for some $x \in V(G)\}$.[9]

The *eccentricity* $e(v)$ of a vertex v is the greatest geodesic distance between v and any other vertex. The *centre* (or Jordan centre) of a graph is the set of all vertices of minimum eccentricity, that is, the set of all

*Corresponding Author: M. Mohammed Abdul Khayyoom
HSST Mathematics, GHSS Pandikkad, Malappuram-676521, Kerala, India

vertices A where the greatest distance $d(A,B)$ to other vertices B is minimal. For basic graph theory notations, refer [4].

II. K-MONOPHONIC NUMBER OF A GRAPH

Definition 2.1 Let G be connected graph of order $n \geq 2$. For an integer $k \geq 1$, a vertex $v \in V(G)$ is *k-monophonic* by a pair $x, y \in V$ if v lies on an $x - y$ monophonic path of length k in G. The minimum cardinality of a k-monophonic set of G is the *k-monophonic number* of G and it is denoted by $m_k(G)$. That set is called *minimum k-monophonic set* or *k-m set.*

Example 2.1 Consider the following graph given in *figure 01*. The minimum k-monophonic set and k-monophonic number of G for different values of k is shown in the following table (Table 1).

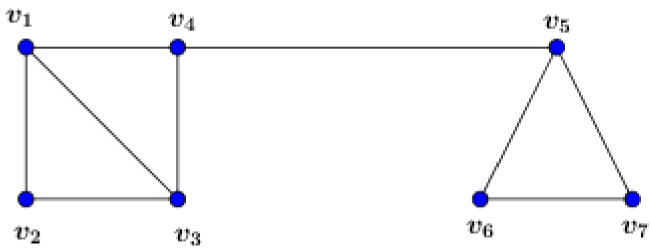

Fig. 1. Graph G with $m_k(G) = 3,4$ *and* 7

k	Minimum k-monophonic set	$m_k(G)$
4	$\{v_2, v_6, v_7\}$	3
3	$\{v_2, v_5, v_6, v_7\}$	4
2	$\{v_2, v_4, v_6, v_7\}$	4
1	$\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$	7

Table 1

Theorem 2.1 Let G be a connected graph. Then each vertex of G belongs to every 1-monophonic set of G
Proof: Let M be a 1-monophonic set of G and $u \in V(G)$ such that $u \notin M$.Then u lies in a $x - y$ monophonic path of length 1 such that both x and y are in M. Then u is either x or y which implies that $u \in M$, a contradiction. Thus $M = V(G)$

Corollary 2.2 If $G = K_p$, the complete graph with p vertices. Then G has only 1-monophonic set and $m_k(G) = p$.
Proof: Since any two vertices of G are adjacent maximum length of any monophonic path is 1. Thus G has only 1-monophonic set. Hence by theorem 2.1 $m_k(G) = p$.

Theorem 2.3 Each extreme vertex belongs to every k-monophonic set.
Proof: Let u be an extreme vertex and M be a k-monophonic set of G. Suppose $u \notin M$. By theorem 2.1, $k \neq 1$.Then u is an internal vertex of an $x - y$ monophonic path P having length k. Let v and w are neighbours of u in P. Then they are not adjacent and u is not an extreme vertex which is a contradiction to the hypothesis. Thus $u \in M$.

Theorem 2.4 Let G be a connected graph and y be a cut vertex. If M is a k-monophonic set, then every component of G contains at least one element of M.
Proof: Let there is a component A of $G - y$ such that A has no vertex of M. Let x be any vertex in A. Since M is a k-monophonic set, there is a pair of vertices u and v in M such that x lies on some pair of $u - v$ k-monophonic path $S: u = x_0, x_1, x_2, \ldots x \ldots, x_n = v$ with $x \neq u, v$. Since y is a cut vertex of G, the $u - x$ sub path S_1 of S and the $x - v$ sub path S_2 of S both contain y, so that S is not a path. This contradicts A has no vertex of M.

Remarks 2.1 No cut vertex of a connected graph G belongs to any minimum monophonic set. (see [10]). Generally, this is not true for k-monophonic sets. Consider the path graph P_4 of four vertices in *Figure 2*. Its $m_k(G)$is given in Table-2. Here v_2 and v_3 are cut vertices.

