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Research Paper

Weakly Commuting Mappings Involving Cubic Terms of $\mathfrak{H}(x, y, z)$ in $\mathfrak{H}$ -Metric space

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ABSTRACT: *In this paper, first we present weak contraction condition that contains cubic and quadratic terms of distance function $\mathfrak{H}(x, y, z)$ and then prove common fixed point theorems for weakly commuting mappings in $\mathfrak{H}$ -metric space. At the end, we provide an example for the support.*
KEYWORDS AND PHRASES: *$\mathfrak{H}$ -metric space, Weakly Commuting mappings.*
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I. INTRODUCTION

The Banach fixed point theorem is the fundamental method for studying fixed point theory, it states that every contraction mapping on a complete metric space has a unique fixed point. Let $(\mathcal{X}, d)$ be a complete metric space. If $\mathcal{T}: \mathcal{X} \rightarrow \mathcal{X}$ satisfies $d(\mathcal{T}(x), \mathcal{T}(y)) \leq k\,(d(x, y))$ for all $x, y \in \mathcal{X}$, $0 \leq k < 1$, then it has a unique fixed point. In 1969, Boyd and Wong [2] replaced the constant k in Banach contraction principle by a implicit function ψ and proved some fixed point theorems.

In 1997, Alber and Gueree-Delabriere [1] introduced the concept of weak contraction in metric space: A map $\mathcal{F}: \mathcal{X} \rightarrow \mathcal{X}$ is said to be weak contraction if for each $x, y \in \mathcal{X}$, there exists a function $\emptyset : [0, \infty) \rightarrow [0, \infty)$, $\emptyset(t) > 0$ for all $t > 0$ and $\emptyset(0) = 0$ such that $d(\mathcal{T}(x), \mathcal{T}(y)) \leq d(x, y) - \emptyset(d(x, y))$.

II. PRELIMINARIES

In 2006, Zead Mustafa and Brailey Sims[6] introduced the notion of $\mathfrak{H}$ -metric space as generalization of the concept of ordinary metric space.

Definition 2.1[6] "A $\mathfrak{H}$ -metric space is a pair $(\mathcal{X}, \mathfrak{H})$, where $\mathcal{X}$ is a non-empty set and $\mathfrak{H}$ is a non-negative real-valued function defined on $\mathcal{X} \times \mathcal{X} \times \mathcal{X}$ such that for all $x, y, z, a \in \mathcal{X}$, we have

(i) $\mathfrak{H}(x, y, z) = 0$ if $x = y = z$,
(ii) $0 < \mathfrak{H}(x, x, y)$, for all $x, y \in \mathcal{X}$, with $x \neq y$,
(iii) $\mathfrak{H}(x, x, y) \leq \mathfrak{H}(x, y, z)$,for all $x, y, z \in \mathcal{X}$, with $z \neq y$,
(iv) $\mathfrak{H}(x, y, z) = \mathfrak{H}(x, z, y) = \mathfrak{H}(y, z, x) = \cdots$, (symmetry in all three variables),
(v) $\mathfrak{H}(x, y, z) \leq \mathfrak{H}(x, a, a) + \mathfrak{H}(a, y, z)$, for all $x, y, z, a \in \mathcal{X}$ (rectangle inequality),

The function $\mathfrak{H}$ is called $\mathfrak{H}$ -metric on $\mathcal{X}$."

Definition 2.2[7] "A sequence x_n in a $\mathfrak{H}$ -metric space $\mathcal{X}$ is said to be convergent if there exist $x \in \mathcal{X}$ such that $\lim_{n,m\rightarrow\infty} \mathfrak{H}(x, x_n, x_m) = 0$ and one says that the sequence (x_n) is $\mathfrak{H}$ -convergent to x. We call x the limit of the sequence (x_n) and write $x_n \rightarrow x$ or $\lim_{n\rightarrow\infty} x_n = x$."

Definition 2.3[7] "In a $\mathfrak{H}$ -metric space $\mathcal{X}$, a sequence (x_n) is said to be $\mathfrak{H}$ -Cauchy if given $\in > 0$, there exists a $n_0 \in \mathbb{N}$ such that $\mathfrak{H}(x_n, x_m, x_l) < \in$,for all $n, m, l \geq n_0$
i.e., $\mathfrak{H}(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$. "

Proposition 2.1[7]" Let $\mathcal{X}$ be $\mathfrak{H}$ -metric space. Then the following statements are equivalent:

(i) (x_n) is $\mathfrak{H}$ -convergent to x,

(ii) $\mathfrak{H}(x_n, x_n, x) \to 0$ as $n \to \infty$,

(iii) $\mathfrak{H}(x_n, x, x) \to 0$ as $n \to \infty$,

(iv) $\mathfrak{H}(x_m, x_n, x) \to 0$ as $n \to \infty$."

Proposition 2.2[7]" Let $\mathcal{X}$ be $\mathfrak{H}$ -metric space. Then the following statements are equivalent:

(i) The sequence (x_n) is $\mathfrak{H}$ -Cauchy;

(ii) For every $\in > 0$, there exists a $n_0 \in \mathbb{N}$ such that $\mathfrak{H}(x_n, x_m, x_m) < \in, \forall\, n, m \geq n_0$

Definition 2.4 "Two self mappings $\mathcal{S}$ and $\mathcal{T}$ of a metric space $(\mathcal{X}, d)$ are said to be commuting if $\mathcal{S}\mathcal{T}t = \mathcal{T}\mathcal{S}t, \forall\, t$ in $\mathcal{X}$."

In 1982, Sessa [13] generalized the notion of commutative mappings to weak commuting maps as follows:

Definition 2.5[13]" Two self mappings $\mathcal{S}$ and $\mathcal{T}$ of a metric space $(\mathcal{X}, d)$ are said to be weakly commuting if $d(\mathcal{S}\mathcal{T}t, \mathcal{T}\mathcal{S}t) \leq d(\mathcal{T}t, \mathcal{S}t)$, $\forall\, t$ in $\mathcal{X}$."

Remark 2.1[13] Commutative mappings are weak commutative mappings, but converse may not be true.

In a similar mode we can define weak commuting in setting of $\mathfrak{H} -$ metric space $(\mathcal{X}, \mathfrak{H})$

2. Fixed Points for Weakly Commuting Mappings

In 2013, Murthy and Prasad [5] introduced a new type of inequality for a map that involves cubic terms of metric function $d(x,y)$ that extended and generalized the results of many cited in the literature of fixed point theory. In this section, we extend the result of Murthy and Prasad [5] for four self weakly commuting mappings satisfying a generalized weak contractive condition involving various combinations of $\mathfrak{H}$ -metric functions.

Theorem 2.1 Let $\mathcal{A}, \mathcal{B}, \mathcal{S}$ and $\mathcal{T}$ are four self mappings of a complete $\mathfrak{H} -$ metric space $(\mathcal{X}, \mathfrak{H})$ satisfying the following conditions:

(2.1) $\mathcal{S}(\mathcal{X}) \subset \mathcal{B}(\mathcal{X}), \mathcal{T}(\mathcal{X}) \subset \mathcal{A}(\mathcal{X})$;

(2.2) One of the $\mathcal{A}, \mathcal{B}, \mathcal{S}$ and $\mathcal{T}$ is continuous;