Theorem 2.5 For any connected $G, 2 \leq m_k(G) \leq n$.
Proof: Any k-monophonic path contains at least two vertices. Therefore $m_k(G) \geq 2$. Since $V(G)$ is always a k-monophonic set for any k, $m_k(G) \leq n$.

Fig. 2. Path graph P_4 with $m_k(P_5)$= 2,3 and 4

k	Minimum k-monophonic set	$m_k(G)$
1	$V(G)$	4
2	$\{v_1, v_3, v_4\}$	3
3	$\{v_1, v_4\}$	2

Table 2

Theorem 2.6 For any connected graph G of order $n, 2 \leq m_k(G) \leq g_k(G) \leq n$.
Proof: Since every k-geodetic path is also a k-monophonic path, every minimum k-monophonic set is also a k-geodetic set. Thus $m_k(G) \leq g_k(G)$. Other inequalities are trivial from theorem 2.5.

Example 2.2(a) Consider the following graph G in *Figure 03*. Note that $g_k(G)$ is not defined for some k where $m_k(G)$ is defined and is equal to 4. (Refer Table 3)

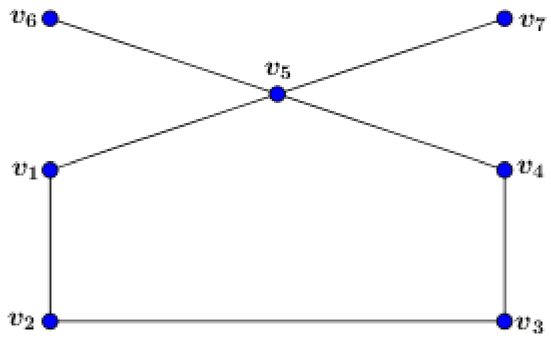

Fig. 3 Graph G with $m_k(G) = 4$ but $g_k(G)$ is not defined for $k = 4$.

k	Minimum k-geodetic set	$g_k(G)$	Minimum k-monophonic set	$m_k(G)$
1	$V(G)$	7	$V(G)$	7
2	$\{v_1, v_2, v_4, v_6, v_7\}$	5	$\{v_1, v_2, v_4, v_6, v_7\}$	5
3	$\{v_2, v_3, v_6, v_7\}$	4	$\{v_2, v_3, v_6, v_7\}$	4
4	Not defined		$\{v_2, v_3, v_6, v_7\}$	4

Table 3

Example 2.2(b) Let G be the complete bipartite graph $K_{m,n}$. Then

$$m_k(K_{m,n}) = \begin{cases} m+n, & \text{if } \quad k = 1 \\ \min\{m,n\}, & \text{if } \quad k \neq 1 \end{cases}$$

Proof: For k =1, the result follows by theorem 2.1, since G has $m + n$ vertices. For $k \neq 1$, maximum length of any monophonic path is 2. For, let $A = \{u_1, u_2, u_3 \ldots u_m\}$ and $B = \{v_1, v_2, \ldots, v_n\}$ are two patricians of G.Then any path $u_i v_j u_k v_t$ of length three does not form a monophonic path. Thus every k-monophonic set is a 2-monophonic set. Then both A and B are monophonic set [7] and minimum of $\{m,n\}$is a minimum 2-monophonic set.

Theorem 2.7: Let $G = C_n$, cycle graph of n vertices. If n is even, there exist some k such that $m_k(G) = 2$. If n is odd, then any k-monophonic set of G contains at least three vertices.
Proof: Let G be the cycle graph of n vertices with closed walk $C : v_1, v_2, \ldots, v_n, v_1$. If n is even, then $n = 2p$ for some p. Consider the set $M = \{v_1, v_{p+1}\}$. Then M is a p-monophonic set. Take $k = p$, then the first part is clear. Let n is odd. On the contrary suppose there is a k-monophonic set M with at most two vertices for some k,

$1 \leq k \leq n$. Since any k-monophonic set contains at least two vertices, M contains exactly two elements, say v_i and v_j. Since v_i and v_j lies in a cycle, there exist two paths P_1 and P_2 connecting v_i and v_j. Since M is a monophonic set, we have $i \geq j+2$ or $i \leq j+2$. Now given n is odd. There for P_1 and P_2 are not of the same length. Thus the vertex v_{i+1} or v_{i-1} does not lies any k- monophonic path connecting v_i and v_j, contradicts the hypothesis that M is a k-monophonic set. Thus M contains at least three vertices.