(2.3) $[1 + \hbar\mathfrak{H}(\mathcal{A}r, \mathcal{B}s, \mathcal{B}s)]\mathfrak{H}^2(\mathcal{S}r, \mathcal{T}s, \mathcal{T}s) \leq$

$$\hbar\, max \begin{Bmatrix} \frac{1}{2}\begin{bmatrix} \mathfrak{H}^2(\mathcal{A}r, \mathcal{S}r, \mathcal{S}r)\mathfrak{H}(\mathcal{B}s, \mathcal{T}s, \mathcal{T}s) \\ +\mathfrak{H}(\mathcal{A}r, \mathcal{S}r, \mathcal{S}r)\mathfrak{H}^2(\mathcal{B}s, \mathcal{T}s, \mathcal{T}s) \end{bmatrix}, \\ \mathfrak{H}(\mathcal{A}r, \mathcal{S}r, \mathcal{S}r)\mathfrak{H}(\mathcal{A}r, \mathcal{T}s, \mathcal{T}s)\mathfrak{H}(\mathcal{B}s, \mathcal{S}r, \mathcal{S}r), \\ \mathfrak{H}(\mathcal{A}r, \mathcal{T}s, \mathcal{T}s)\mathfrak{H}(\mathcal{B}s, \mathcal{S}r, \mathcal{S}r)\mathfrak{H}(\mathcal{B}s, \mathcal{T}s, \mathcal{T}s) \end{Bmatrix} + \sigma(\mathcal{A}r, \mathcal{B}s) - \emptyset(\sigma(\mathcal{A}r, \mathcal{B}s)),$$

where
$$\sigma(\mathcal{A}r, \mathcal{B}s) = max \begin{Bmatrix} \mathfrak{H}^2(\mathcal{A}r, \mathcal{B}s, \mathcal{B}s), \\ \mathfrak{H}(\mathcal{A}r, \mathcal{S}r, \mathcal{S}r)\mathfrak{H}(\mathcal{B}s, \mathcal{T}s, \mathcal{T}s), \\ \mathfrak{H}(\mathcal{A}r, \mathcal{T}s, \mathcal{T}s)\mathfrak{H}(\mathcal{B}s, \mathcal{S}r, \mathcal{S}r), \\ \frac{1}{2}\begin{bmatrix} \mathfrak{H}(\mathcal{A}r, \mathcal{S}r, \mathcal{S}r)\mathfrak{H}(\mathcal{A}r, \mathcal{T}s, \mathcal{T}s) + \\ \mathfrak{H}(\mathcal{B}s, \mathcal{S}r, \mathcal{S}r)\mathfrak{H}(\mathcal{B}s, \mathcal{T}s, \mathcal{T}s) \end{bmatrix} \end{Bmatrix}$$

$\hbar \geq 0$ is a real number and $\emptyset$: $[0, \infty) \to [0, \infty)$ is a continuous function with $\emptyset(t) = 0$ iff $t = 0$ and $\emptyset(t) > 0$ for each $t > 0$. Then $\mathcal{A}, \mathcal{B}, \mathcal{S}$ and $\mathcal{T}$ have a unique common fixed point in $\mathcal{X}$, provided that the pairs $(\mathcal{A}, \mathcal{S})$ and $(\mathcal{B}, \mathcal{T})$ are weakly commuting.

Proof. Let $r_0 \in \mathcal{X}$. Using (2.1), we can find a point $r_1 \in \mathcal{X}$ s.t $\mathcal{S}(r_0) = \mathcal{B}(r_1) = s_0$. For this point r_1, we can find another point $r_2 \in \mathcal{X}$ such that $s_1 = \mathcal{A}(r_2) = \mathcal{T}(r_1)$. In general, one can construct a sequence $\{s_n\}$ in $\mathcal{X}$ s.t.

$$s_{2n} = \mathcal{S}(r_{2n}) = \mathcal{B}(r_{2n+1});$$
$$s_{2n+1} = \mathcal{T}(r_{2n+1}) = \mathcal{A}(r_{2n+2}) \text{ for each } n \geq 0. \qquad \textbf{(2.4)}$$

For brevity, we write $m_n = \mathfrak{H}(s_{n-1}, s_n, s_n)$.

Firstly, we will prove that m_n is non-increasing sequence and converges to 0.

Case I. If *n* is even, taking $r = r_{2n}$ and $s = r_{2n+1}$ in (2.3), we get .

$$[1 + \hbar\mathfrak{H}(\mathcal{A}r_{2n}, \mathcal{B}r_{2n+1}, \mathcal{B}r_{2n+1})]\mathfrak{H}^2(\mathcal{S}r_{2n}, \mathcal{T}r_{2n+1}, \mathcal{T}r_{2n+1}) \leq$$

$$\hbar\, max \begin{Bmatrix} \frac{1}{2}\begin{bmatrix} \mathfrak{H}^2(\mathcal{A}r_{2n}, \mathcal{S}r_{2n}, \mathcal{S}r_{2n})\mathfrak{H}(\mathcal{B}r_{2n+1}, \mathcal{T}r_{2n+1}, \mathcal{T}r_{2n+1}) \\ +\mathfrak{H}(\mathcal{A}r_{2n}, \mathcal{S}r_{2n}, \mathcal{S}r_{2n})\mathfrak{H}^2(\mathcal{B}r_{2n+1}, \mathcal{T}r_{2n+1}, \mathcal{T}r_{2n+1}) \end{bmatrix}, \\ \mathfrak{H}(\mathcal{A}r_{2n}, \mathcal{S}r_{2n}, \mathcal{S}r_{2n})\mathfrak{H}(\mathcal{A}r_{2n}, \mathcal{T}r_{2n+1}, \mathcal{T}r_{2n+1})\mathfrak{H}(\mathcal{B}r_{2n+1}, \mathcal{S}r_{2n}, \mathcal{S}r_{2n}), \\ \mathfrak{H}(\mathcal{A}r_{2n}, \mathcal{T}r_{2n+1}, \mathcal{T}r_{2n+1})\mathfrak{H}(\mathcal{B}r_{2n+1}, \mathcal{S}r_{2n}, \mathcal{S}r_{2n})\mathfrak{H}(\mathcal{B}r_{2n+1}, \mathcal{T}r_{2n+1}, \mathcal{T}r_{2n+1}) \end{Bmatrix} +$$
$\sigma(\mathcal{A}r_{2n}, \mathcal{B}r_{2n+1}) - \emptyset(\sigma(\mathcal{A}r_{2n}, \mathcal{B}r_{2n+1}),$

where $\quad \sigma(\mathcal{A}r_{2n},\mathcal{B}r_{2n+1}) = max\left\{\begin{matrix} \mathfrak{H}^2(\mathcal{A}r_{2n},\mathcal{B}r_{2n+1},\mathcal{B}r_{2n+1}), \\ \mathfrak{H}(\mathcal{A}r_{2n},\mathcal{S}r_{2n},\mathcal{S}r_{2n})\mathfrak{H}(\mathcal{B}r_{2n+1},\mathcal{T}r_{2n+1},\mathcal{T}r_{2n+1}), \\ \mathfrak{H}(\mathcal{A}r_{2n},\mathcal{T}r_{2n+1},\mathcal{T}r_{2n+1})\mathfrak{H}(\mathcal{B}r_{2n+1},\mathcal{S}r_{2n},\mathcal{S}r_{2n}), \\ \frac{1}{2}\begin{bmatrix} \mathfrak{H}(\mathcal{A}r_{2n},\mathcal{S}r_{2n},\mathcal{S}r_{2n})\mathfrak{H}(\mathcal{A}r_{2n},\mathcal{T}r_{2n+1},\mathcal{T}r_{2n+1}) + \\ \mathfrak{H}(\mathcal{B}r_{2n+1},\mathcal{S}r_{2n},\mathcal{S}r_{2n})\mathfrak{H}(\mathcal{B}r_{2n+1},\mathcal{T}r_{2n+1},\mathcal{T}r_{2n+1}) \end{bmatrix} \end{matrix}\right\}$

Using (2.4), we get

$$[1 + \hbar\mathfrak{H}(s_{2n-1},s_{2n},s_{2n})]\mathfrak{H}^2(s_{2n},s_{2n+1},s_{2n+1}) \leq$$

$$\hbar\, max\left\{\begin{matrix} \frac{1}{2}\begin{bmatrix} \mathfrak{H}^2(s_{2n-1},s_{2n},s_{2n})\mathfrak{H}(s_{2n},s_{2n+1},s_{2n+1}) \\ +\mathfrak{H}(s_{2n-1},s_{2n},s_{2n})\mathfrak{H}^2(s_{2n},s_{2n+1},s_{2n+1}) \end{bmatrix}, \\ \mathfrak{H}(s_{2n-1},s_{2n},s_{2n})\mathfrak{H}(s_{2n-1},s_{2n+1},s_{2n+1})\mathfrak{H}(s_{2n},s_{2n},s_{2n}), \\ \mathfrak{H}(s_{2n-1},s_{2n+1},s_{2n+1})\mathfrak{H}(s_{2n},s_{2n},s_{2n})\mathfrak{H}(s_{2n},s_{2n+1},s_{2n+1}) \end{matrix}\right\}$$

$$+\sigma(s_{2n-1},s_{2n}) - \emptyset(\sigma(s_{2n-1},s_{2n}),$$

where $\quad \sigma(s_{2n-1},s_{2n}) = max\left\{\begin{matrix} \mathfrak{H}^2(s_{2n-1},s_{2n},s_{2n}), \\ \mathfrak{H}(s_{2n-1},s_{2n},s_{2n})\mathfrak{H}(s_{2n},s_{2n+1},s_{2n+1}), \\ \mathfrak{H}(s_{2n-1},s_{2n+1},s_{2n+1})\mathfrak{H}(s_{2n},s_{2n},s_{2n}), \\ \frac{1}{2}\begin{bmatrix} \mathfrak{H}(s_{2n-1},s_{2n},s_{2n})\mathfrak{H}(s_{2n-1},s_{2n+1},s_{2n+1}) + \\ \mathfrak{H}(s_{2n},s_{2n},s_{2n})\mathfrak{H}(s_{2n},s_{2n+1},s_{2n+1}) \end{bmatrix} \end{matrix}\right\}$

On putting $m_{2n} = \mathfrak{H}(s_{2n-1},s_{2n},s_{2n})$ we have

$$[1 + \hbar m_{2n}]m_{2n+1}^2 \leq \hbar max\left\{\frac{1}{2}[m_{2n}^2 m_{2n+1} + m_{2n}m_{2n+1}^2], 0,0\right\}$$

$$+ \sigma(s_{2n-1},s_{2n}) - \emptyset(\sigma(s_{2n-1},s_{2n})),$$

where $\quad \sigma(s_{2n-1},s_{2n}) = max\left\{m_{2n}^2, m_{2n}m_{2n+1}, 0, \frac{1}{2}[m_{2n}\mathfrak{H}(s_{2n-1},s_{2n+1},s_{2n+1}) + 0]\right\}$. .

By using rectangular inequality and property of $\emptyset$, we get

$$\mathfrak{H}(s_{2n-1},s_{2n+1},s_{2n+1}) \leq \mathfrak{H}(s_{2n-1},s_{2n},s_{2n}) + \mathfrak{H}(s_{2n},s_{2n+1},s_{2n+1})$$

$= m_{2n} + m_{2n+1}$ and

$\sigma(s_{2n-1},s_{2n}) \leq m_1{}'(x,y) = max\left\{m_{2n}^2, m_{2n}m_{2n+1}, 0, \frac{1}{2}[m_{2n}(m_{2n} + m_{2n+1}), 0]\right\}$.

If $m_{2n} < m_{2n+1}$, then we get

$\hbar m_{2n+1}^2 \leq \hbar m_{2n+1}^2 - \emptyset(m_{2n+1}^2)$, a contradiction.

Therefore, $m_{2n+1}^2 \leq m_{2n}^2$ i.e., $m_{2n+1} \leq m_{2n}$.

Similarly, if *n* is odd, then we can obtain $m_{2n+2} < m_{2n+1}$.

It follows that the sequence $\{m_n\}$ is decreasing.

Let $\lim_{n\to\infty} m_n = x$, for some $x \geq 0$.

Suppose $x > 0$; then putting $r = r_{2n}$ and $s = r_{2n+1}$ in (2.3), we have

$$[1 + \hbar\mathfrak{H}(\mathcal{A}r_{2n},\mathcal{B}r_{2n+1},\mathcal{B}r_{2n+1})]\mathfrak{H}^2(\mathcal{S}r_{2n},\mathcal{T}r_{2n+1},\mathcal{T}r_{2n+1}) \leq$$

$$\hbar\, max\left\{\begin{matrix} \frac{1}{2}\begin{bmatrix} \mathfrak{H}^2(\mathcal{A}r_{2n},\mathcal{S}r_{2n},\mathcal{S}r_{2n})\mathfrak{H}(\mathcal{B}r_{2n+1},\mathcal{T}r_{2n+1},\mathcal{T}r_{2n+1}) \\ +\mathfrak{H}(\mathcal{A}r_{2n},\mathcal{S}r_{2n},\mathcal{S}r_{2n})\mathfrak{H}^2(\mathcal{B}r_{2n+1},\mathcal{T}r_{2n+1},\mathcal{T}r_{2n+1}) \end{bmatrix}, \\ \mathfrak{H}(\mathcal{A}r_{2n},\mathcal{S}r_{2n},\mathcal{S}r_{2n})\mathfrak{H}(\mathcal{A}r_{2n},\mathcal{T}r_{2n+1},\mathcal{T}r_{2n+1})\mathfrak{H}(\mathcal{B}r_{2n+1},\mathcal{S}r_{2n},\mathcal{S}r_{2n}), \\ \mathfrak{H}(\mathcal{A}r_{2n},\mathcal{T}r_{2n+1},\mathcal{T}r_{2n+1})\mathfrak{H}(\mathcal{B}r_{2n+1},\mathcal{S}r_{2n},\mathcal{S}r_{2n})\mathfrak{H}(\mathcal{B}r_{2n+1},\mathcal{T}r_{2n+1},\mathcal{T}r_{2n+1}) \end{matrix}\right\} +$$

$$\sigma(\mathcal{A}r_{2n},\mathcal{B}r_{2n+1}) - \emptyset(\sigma(\mathcal{A}r_{2n},\mathcal{B}r_{2n+1}),$$

where $\quad \sigma(\mathcal{A}r_{2n},\mathcal{B}r_{2n+1}) = max\left\{\begin{matrix} \mathfrak{H}^2(\mathcal{A}r_{2n},\mathcal{B}r_{2n+1},\mathcal{B}r_{2n+1}), \\ \mathfrak{H}(\mathcal{A}r_{2n},\mathcal{S}r_{2n},\mathcal{S}r_{2n})\mathfrak{H}(\mathcal{B}r_{2n+1},\mathcal{T}r_{2n+1},\mathcal{T}r_{2n+1}), \\ \mathfrak{H}(\mathcal{A}r_{2n},\mathcal{T}r_{2n+1},\mathcal{T}r_{2n+1})\mathfrak{H}(\mathcal{B}r_{2n+1},\mathcal{S}r_{2n},\mathcal{S}r_{2n}), \\ \frac{1}{2}\begin{bmatrix} \mathfrak{H}(\mathcal{A}r_{2n},\mathcal{S}r_{2n},\mathcal{S}r_{2n})\mathfrak{H}(\mathcal{A}r_{2n},\mathcal{T}r_{2n+1},\mathcal{T}r_{2n+1}) + \\ \mathfrak{H}(\mathcal{B}r_{2n+1},\mathcal{S}r_{2n},\mathcal{S}r_{2n})\mathfrak{H}(\mathcal{B}r_{2n+1},\mathcal{T}r_{2n+1},\mathcal{T}r_{2n+1}) \end{bmatrix} \end{matrix}\right\}$

Using (2.4), we get

$$[1 + \hbar\mathfrak{H}(s_{2n-1},s_{2n},s_{2n})]\mathfrak{H}^2(s_{2n},s_{2n+1},s_{2n+1}) \leq$$

$$\hbar\, max\left\{\begin{matrix} \frac{1}{2}\begin{bmatrix} \mathfrak{H}^2(s_{2n-1},s_{2n},s_{2n})\mathfrak{H}(s_{2n},s_{2n+1},s_{2n+1}) \\ +\mathfrak{H}(s_{2n-1},s_{2n},s_{2n})\mathfrak{H}^2(s_{2n},s_{2n+1},s_{2n+1}) \end{bmatrix}, \\ \mathfrak{H}(s_{2n-1},s_{2n},s_{2n})\mathfrak{H}(s_{2n-1},s_{2n+1},s_{2n+1})\mathfrak{H}(s_{2n},s_{2n},s_{2n}), \\ \mathfrak{H}(s_{2n-1},s_{2n+1},s_{2n+1})\mathfrak{H}(s_{2n},s_{2n},s_{2n})\mathfrak{H}(s_{2n},s_{2n+1},s_{2n+1}) \end{matrix}\right\}$$

$$+\sigma(s_{2n-1},s_{2n}) - \emptyset(\sigma(s_{2n-1},s_{2n}),$$

where $\quad \sigma(s_{2n-1},s_{2n}) = max\left\{\begin{array}{c}\mathfrak{H}^2(s_{2n-1},s_{2n},s_{2n}),\\ \mathfrak{H}(s_{2n-1},s_{2n},s_{2n})\mathfrak{H}(s_{2n},s_{2n+1},s_{2n+1}),\\ \mathfrak{H}(s_{2n-1},s_{2n+1},s_{2n+1})\mathfrak{H}(s_{2n},s_{2n},s_{2n}),\\ \frac{1}{2}\left[\begin{array}{c}\mathfrak{H}(s_{2n-1},s_{2n},s_{2n})\mathfrak{H}(s_{2n-1},s_{2n+1},s_{2n+1})+\\ \mathfrak{H}(s_{2n},s_{2n},s_{2n})\mathfrak{H}(s_{2n},s_{2n+1},s_{2n+1})\end{array}\right]\end{array}\right\}$