III 2-MONOPHONIC SETS IN $G \times K_2$

Let G be a non-trivial connected graph. Take H_1 and H_2 as two copies of G such that v_1v_2 is an edge of $G \times K_2$ and $v_i \in V(H_i)$ for $i = 1,2$. Note that, if both u and v are in $V(H_i)$ for $i = 1,2$ then there are minimum $u - v$ monophonic path that completely lies in H_i. Thus:

Lemma 3.1 Let G be a non-trivial connected graph and let H_1 and H_2 as two copies of G. If M is a 2-monophonic set in $G \times K_2$, then $M \cap V(H_1)$ and $M \cap V(H_2)$ are non-empty.

Theorem 3.2 If G is a connected graph of order $n \geq 4$, then $4 \leq m_2(G \times K_2) \leq n$.

Proof: First, prove $4 \leq m_2(G \times K_2)$. On the contrary assume that there is a connected graph G of order $n \geq 4$ such that $m_2(G \times K_2) \leq 3$. Then $G \times K_2$ contain a 2-monophonic set M of cardinality 3, say $\{a, b, c\}$. In $G \times K_2$ let H_1 and H_2 are two copies of G with $V(H_1) = \{u_1, u_2, u_3 \ldots u_n\}$ and $V(H_2) = \{v_1, v_2, \ldots, v_n\}$ so that u_iv_i is an edge in $G \times K_2$ for $1 \leq i \leq n$. By lemma 3.1 $M \cap V(H_1)$ and $M \cap V(H_2)$ are non-empty. Thus assume that $a, b \in V(H_1)$ and $c \in V(H_2)$, say $a = u_1, b = u_2$ and $c = v_i$ for $1 \leq i \leq n$. Since $n \geq 4$, the set $\{1,2, \ldots, n\} - \{1,2, i\}$ is non-empty and let j belongs to this set. Then $v_j \notin M$. Now v_j is 2-monophonic by u_j, v_i and u_j are not in M which follows that v_j does not lie on 2-monophonic path of any vertices of M and it is a contradiction. Thus $m_2(G \times K_2) \geq 4$.

Next, prove $m_2(G \times K_2) \leq n$. Let $diam\, G, d \geq 2$ and let $u_1 \in V(H_1)$ such that $e(u_1) = d$. Let $v_1 \in V(H_2)$ such that v_1 corresponds to u_1 in $G \times K_2$. For each integer $1 \leq i \leq d$, let $X_i = \{x \in V(H_1): d(u_1, x) = i\}$ and $Y_i = \{y \in V(H_2): d(v_1, y) = i\}$. Then $X_0 = \{u_1\}$ and $Y_0 = \{v_1\}$. Take the set M as the union of the sets $X_0, X_2 \ldots X_d, Y_1, Y_3 \ldots Y_{d-1}$ if d is even and union of the sets $X_0, X_2 \ldots X_{d-1}$, $Y_1, Y_3 \ldots Y_d$, if d is odd. Then M is a 2-monophonic set of $G \times K_2$. Let $v \in V(G \times K_2) - M$. If d is even, then either $v \in X_i$ for odd i or $v \in V_j$ for even j. Suppose the first. Let v' be the vertex of H_2 that corresponds to v in $G \times K_2$ and so $v' \in V_i \subseteq M$. Let u be a vertex that is either in X_{i-1} or in X_{i+1} such that u is adjacent to v. Then $u \in M$ by the definition of M and v is 2-monophonic by v' and u. Thus M is a 2-monophonic set of $G \times K_2$ if d is even. Similarly M is 2-monophonic when d is odd. Thus, $m_2(G \times K_2) \leq n$. Hence the theorem.

Example 3.1 Consider the product graph $C_4 \times K_2$ given in *Figure 04*. Its minimum 2-monophonic set contains four elements. That is $m_2(G \times K_2) = 4$. The sets $\{u_1, u_3, v_2, v_4\}$ or $\{u_2, u_4, v_1, v_3\}$ are minimum 2-monophonic set of $C_4 \times K_2$.