Now by using triangular inequality and property of $\emptyset$ and proceeds limit $n\to\infty$, we get
$[1+hx]x^2 \le hx^3 + x^2 - \emptyset(x^2)$.
This implies that $\emptyset(x^2)\le 0$. Since x is positive, then by using the property of $\emptyset$, we get $x=0$. Therefore, we conclude that
$\lim_{n\to\infty} m_n = \lim_{n\to\infty}\mathfrak{H}(s_{n-1},s_n,s_n) = x = 0.$ **(2.5)**
Next, we show that $\{s_n\}$ is a Cauchy sequence. Suppose we assume that $\{s_n\}$ is not a Cauchy sequence. For a given $\in > 0$, we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that

$\mathfrak{H}(s_{m(k)},s_{n(k)},s_{n(k)}) \ge \in,\ \mathfrak{H}(s_{m(k)},s_{n(k)-1},s_{n(k)-1}) < \in$ **(2.6)**

and $\qquad n(k) > m(k) > k.$

Now $\in\ \le \mathfrak{H}(s_{m(k)},s_{n(k)},s_{n(k)})$
$\qquad \le \mathfrak{H}(s_{m(k)},s_{n(k)-1},s_{n(k)-1}) + \mathfrak{H}(s_{n(k)-1},s_{n(k)},s_{n(k)})$

Letting $k\to\infty$, and using (2.5) and (2.6), we get $\lim_{k\to\infty}\mathfrak{H}(s_{m(k)},s_{n(k)},s_{n(k)}) = \in$ **(2.7)**

Now from the rectangular inequality, we have,

$$\left|\mathfrak{H}(s_{n(k)},s_{m(k)+1},s_{m(k)+1}) - \mathfrak{H}(s_{m(k)},s_{n(k)},s_{n(k)})\right| \le \mathfrak{H}(s_{m(k)},s_{m(k)+1},s_{m(k)+1})$$

Letting $k\to\infty$, and using (2.5) and (2.7), we get $\lim_{k\to\infty}\mathfrak{H}(s_{n(k)},s_{m(k)+1},s_{m(k)+1}) = \in$ **(2.8)**

Again from the rectangular inequality, we have,

$$\left|\mathfrak{H}(s_{m(k)},s_{n(k)+1},s_{n(k)+1}) - \mathfrak{H}(s_{m(k)},s_{n(k)},s_{n(k)})\right| \le \mathfrak{H}(s_{n(k)},s_{n(k)+1},s_{n(k)+1})$$

Letting $k\to\infty$, and using (2.5) and (2.7), we get $\lim_{k\to\infty}\mathfrak{H}(s_{m(k)},s_{n(k)+1},s_{n(k)+1}) = \in$ **(2.9)**

Now again from the rectangular inequality, we have,

$$\left|\mathfrak{H}(s_{m(k)+1},s_{n(k)+1},s_{n(k)+1}) - \mathfrak{H}(s_{m(k)},s_{n(k)},s_{n(k)})\right|$$
$$\le \mathfrak{H}(s_{m(k)},s_{m(k)+1},s_{m(k)+1}) + \mathfrak{H}(s_{n(k)},s_{n(k)+1},s_{n(k)+1})$$

Letting $k\to\infty$, and using (2.5) and (2.7), we get $\lim_{k\to\infty}\mathfrak{H}(s_{n(k)+1},s_{m(k)+1},s_{m(k)+1}) = \in$ **(2.10)**

On putting $r = r_{m(k)}$ and $s = r_{n(k)}$ in (2.3), we get

$$[1+h\mathfrak{H}(\mathcal{A}r_{m(k)},\mathcal{B}r_{n(k)},\mathcal{B}r_{n(k)})]\mathfrak{H}^2(\mathcal{S}r_{m(k)},\mathcal{T}r_{n(k)},\mathcal{T}r_{n(k)}) \le$$
$$h\, max\left\{\begin{array}{c}\frac{1}{2}\left[\begin{array}{c}\mathfrak{H}^2(\mathcal{A}r_{m(k)},\mathcal{S}r_{m(k)},\mathcal{S}r_{m(k)})\mathfrak{H}(\mathcal{B}r_{n(k)},\mathcal{T}r_{n(k)},\mathcal{T}r_{n(k)})\\ +\mathfrak{H}(\mathcal{A}r_{m(k)},\mathcal{S}r_{m(k)},\mathcal{S}r_{m(k)})\mathfrak{H}^2(\mathcal{B}r_{n(k)},\mathcal{T}r_{n(k)},\mathcal{T}r_{n(k)})\end{array}\right],\\ \mathfrak{H}(\mathcal{A}r_{m(k)},\mathcal{S}r_{m(k)},\mathcal{S}r_{m(k)})\mathfrak{H}(\mathcal{A}r_{m(k)},\mathcal{T}r_{n(k)},\mathcal{T}r_{n(k)})\mathfrak{H}(\mathcal{B}r_{n(k)},\mathcal{S}r_{m(k)},\mathcal{S}r_{m(k)}),\\ \mathfrak{H}(\mathcal{A}r_{m(k)},\mathcal{T}r_{n(k)},\mathcal{T}r_{n(k)})\mathfrak{H}(\mathcal{B}r_{n(k)},\mathcal{S}r_{m(k)},\mathcal{S}r_{m(k)})\mathfrak{H}(\mathcal{B}r_{n(k)},\mathcal{T}r_{n(k)},\mathcal{T}r_{n(k)})\end{array}\right\}+$$
$$\sigma(\mathcal{A}r_{m(k)},\mathcal{B}r_{n(k)}) - \emptyset(\sigma(\mathcal{A}r_{m(k)},\mathcal{B}r_{n(k)})),$$

where $\quad \sigma(\mathcal{A}r_{m(k)},\mathcal{B}r_{n(k)}) = max\left\{\begin{array}{c}\mathfrak{H}^2(\mathcal{A}r_{m(k)},\mathcal{B}r_{n(k)},\mathcal{B}r_{n(k)}),\\ \mathfrak{H}(\mathcal{A}r_{m(k)},\mathcal{S}r_{m(k)},\mathcal{S}r_{m(k)})\mathfrak{H}(\mathcal{B}r_{n(k)},\mathcal{T}r_{n(k)},\mathcal{T}r_{n(k)}),\\ \mathfrak{H}(\mathcal{A}r_{m(k)},\mathcal{T}r_{n(k)},\mathcal{T}r_{n(k)})\mathfrak{H}(\mathcal{B}r_{n(k)},\mathcal{S}r_{m(k)},\mathcal{S}r_{m(k)}),\\ \frac{1}{2}\left[\begin{array}{c}\mathfrak{H}(\mathcal{A}r_{m(k)},\mathcal{S}r_{m(k)},\mathcal{S}r_{m(k)})\mathfrak{H}(\mathcal{A}r_{m(k)},\mathcal{T}r_{n(k)},\mathcal{T}r_{n(k)})+\\ \mathfrak{H}(\mathcal{B}r_{n(k)},\mathcal{S}r_{m(k)},\mathcal{S}r_{m(k)})\mathfrak{H}(\mathcal{B}r_{n(k)},\mathcal{T}r_{n(k)},\mathcal{T}r_{n(k)})\end{array}\right]\end{array}\right\}$