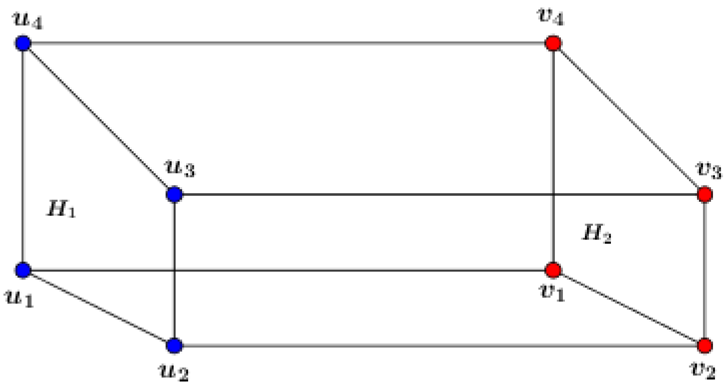

Fig 4. Graph of $C_4 \times K_2$ with $m_2(G) = 4$

Note that $m_2(K_n \times K_2) = n$ for all $n \geq 2$. That is the upper bond in the above theorem is sharp. The next results show the lower bond is also sharp.

Theorem 3.3 Let H_1 and H_2 are two copies of $K_{m,n}$ where $2 \leq m \leq n$ in $K_{m,n} \times K_2$. If M is a 2-monoponic set of $K_{m,n} \times K_2$ then $M \cap V(H_1)$ and $M \cap V(H_2)$ contains at least two vertices.

Proof: On the contrary suppose $M \cap V(H_1)$ and $M \cap V(H_2)$ contain at most one element. By lemma 3.1 they contain exactly one element. Let $U = \{u_1, u_2, u_3 \ldots u_m\}$ and $V = \{v_1, v_2, \ldots, v_n\}$ be two patricians of H_1. Thus $M \cap V(H_1) = \{u_i\}$, $1 \le i \le m$ or $M \cap V(H_2) = \{v_j\}$ for $1 \le j \le n$. If $M \cap V(H_1) = \{u_i\}$, then $U - \{u_i\}$ is not a 2-monophonic set which leads to a contradiction. Similarly the second case also leads to a contradiction..

Theorem 3.4 If $G = K_{m,n}$, $1 \le m \le n$, then

$$m_2(G \times K_2) = \begin{cases} m+n & \text{if } m = 1 \\ \min\{2m, 8\} & \text{if } m \ne 1, 4, 5 \\ m+2 & \text{if } m = 4,5 \end{cases}$$

Proof: Let $G = K_{m,n}$ and let H_1 and H_2 are two copies of $K_{m,n}$. Let $U = \{u_1, u_2, u_3 \ldots u_m\}$ and $V = \{v_1, v_2, \ldots, v_n\}$ be two particians of H_1 and $X = \{x_1, x_2 \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be the particians of H_2.

***Case 1*:** Let $m = 1$. Then u_1 and x_1 are respectively central vertices of H_1 and H_2. Take M as the m_2-set of $G \times K_2$. If $v_i \notin M$, $1 \le i \le n$, then v_i is 2-monophonic by u_1 and y_I and so u_1, y_i lies in M. Similarly if y_i is not in M then x_1, v_i are in M. Thus M contains at least one vertex from $\{u_1, x_1\}$ and each set $\{v_i, y_i\}$ for $1 \le i \le n$. Thus M contains at least $1 + n$ vertices. By theorem 3.2, $m_2(G \times K_2) = 1 + n = m + n$.

Case 2: For $m \ne 4,5$ take $M_1 = \{u_1, u_2, \ldots, v_1, v_2, x_1, x_2, y_1, y_2\}$ and $M_2 = U \cup X$ are 2-monophonic sets of $G \times K_2$, it follows that $m_2(G \times K_2) \le |M_1| = 8$ and $m_2(G \times K_2) = |U \cup X| = 2m$. Thus $m_2(G \times K_2) \le \min\{8, 2m\}$.

Next, it is enough to prove that $m_2(G \times K_2) \ge \min\{8, 2m\}$. On the contrary suppose $m_2(G \times K_2) < \min\{8, 2m\}$. Let M be a 2-monophonic set of $(G \times K_2)$ with $M = \min\{8, 2m\} - 1$. Then there exists three cases.