Using (2.4), we get

$$[1+h\mathfrak{H}(s_{m(k)-1},s_{n(k)-1},s_{n(k)-1})]\mathfrak{H}^2(s_{m(k)},s_{n(k)},s_{n(k)}) \le$$
$$h\, max\left\{\begin{array}{c}\frac{1}{2}\left[\begin{array}{c}\mathfrak{H}^2(s_{m(k)-1},s_{m(k)},s_{m(k)})\mathfrak{H}(s_{n(k)-1},s_{n(k)},s_{n(k)})\\ +\mathfrak{H}(s_{m(k)-1},s_{m(k)},s_{m(k)})\mathfrak{H}^2(s_{n(k)-1},s_{n(k)},s_{n(k)})\end{array}\right],\\ \mathfrak{H}(s_{m(k)-1},s_{m(k)},s_{m(k)})\mathfrak{H}(s_{m(k)-1},s_{n(k)},s_{n(k)})\mathfrak{H}(s_{n(k)-1},s_{m(k)},s_{m(k)}),\\ \mathfrak{H}(s_{m(k)-1},s_{n(k)},s_{n(k)})\mathfrak{H}(s_{n(k)-1},s_{m(k)},s_{m(k)})\mathfrak{H}(s_{n(k)-1},s_{n(k)},s_{n(k)})\end{array}\right\}$$

$$+\sigma(s_{m(k)-1},s_{n(k)-1})-\emptyset(\sigma(s_{m(k)-1},s_{n(k)-1}),$$

where $$\sigma(s_{m(k)-1},s_{n(k)-1})=max\left\{\begin{array}{c}\mathfrak{H}^2(s_{m(k)-1},s_{n(k)-1},s_{n(k)-1}),\\ \mathfrak{H}(s_{m(k)-1},s_{s_{m(k)}},s_{m(k)})\mathfrak{H}(s_{n(k)-1},s_{n(k)},s_{n(k)}),\\ \mathfrak{H}(s_{m(k)-1},s_{n(k)},s_{n(k)})\mathfrak{H}(s_{n(k)-1},s_{m(k)},s_{m(k)}),\\ \frac{1}{2}\left[\begin{array}{c}\mathfrak{H}(s_{m(k)-1},s_{m(k)},s_{m(k)})\mathfrak{H}(s_{m(k)-1},s_{n(k)},s_{n(k)})+\\ \mathfrak{H}(s_{n(k)-1},s_{m(k)},s_{m(k)})\mathfrak{H}(s_{n(k)-1},s_{n(k)},s_{n(k)})\end{array}\right]\end{array}\right\}$$

Letting $k\to\infty$, we get

$$[1+k\in]\in^2\le k max\left\{\frac{1}{2}[0+0],0,0\right\}+\in^2-\emptyset(\in^2)$$
$$=\in^2-\emptyset(\in^2),\text{ a contradiction.}$$

Thus$\{s_n\}$ is a Cauchy sequence in $\mathcal{X}$ Since $(\mathcal{X},\mathfrak{H})$ is a complete $\mathfrak{H}$-metric space, therefore, $\{s_n\}$ converges to a point say v as $n\to\infty$, Consequently, the subsequences $\{\mathcal{S}r_{2n}\},\{\mathcal{A}r_{2n}\}$, $\{\mathcal{T}r_{2n+1}\}$, and $\{\mathcal{B}r_{2n+1}\}$ also converges to same point v.

Case -1. Suppose that $\mathcal{A}$ is continuous. Then $\{\mathcal{A}\mathcal{S}r_{2n}\}$,$\{\mathcal{A}\mathcal{A}r_{2n}\}$ converges to $\mathcal{A}v$ as $n\to\infty$,

Since the mappings $\mathcal{A}$ and $\mathcal{S}$ are weakly commuting on $\mathcal{X}$, therefore,

$\mathfrak{H}(\mathcal{S}\mathcal{A}r_{2n},\mathcal{A}\mathcal{S}r_{2n},\mathcal{A}\mathcal{S}r_{2n})\le\mathfrak{H}(\mathcal{A}r_{2n},\mathcal{S}r_{2n},\mathcal{S}r_{2n})$.

Proceeding the limit as $n\to\infty$, we get

$\lim_{n\to\infty}\mathfrak{H}(\mathcal{S}\mathcal{A}r_{2n},\mathcal{A}v,\mathcal{A}v)\le\mathfrak{H}(v,v,v)=0$ i.e., $\lim_{n\to\infty}\mathcal{S}\mathcal{A}r_{2n}=\mathcal{A}v$.

Now we show that $v=\mathcal{A}v$. On putting $r=\mathcal{A}r_{2n}$ and $s=r_{2n+1}$ in (2.3), we have

$$[1+k\mathfrak{H}(\mathcal{A}\mathcal{A}r_{2n},\mathcal{B}r_{2n+1},\mathcal{B}r_{2n+1})]\mathfrak{H}^2(\mathcal{S}\mathcal{A}r_{2n},\mathcal{T}r_{2n+1},\mathcal{T}r_{2n+1})\le$$
$$k\,max\left\{\begin{array}{c}\frac{1}{2}\left[\begin{array}{c}\mathfrak{H}^2(\mathcal{A}\mathcal{A}r_{2n},\mathcal{S}\mathcal{A}r_{2n},\mathcal{S}\mathcal{A}r_{2n})\mathfrak{H}(\mathcal{B}r_{2n+1},\mathcal{T}r_{2n+1},\mathcal{T}r_{2n+1})\\ +\mathfrak{H}(\mathcal{A}\mathcal{A}r_{2n},\mathcal{S}\mathcal{A}r_{2n},\mathcal{S}\mathcal{A}r_{2n})\mathfrak{H}^2(\mathcal{B}r_{2n+1},\mathcal{T}r_{2n+1},\mathcal{T}r_{2n+1})\end{array}\right],\\ \mathfrak{H}(\mathcal{A}\mathcal{A}r_{2n},\mathcal{S}\mathcal{A}r_{2n},\mathcal{S}\mathcal{A}r_{2n})\mathfrak{H}(\mathcal{A}\mathcal{A}r_{2n},\mathcal{T}r_{2n+1},\mathcal{T}r_{2n+1})\mathfrak{H}(\mathcal{B}r_{2n+1},\mathcal{S}\mathcal{A}r_{2n},\mathcal{S}\mathcal{A}r_{2n}),\\ \mathfrak{H}(\mathcal{A}\mathcal{A}r_{2n},\mathcal{T}r_{2n+1},\mathcal{T}r_{2n+1})\mathfrak{H}(\mathcal{B}r_{2n+1},\mathcal{S}\mathcal{A}r_{2n},\mathcal{S}\mathcal{A}r_{2n})\mathfrak{H}(\mathcal{B}r_{2n+1},\mathcal{T}r_{2n+1},\mathcal{T}r_{2n+1})\end{array}\right\}$$
$$+\sigma(\mathcal{A}\mathcal{A}r_{2n},\mathcal{B}r_{2n+1})-\emptyset(\sigma(\mathcal{A}\mathcal{A}r_{2n},\mathcal{B}r_{2n+1}),$$

where $$\sigma(\mathcal{A}\mathcal{A}r_{2n},\mathcal{B}r_{2n+1})=max\left\{\begin{array}{c}\mathfrak{H}^2(\mathcal{A}\mathcal{A}r_{2n},\mathcal{B}r_{2n+1},\mathcal{B}r_{2n+1}),\\ \mathfrak{H}(\mathcal{A}\mathcal{A}r_{2n},\mathcal{S}\mathcal{A}r_{2n},\mathcal{S}\mathcal{A}r_{2n})\mathfrak{H}(\mathcal{B}r_{2n+1},\mathcal{T}r_{2n+1},\mathcal{T}r_{2n+1}),\\ \mathfrak{H}(\mathcal{A}\mathcal{A}r_{2n},\mathcal{T}r_{2n+1},\mathcal{T}r_{2n+1})\mathfrak{H}(\mathcal{B}r_{2n+1},\mathcal{S}\mathcal{A}r_{2n},\mathcal{S}\mathcal{A}r_{2n}),\\ \frac{1}{2}\left[\begin{array}{c}\mathfrak{H}(\mathcal{A}\mathcal{A}r_{2n},\mathcal{S}\mathcal{A}r_{2n},\mathcal{S}\mathcal{A}r_{2n})\mathfrak{H}(\mathcal{A}\mathcal{A}r_{2n},\mathcal{T}r_{2n+1},\mathcal{T}r_{2n+1})+\\ \mathfrak{H}(\mathcal{B}r_{2n+1},\mathcal{S}\mathcal{A}r_{2n},\mathcal{S}\mathcal{A}r_{2n})\mathfrak{H}(\mathcal{B}r_{2n+1},\mathcal{T}r_{2n+1},\mathcal{T}r_{2n+1})\end{array}\right]\end{array}\right\}$$