*Sub Case 1***:** Let $m = 2$. Then $\min\{8, 2m\} = 4$ implies M has three vertices. This contradicts theorem 3.2

Sub Case 2: Let $m = 3$. Then M contain six elements. By theorem 3.3 M contains two vertices from H_1 and H_2. There for $U - M \cap V(H_1)$ and $V - M \cap V(H_2)$ are non-empty. If $M \cap V(H_1) \subseteq U$ then no vertices in $U - M \cap V(H_1)$ can be 2-monophonic by M. Similarly if $M \cap V(H_2) \subseteq V$, then no vertices in $V - M \cap V(H_2)$ can be 2-monophonic. Thus $M \cap U$ and $M \cap V$ are non-empty. Suppose $M \cap V(H_1) = \{u_1, v_1\}$. Each vertex u_j is 2- monophonic by x_i and a vertex in V. There for $X - \{x_i\} \subseteq M$. Similarly $Y - \{y_i\} \subseteq M$ implies that M contains more than six vertices and is a contradiction.

Sub Case 3: Let $m \ge 6$. Then, $\min\{8, 2m\} = 8$. Therefore M contains seven vertices. Suppose M contains at most three vertices of H_1. Then as in sub case 2, M contains minimum 12 vertices which leads to a contradiction. Thus M contains exactly three vertices of H_1 and let it be $\{u_1, u_2, v_1\}$ or $\{u_1, v_1, v_2\}$. In first case $X - \{x_1, x_2\}$ lies in M. Also x_1 is 2-monophonic by u_1 and a vertex in Y and x_2 is 2-monophonic by u_2 and a vertex in Y. Thus either $x_1, x_2 \in M$ or there is a vertex $y \in Y$ that also in M. Then M contains at least eight elements that also leads to a contradiction. Similar arguments lead to a contradiction in the second case. Hence $m_2(G \times K_2) = \min\{8, 2m\}$ for $m \ne 4,5$.

Case 3: Let $m = 4$ or 5. First show that $m_2(G \times K_2) \le m + 2$. Take $M_1 = \{u_1, u_2, v_1, y_1\} \cup (X - \{x_1, x_2\}$. Then M_1 is a 2-monophonic set of $(G \times K_2)$ and $m_2(G \times K_2) \le 4 + (m - 2) = m + 2$. To prove the lower limits, consider two cases.

Sub Case A: Let $m = 4$. Then, $m_2(G \times K_2) \ge 6$. On the contrary let $m_2(G \times K_2) \le 5$. Take M as a 2-monophonic set with five vertices.Suppose M contains exactly two vertices of H_1. Then $M \cap U$ and $M \cap V$ containes common vertices. Let $M \cap V(H_1) = \{u_1, v_1\}$. Since each u_i is 2-monophonic by x_i and v_1, $X - \{x_i\}$ lies in M. Similarly $Y - \{y_i\}$ lies in M. Thus M containes more than two vertices of H_1.This is a contradiction.

Sub Case B: Let $m = 5$. Clearly $m_2(G \times K_2) \ge 7$. On the contrary suppose $m_2(G \times K_2) \le 6$. Let M be a 2-monophonic set of $(G \times K_2)$ with six vertices and suppose at most three vertices are from H_1. If M contains exactly two vertices of H_1, as in sub case A, M contains more than eight elements and is a contradiction. Hence M contains exactly three vertices of H_1. Since $M \cap U$ and $M \cap V$ are non - empty, assume $\{u_1, u_2, v_1\}$ or $\{u_1, v_1, v_2\}$ lies in $M \cap V(H_1)$. In first case, $M = \{u_1, u_2, v_1, x_3, x_4, x_5\}$ and x_1 and x_2 are not 2-monophonic by M. In second case, $M = \{u_1, v_1, v_2\} \cup Y - \{y_1, y_2\}$ and y_1 and y_2 are not 2-monophonic by M. This contradicts the fact that M is a 2- monophonic set. Hence $m_2(G \times K_2) = m + 2$ when $m = 4$ and 5.

Theorem 3.5: Let $n \ge 3$. If $G = P_n$ or C_n, then $m_2(G \times K_2) = n$

Proof: Theorem 3.2 gives the upper bond of $m_2(G \times K_2)$. That is $m_2(G \times K_2) \le n$ for $G = P_n$ or C_n. For $G = P_n$, take $V(H_1) = \{x_1, x_2, x_3 \ldots, x_n\}$ and $V(H_2) = \{y_1, y_2, \ldots, y_n\}$, two copies of G. It is enough to show that $m_2(G \times K_2) \ge n$. Consider two cases.

Case 1: If n is even. Let $n = 2p$, $p \ge 2$. Take M as m_2 set of $(G \times K_2)$. Then M contains at least two elements from the set $\{u_{2i-1}, u_{2i}, v_{2i-1}, v_{2i}\}$. If not, M contains at most one vertex. Suppose $\{u_{2i-1}, u_{2i}, v_{2i-1}\}$

are not in M or $\{u_{2i-1}, u_{2i}, v_{2i}\} \notin M$. In first case u_{2i-1} does not lies in 2-monophonic set and in second case v_{2i-1} is not 2-monophonic. Thus there is a contradiction. So M contains at least two elements as desired. Hence $m_2(G \times K_2) = n$

Case 2: If n is odd, then $n-1$ is even. Take $n-1=2p$. If M is a 2-monophonic set as in case 1 it contains at least two vertices from the sets $\{u_{2i-1}, u_{2i}, v_{2i-1}, v_{2i}\}$ for each $i, 1 \le i \le p$. Clearly, u_n is 2-monophonic by u_{n-1} and v_n and v_n by u_n and v_{n-1}. There for M contains at least one vertex from $\{u_n, v_n\}$. Hence, $m_2(G \times K_2) = n$. The proof of $G = C_n$ is similar to this steps and leave to an exercise.

Theorem 3.6 For each pair k, n of integers with $4 \le k \le n$, there is a connected graph G of order n such that $m_2(G \times K_2) = k$.

Proof: Let the inequality were strict. That is $4 < k < n$. Take the path P_{n-2}: $x_1, x_2 \dots x_{n-2}$. G be the graph derived from P_{n-2} by adding x_{n-1} and x_n. Then join each x_{n-1} and x_n to the vertices x_i for all i, $k-3 \le i \le n-2$. Thus G contains n vertices. Let H be the other copy of G in $(G \times K_2)$ and let $V(H) = \{y_1, y_2 \dots y_n\}$ such that x_i is adjacent with y_I (see *Figure 05*). Then $m_2(G \times K_2) = k$. Take M is the set $\{x_1, y_2, x_3, y_4 \dots, y_{k-5}, x_{k-4}, x_{n-1}, y_{n-1}, x_n, y_n\}$ if k is odd and the set $\{x_1, y_2, x_3, y_4, \dots, x_{k-5}, y_{k-4}, x_{n-1}, y_{n-1}, x_n, y_n\}$ if k is even. Since M is 2-monophonic of $(G \times K_2)$, $m_2(G \times K_2) \le k$.

For the converse, assume the contrary. That is $m_2(G \times K_2) < k$. Let M be a 2-m set of $(G \times K_2)$ having $k-1$ vertices. Then for each $i, 1 \le i \le k-4$, the vertex x_i is 2-monophonic by itself, then by x_{i-1} and by y_i. Hence M contains the vertices $\{x_i, y_I\}$. If $A = \{x_{k-3}, x_{k-2} \dots x_n\}$ and $B = \{y_{k-3}, y_{k-2} \dots y_n\}$, then M contains four vertices from $A \cup B$. Otherwise M contains at most three vertices from $A \cup B$. So M contains one vertex from A and one from B. Then there exist the following cases.

***Case 1*:** If M contains no elements of A, then each x_i is 2-monophonic by a pair u, v such that u and v belongs to A so that x_i is not 2-monophonic leads to a contradiction.

Case 2: M contains one element of A. For $x = x_n$ or $x = x_{n-1}$, then x_{n-1} is not 2-monophonic which is not true. Hence, take $x = x_i$ for some i with $k-3 \le i \le n-2$. When $n-k \ge 4$, either x_{i-2} lies in A or x_{i+2} lies in A, say x_{i-2}. Then also, x_{i-2} not 2-monophonic by M and is false. For $1 \le n-k \le 3$, each vertex x_i is 2-monophonic by x_i and y_i so that $B - \{y_i\}$ lies in M. Since $B - \{v_i\}$ contain at least three vertices, M contains at least three vertices of B so that M contains at least four vertices of AUB. This is also a contradiction. There for M contains four vertices of AUB. Thus we have $m_2(G \times K_2) \ge k-4+4 = k$ vertices. Combining these two we get $m_2(G \times K_2) = k$. When $k=4$, take the bipartite graph $K_{2, n-2}$. Then, by theorem 3.5 $m_2(G \times K_2) = k$. To prove the upper limit, take $G = K_n$. Then we get $m_2(G \times K_2) = n$. Hence the theorem is proved.