Proceeding limit as $n\to\infty$, we get

$$[1+k\mathfrak{H}(\mathcal{A}v,v,v)]\mathfrak{H}^2(\mathcal{A}v,v,v)\le$$
$$k\,max\left\{\begin{array}{c}\frac{1}{2}\left[\begin{array}{c}\mathfrak{H}^2(\mathcal{A}v,\mathcal{A}v,\mathcal{A}v)\mathfrak{H}(v,v,v)\\ +\mathfrak{H}(\mathcal{A}v,\mathcal{A}v,\mathcal{A}v)\mathfrak{H}^2(v,v,v)\end{array}\right],\\ \mathfrak{H}(\mathcal{A}v,\mathcal{A}v,\mathcal{A}v)\mathfrak{H}(\mathcal{A}v,v,v)\mathfrak{H}(v,\mathcal{A}v,\mathcal{A}v),\\ \mathfrak{H}(\mathcal{A}v,v,v)\mathfrak{H}(v,\mathcal{A}v,\mathcal{A}v)\mathfrak{H}(v,v,v)\end{array}\right\}$$
$$+\sigma(\mathcal{A}v,v)-\emptyset(\sigma(\mathcal{A}v,v),$$

where $$\sigma(\mathcal{A}v,v)=max\left\{\begin{array}{c}\mathfrak{H}^2(\mathcal{A}v,v,v),\\ \mathfrak{H}(\mathcal{A}v,\mathcal{A}v,\mathcal{A}v)\mathfrak{H}(v,v,v),\\ \mathfrak{H}(\mathcal{A}v,v,v)\mathfrak{H}(v,\mathcal{A}v,\mathcal{A}v),\\ \frac{1}{2}\left[\begin{array}{c}\mathfrak{H}(\mathcal{A}v,\mathcal{A}v,\mathcal{A}v)\mathfrak{H}(\mathcal{A}v,v,v)+\\ \mathfrak{H}(v,\mathcal{A}v,\mathcal{A}v)\mathfrak{H}(v,v,v)\end{array}\right]\end{array}\right\}$$

$$[1+k\mathfrak{H}(\mathcal{A}v,v,v)]\mathfrak{H}^2(\mathcal{A}v,v,v)\le k\,max\left\{\begin{array}{c}\frac{1}{2}[0+0],\\ 0,\\ 0\end{array}\right\}+\sigma(\mathcal{A}v,v)-\emptyset(\sigma(\mathcal{A}v,v)),$$

$$[1+k\mathfrak{H}(\mathcal{A}v,v,v)]\mathfrak{H}^2(\mathcal{A}v,v,v)\le\mathfrak{H}^2(\mathcal{A}v,v,v)-\emptyset(\mathfrak{H}^2(\mathcal{A}v,v,v)),$$

On simplification, we get $\mathfrak{H}^2(\mathcal{A}v,v,v)=0$. This implies that $\mathcal{A}v=v$.

Next, we will show that $\mathcal{S}v=v$. On putting $r=v$ and $s=r_{2n+1}$ in (2.3), we have

$$[1+k\mathfrak{H}(\mathcal{A}v,\mathcal{B}r_{2n+1},\mathcal{B}r_{2n+1})]\mathfrak{H}^2(\mathcal{S}v,\mathcal{T}r_{2n+1},\mathcal{T}r_{2n+1})\le$$
$$k\,max\left\{\begin{array}{c}\frac{1}{2}\left[\begin{array}{c}\mathfrak{H}^2(\mathcal{A}v,\mathcal{S}v,\mathcal{S}v)\mathfrak{H}(\mathcal{B}r_{2n+1},\mathcal{T}r_{2n+1},\mathcal{T}r_{2n+1})\\ +\mathfrak{H}(\mathcal{A}v,\mathcal{S}v,\mathcal{S}v)\mathfrak{H}^2(\mathcal{B}r_{2n+1},\mathcal{T}r_{2n+1},\mathcal{T}r_{2n+1})\end{array}\right],\\ \mathfrak{H}(\mathcal{A}v,\mathcal{S}v,\mathcal{S}v)\mathfrak{H}(\mathcal{A}v,\mathcal{T}r_{2n+1},\mathcal{T}r_{2n+1})\mathfrak{H}(\mathcal{B}r_{2n+1},\mathcal{S}v,\mathcal{S}v),\\ \mathfrak{H}(\mathcal{A}v,\mathcal{T}r_{2n+1},\mathcal{T}r_{2n+1})\mathfrak{H}(\mathcal{B}r_{2n+1},\mathcal{S}v,\mathcal{S}v)\mathfrak{H}(\mathcal{B}r_{2n+1},\mathcal{T}r_{2n+1},\mathcal{T}r_{2n+1})\end{array}\right\}+\sigma(\mathcal{A}v,\mathcal{B}r_{2n+1})-$$
$$\emptyset(\sigma(\mathcal{A}v,\mathcal{B}r_{2n+1}),$$

where $\sigma(\mathcal{A}v,\mathcal{B}r_{2n+1}) = max\left\{\begin{matrix} \mathfrak{H}^2(\mathcal{A}v,\mathcal{B}r_{2n+1},\mathcal{B}r_{2n+1}), \\ \mathfrak{H}(\mathcal{A}v,\mathcal{S}v,\mathcal{S}v)\mathfrak{H}(\mathcal{B}r_{2n+1},\mathcal{T}r_{2n+1},\mathcal{T}r_{2n+1}), \\ \mathfrak{H}(\mathcal{A}v,\mathcal{T}r_{2n+1},\mathcal{T}r_{2n+1})\mathfrak{H}(\mathcal{B}r_{2n+1},\mathcal{S}v,\mathcal{S}v), \\ \frac{1}{2}\left[\begin{matrix}\mathfrak{H}(\mathcal{A}v,\mathcal{S}v,\mathcal{S}v)\mathfrak{H}(\mathcal{A}v,\mathcal{T}r_{2n+1},\mathcal{T}r_{2n+1}) + \\ \mathfrak{H}\mathcal{B}r_{2n+1},\mathcal{S}v,\mathcal{S}v)\mathfrak{H}(\mathcal{B}r_{2n+1},\mathcal{T}r_{2n+1},\mathcal{T}r_{2n+1})\end{matrix}\right]\end{matrix}\right\}$

Proceeding limit as $n \to \infty$, we get

$$[1 + \hbar\mathfrak{H}(\mathcal{A}v,v,v)]\mathfrak{H}^2(\mathcal{S}v,v,v) \le$$
$$\hbar\, max\left\{\begin{matrix} \frac{1}{2}\left[\begin{matrix}\mathfrak{H}^2(\mathcal{A}v,\mathcal{S}v,\mathcal{S}v)\mathfrak{H}(v,v,v) \\ +\mathfrak{H}(\mathcal{A}v,\mathcal{S}v,\mathcal{S}v)\mathfrak{H}^2(v,v,v)\end{matrix}\right], \\ \mathfrak{H}(\mathcal{A}v,\mathcal{S}v,\mathcal{S}v)\mathfrak{H}(\mathcal{A}v,v,v)\mathfrak{H}(v,\mathcal{S}v,\mathcal{S}v), \\ \mathfrak{H}(\mathcal{A}v,v,v)\mathfrak{H}(v,\mathcal{S}v,\mathcal{S}v)\mathfrak{H}(v,v,v)\end{matrix}\right\}$$
$+\sigma(\mathcal{A}v,v) - \emptyset(\sigma(\mathcal{A}v,v),$

where $\sigma(\mathcal{A}v,v) = max\left\{\begin{matrix} \mathfrak{H}^2(\mathcal{A}v,v,v), \\ \mathfrak{H}(\mathcal{A}v,\mathcal{S}v,\mathcal{S}v)\mathfrak{H}(v,v,v), \\ \mathfrak{H}(\mathcal{A}v,v,v)\mathfrak{H}(v,\mathcal{S}v,\mathcal{S}v), \\ \frac{1}{2}\left[\begin{matrix}\mathfrak{H}(\mathcal{A}v,\mathcal{S}v,\mathcal{S}v)\mathfrak{H}(\mathcal{A}v,v,v) + \\ \mathfrak{H}(v,\mathcal{S}v,\mathcal{S}v)\mathfrak{H}(v,v,v)\end{matrix}\right]\end{matrix}\right\} = 0.$

$$[1 + \hbar\mathfrak{H}(\mathcal{A}v,v,v)]\mathfrak{H}^2(\mathcal{S}v,v,v) \le \hbar\, max\left\{\begin{matrix}\frac{1}{2}[0+0], \\ 0, \\ 0\end{matrix}\right\} + 0 - \emptyset(0).$$

Thus we get $\mathfrak{H}^2(\mathcal{S}v,v,v) = 0$. This implies that $\mathcal{S}v = v$. Since $\mathcal{S}(\mathcal{X}) \subset \mathcal{B}(\mathcal{X})$, therefore, there exists a point $u \in \mathcal{X}$ such that $\mathcal{S}v = v = \mathcal{B}u$.

Now we show that $v = \mathcal{T}u$. On putting $r = v$ and $s = u$ in (2.3), we have

$$[1 + \hbar\mathfrak{H}(\mathcal{A}v,\mathcal{B}u,\mathcal{B}u)]\mathfrak{H}^2(\mathcal{S}v,\mathcal{T}u,\mathcal{T}u) \le$$
$$\hbar\, max\left\{\begin{matrix} \frac{1}{2}\left[\begin{matrix}\mathfrak{H}^2(\mathcal{A}v,\mathcal{S}v,\mathcal{S}v)\mathfrak{H}(\mathcal{B}u,\mathcal{T}u,\mathcal{T}u) \\ +\mathfrak{H}(\mathcal{A}v,\mathcal{S}v,\mathcal{S}v)\mathfrak{H}^2(\mathcal{B}u,\mathcal{T}u,\mathcal{T}u)\end{matrix}\right], \\ \mathfrak{H}(\mathcal{A}v,\mathcal{S}v,\mathcal{S}v)\mathfrak{H}(\mathcal{A}v,\mathcal{T}u,\mathcal{T}u)\mathfrak{H}(\mathcal{B}u,\mathcal{S}v,\mathcal{S}v), \\ \mathfrak{H}(\mathcal{A}v,\mathcal{T}u,\mathcal{T}u)\mathfrak{H}(\mathcal{B}u,\mathcal{S}v,\mathcal{S}v)\mathfrak{H}(\mathcal{B}u,\mathcal{T}u,\mathcal{T}u)\end{matrix}\right\} + \sigma(\mathcal{A}v,\mathcal{B}u) - \emptyset(\sigma(\mathcal{A}v,\mathcal{B}u),$$

where $\sigma(\mathcal{A}v,\mathcal{B}s) = max\left\{\begin{matrix} \mathfrak{H}^2(\mathcal{A}v,\mathcal{B}u,\mathcal{B}u), \\ \mathfrak{H}(\mathcal{A}v,\mathcal{S}v,\mathcal{S}v)\mathfrak{H}(\mathcal{B}u,\mathcal{T}u,\mathcal{T}u), \\ \mathfrak{H}(\mathcal{A}v,\mathcal{T}u,\mathcal{T}u)\mathfrak{H}(\mathcal{B}u,\mathcal{S}v,\mathcal{S}v), \\ \frac{1}{2}\left[\begin{matrix}\mathfrak{H}(\mathcal{A}v,\mathcal{S}v,\mathcal{S}v)\mathfrak{H}(\mathcal{A}v,\mathcal{T}u,\mathcal{T}u) + \\ \mathfrak{H}(\mathcal{B}u,\mathcal{S}v,\mathcal{S}v)\mathfrak{H}(\mathcal{B}u,\mathcal{T}u,\mathcal{T}u)\end{matrix}\right]\end{matrix}\right\}$

$$[1 + \hbar\mathfrak{H}(v,v,v)]\mathfrak{H}^2(v,\mathcal{T}u,\mathcal{T}u) \le$$
$$\hbar\, max\left\{\begin{matrix} \frac{1}{2}\left[\begin{matrix}\mathfrak{H}^2(v,v,v)\mathfrak{H}(v,\mathcal{T}u,\mathcal{T}u) \\ +\mathfrak{H}(v,v,v)\mathfrak{H}^2(v,\mathcal{T}u,\mathcal{T}u)\end{matrix}\right], \\ \mathfrak{H}(v,v,v)\mathfrak{H}(v,\mathcal{T}u,\mathcal{T}u)\mathfrak{H}(v,\mathcal{S}v,\mathcal{S}v), \\ \mathfrak{H}(v,\mathcal{T}u,\mathcal{T}u)\mathfrak{H}\big((v,v,v)\big)\mathfrak{H}(v,\mathcal{T}u,\mathcal{T}u)\end{matrix}\right\} + \sigma(v,v) - \emptyset(\sigma(v,v),$$

where $\sigma(v,v) = max\left\{\begin{matrix} \mathfrak{H}^2(v,v,v), \\ \mathfrak{H}(v,v,v)\mathfrak{H}(v,\mathcal{T}u,\mathcal{T}u), \\ \mathfrak{H}(v,\mathcal{T}u,\mathcal{T}u)\mathfrak{H}(v,v,v), \\ \frac{1}{2}\left[\begin{matrix}\mathfrak{H}(v,v,v)\mathfrak{H}(v,\mathcal{T}u,\mathcal{T}u) + \\ \mathfrak{H}(v,v,v)\mathfrak{H}(v,\mathcal{T}u,\mathcal{T}u)\end{matrix}\right]\end{matrix}\right\} = 0.$

$$[1 + \hbar\mathfrak{H}(v,v,v)]\mathfrak{H}^2(v,\mathcal{T}u,\mathcal{T}u) \le \hbar\, max\left\{\begin{matrix}\frac{1}{2}[0+0], \\ 0, \\ 0\end{matrix}\right\} + 0 - \emptyset(0).$$

Thus we get $\mathfrak{H}^2(v,\mathcal{T}u,\mathcal{T}u) = 0$. This implies that $\mathcal{T}u = v$. Since the pair $(\mathcal{B},\mathcal{T})$ is weak commutative, therefore, we have

$$\mathfrak{H}(\mathcal{B}v,\mathcal{T}v,\mathcal{T}v) = \mathfrak{H}(\mathcal{B}\mathcal{T}u,\mathcal{T}\mathcal{B}u,\mathcal{T}\mathcal{B}u) \le \mathfrak{H}(\mathcal{B}u,\mathcal{T}u,\mathcal{T}u) = \mathfrak{H}(v,v,v) = 0.$$

Thus $\mathcal{B}v = \mathcal{T}v$.

Now we show that $v = \mathcal{T}v$. On putting $r = v$ and $s = v$ in (2.3), we have

$$[1 + \hbar\mathfrak{H}(\mathcal{A}v,\mathcal{B}v,\mathcal{B}v)]\mathfrak{H}^2(\mathcal{S}v,\mathcal{T}v,\mathcal{T}v) \le$$

$$\hbar\, max\left\{\begin{array}{c}\frac{1}{2}\left[\begin{array}{c}\mathfrak{H}^2(\mathcal{A}v,\mathcal{S}v,\mathcal{S}v)\mathfrak{H}(\mathcal{B}v,\mathcal{T}v,\mathcal{T}v)\\+\mathfrak{H}(\mathcal{A}v,\mathcal{S}v,\mathcal{S}v)\mathfrak{H}^2\big((\mathcal{B}v,\mathcal{T}v,\mathcal{T}v)\big)\end{array}\right],\\ \mathfrak{H}(\mathcal{A}v,\mathcal{S}v,\mathcal{S}v)\mathfrak{H}(\mathcal{A}v,\mathcal{T}v,\mathcal{T}v)\mathfrak{H}(\mathcal{B}v,\mathcal{S}v,\mathcal{S}v),\\ \mathfrak{H}(\mathcal{A}v,\mathcal{T}v,\mathcal{T}v)\mathfrak{H}(\mathcal{B}v,\mathcal{S}v,\mathcal{S}v)\mathfrak{H}\big((\mathcal{B}v,\mathcal{T}v,\mathcal{T}v)\big)\end{array}\right\}+\sigma(\mathcal{A}v,\mathcal{B}v)-\emptyset(\sigma(\mathcal{A}v,\mathcal{B}v),$$

where $$\sigma(\mathcal{A}v,\mathcal{B}v)=max\left\{\begin{array}{c}\mathfrak{H}^2(\mathcal{A}v,\mathcal{B}v,\mathcal{B}v),\\ \mathfrak{H}(\mathcal{A}v,\mathcal{S}v,\mathcal{S}v)\mathfrak{H}\big((\mathcal{B}v,\mathcal{T}v,\mathcal{T}v)\big),\\ \mathfrak{H}(\mathcal{A}v,\mathcal{T}v,\mathcal{T}v)\mathfrak{H}(\mathcal{B}v,\mathcal{S}v,\mathcal{S}v),\\ \frac{1}{2}\left[\begin{array}{c}\mathfrak{H}(\mathcal{A}v,\mathcal{S}v,\mathcal{S}v)\mathfrak{H}(\mathcal{A}v,\mathcal{T}v,\mathcal{T}v)+\\ \mathfrak{H}(\mathcal{B}v,\mathcal{S}v,\mathcal{S}v)\mathfrak{H}\big((\mathcal{B}v,\mathcal{T}v,\mathcal{T}v)\big)\end{array}\right]\end{array}\right\}=\mathfrak{H}^2(v,\mathcal{T}v,\mathcal{T}v).$$

Therefore, we get

$$[1+\hbar\mathfrak{H}(v,\mathcal{T}v,\mathcal{T}v)]\mathfrak{H}^2(v,\mathcal{T}v,\mathcal{T}v)\le\ \hbar max\left\{\begin{array}{c}\frac{1}{2}[0+0],\\ 0,\\ 0\end{array}\right\}+\mathfrak{H}^2(v,\mathcal{T}v,\mathcal{T}v)-\emptyset(\mathfrak{H}^2(v,\mathcal{T}v,\mathcal{T}v))\ .$$ This implies that $v=\mathcal{T}v$.