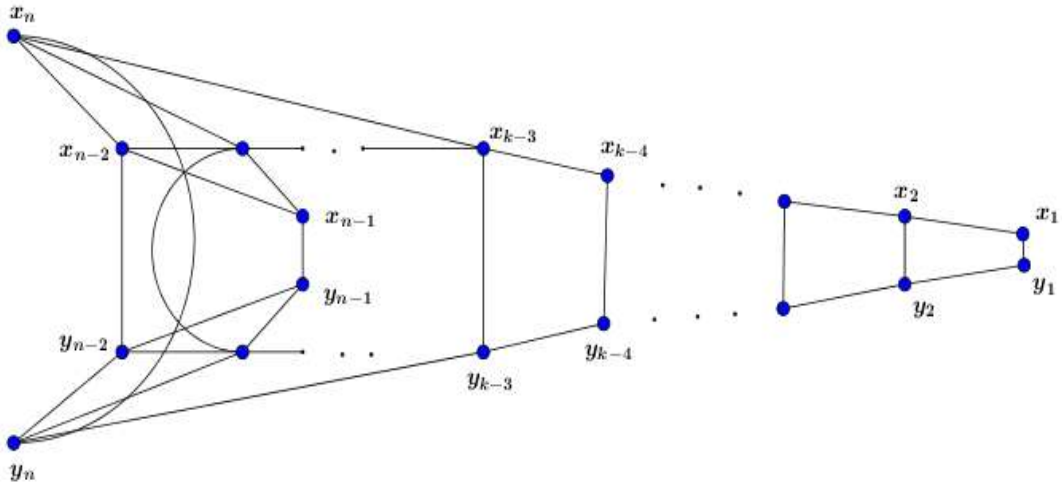

Fig. 5 Graph G with $m_2(G \times K_2) = k$

IV CONCLUSION

The concept of k- monophonic set and k-monophonic number of graphs can extend to find k-edge monophonic number of a graph, k-monophonic domination number of a graph and k-edge monophonic domination number of graphs.

REFERENCES

[1]. P. Arul Paul Sudhahar, M Mohammed Abdul Khayyoom and A Sadiquali. Edge Monophonic Domination Number of Graphs. J. Adv. in Mathematics. Vol. 11. 10 pp 5781-5785 (Jan 2016)

[2]. P. Arul Paul Sudhahar, M Mohammed Abdul Khayyoom and A Sadiquali. The Connected Edge Monophonic Domination Number of Graphs. Int.J Comp. Applications, Vol. 145. No 12, July 2016, pp 18-21

[3]. P. Arul Paul Sudhahar, A. Sadiquali and M Mohammed Abdul Khayyoom. The Monophonic Geodetic Domination Number of Graphs. J. Comp. Math. Sci. Vol 7(1). Pp 27-38 (Jan 2016)

[4]. Gary Chartrand and P.Zhang. Introduction to Graph Theory. MacGraw Hill (2005)

[5]. F.Harary, E.Loukkas and C Tsouros. The Geodetic Number of a Graph. Math. comp Mod. Vol.17 No.11.(1993)pp89-95

[6]. A.A Kinsley and K Karthika. Algorithmic Aspects of k-Geodetic Sets in Graphs. Int. J Math and Appl. Vol. (3),1B(2016) pp141-144

[7]. J. John and P.Arul Paul Sudhahar. On The Edge Monophonic Number of a Graph. Filomat. Vol.26.6 pp 1081-1089(2012).

[8]. J. John and P.Arul Paul Sudhahar. The Monophonic Domination Number of a Graph. Proceedings of the International Conference on Mathematics and Business Management. (2012) pp 142-145.

[9]. Ralucca Gera and Ping Zhang. On k-geodomination in Cartesian Products Congressus Numerantium 158(2002) pp.163-178

[10]. A.P Santhakumaran, P. Titus and R. Ganesamoorthy. On The Monophonic Number of a Graph Applied Math and Informatics. Vol 32,pp 255-266 (2014).