Case 2. Suppose that $\mathcal{B}$ is continuous; we can obtain the same result by way of Case 1.

Case 3. Suppose that $\mathcal{S}$ is continuous. Then $\{\mathcal{S}\mathcal{S}r_{2n}\}$,$\{\mathcal{S}\mathcal{A}r_{2n}\}$ converges to $\mathcal{S}v$ as $n\to\infty$. Since the mappings $\mathcal{A}$ and $\mathcal{S}$ are weakly commuting on $\mathcal{X}$, therefore,

$\mathfrak{H}(\mathcal{A}\mathcal{S}r_{2n},\mathcal{S}\mathcal{A}r_{2n},\mathcal{S}\mathcal{A}r_{2n})\le\mathfrak{H}(\mathcal{S}r_{2n},\mathcal{A}r_{2n},\mathcal{A}r_{2n})$.

Proceeding the limit as $n\to\infty$, we get

$\lim_{n\to\infty}\mathfrak{H}(\mathcal{A}\mathcal{S}r_{2n},\mathcal{A}v,\mathcal{A}v)\le\mathfrak{H}(v,v,v)=0$ i.e., $\lim_{n\to\infty}\mathcal{A}\mathcal{S}r_{2n}=\mathcal{S}v$.

Now we show that $v=\mathcal{S}v$. On putting $r=\mathcal{S}r_{2n}$ and $s=r_{2n+1}$ in (2.3), we have

$$[1+\hbar\mathfrak{H}(\mathcal{A}\mathcal{S}r_{2n},\mathcal{B}r_{2n+1},\mathcal{B}r_{2n+1})]\mathfrak{H}^2(\mathcal{S}\mathcal{S}r_{2n},\mathcal{T}r_{2n+1},\mathcal{T}r_{2n+1})\le$$

$$\hbar\, max\left\{\begin{array}{c}\frac{1}{2}\left[\begin{array}{c}\mathfrak{H}^2(\mathcal{A}\mathcal{S}r_{2n},\mathcal{S}\mathcal{S}r_{2n},\mathcal{S}\mathcal{S}r_{2n})\mathfrak{H}(\mathcal{B}r_{2n+1},\mathcal{T}r_{2n+1},\mathcal{T}r_{2n+1})\\+\mathfrak{H}(\mathcal{A}\mathcal{S}r_{2n},\mathcal{S}\mathcal{S}r_{2n},\mathcal{S}\mathcal{S}r_{2n})\mathfrak{H}^2(\mathcal{B}r_{2n+1},\mathcal{T}r_{2n+1},\mathcal{T}r_{2n+1})\end{array}\right],\\ \mathfrak{H}(\mathcal{A}\mathcal{S}r_{2n},\mathcal{S}\mathcal{S}r_{2n},\mathcal{S}\mathcal{S}r_{2n})\mathfrak{H}(\mathcal{A}\mathcal{S}r_{2n},\mathcal{T}r_{2n+1},\mathcal{T}r_{2n+1})\mathfrak{H}(\mathcal{B}r_{2n+1},\mathcal{S}\mathcal{S}r_{2n},\mathcal{S}\mathcal{S}r_{2n}),\\ \mathfrak{H}(\mathcal{A}\mathcal{S}r_{2n},\mathcal{T}r_{2n+1},\mathcal{T}r_{2n+1})\mathfrak{H}(\mathcal{B}r_{2n+1},\mathcal{S}\mathcal{S}r_{2n},\mathcal{S}\mathcal{S}r_{2n})\mathfrak{H}(\mathcal{B}r_{2n+1},\mathcal{T}r_{2n+1},\mathcal{T}r_{2n+1})\end{array}\right\}$$

$+\sigma(\mathcal{A}\mathcal{S}r_{2n},\mathcal{B}r_{2n+1})-\emptyset(\sigma(\mathcal{A}\mathcal{S}r_{2n},\mathcal{B}r_{2n+1}),$

where $$\sigma(\mathcal{A}\mathcal{S}r_{2n},\mathcal{B}r_{2n+1})=max\left\{\begin{array}{c}\mathfrak{H}^2(\mathcal{A}\mathcal{S}r_{2n},\mathcal{B}r_{2n+1},\mathcal{B}r_{2n+1}),\\ \mathfrak{H}(\mathcal{A}\mathcal{S}r_{2n},\mathcal{S}\mathcal{S}r_{2n},\mathcal{S}\mathcal{S}r_{2n})\mathfrak{H}(\mathcal{B}r_{2n+1},\mathcal{T}r_{2n+1},\mathcal{T}r_{2n+1}),\\ \mathfrak{H}(\mathcal{A}\mathcal{S}r_{2n},\mathcal{T}r_{2n+1},\mathcal{T}r_{2n+1})\mathfrak{H}(\mathcal{B}r_{2n+1},\mathcal{S}\mathcal{S}r_{2n},\mathcal{S}\mathcal{S}r_{2n}),\\ \frac{1}{2}\left[\begin{array}{c}\mathfrak{H}(\mathcal{A}\mathcal{S}r_{2n},\mathcal{S}\mathcal{S}r_{2n},\mathcal{S}\mathcal{S}r_{2n})\mathfrak{H}(\mathcal{A}\mathcal{S}r_{2n},\mathcal{T}r_{2n+1},\mathcal{T}r_{2n+1})+\\ \mathfrak{H}(\mathcal{B}r_{2n+1},\mathcal{S}\mathcal{S}r_{2n},\mathcal{S}\mathcal{S}r_{2n})\mathfrak{H}(\mathcal{B}r_{2n+1},\mathcal{T}r_{2n+1},\mathcal{T}r_{2n+1})\end{array}\right]\end{array}\right\}$$

Proceeding limit as $n\to\infty$, we get

$$[1+\hbar\mathfrak{H}(\mathcal{S}v,v,v)]\mathfrak{H}^2(\mathcal{S}v,v,v)\le$$

$$\hbar\, max\left\{\begin{array}{c}\frac{1}{2}\left[\begin{array}{c}\mathfrak{H}^2(\mathcal{S}v,\mathcal{S}v,\mathcal{S}v)\mathfrak{H}(v,v,v)\\+\mathfrak{H}\big((\mathcal{S}v,\mathcal{S}v,\mathcal{S}v)\big)\mathfrak{H}^2(v,v,v)\end{array}\right],\\ \mathfrak{H}\big((\mathcal{S}v,\mathcal{S}v,\mathcal{S}v)\big)\mathfrak{H}(\mathcal{S}v,v,v)\mathfrak{H}(v,\mathcal{S}v,\mathcal{S}v),\\ \mathfrak{H}(\mathcal{S}v,v,v)\mathfrak{H}(v,\mathcal{S}v,\mathcal{S}v)\mathfrak{H}(v,v,v)\end{array}\right\}$$

$+\sigma(\mathcal{S}v,v)-\emptyset(\sigma(\mathcal{S}v,v),$

where $$\sigma(\mathcal{S}v,v)=max\left\{\begin{array}{c}\mathfrak{H}^2(\mathcal{S}v,v,v),\\ \mathfrak{H}\big((\mathcal{S}v,\mathcal{S}v,\mathcal{S}v)\big)\mathfrak{H}(v,v,v),\\ \mathfrak{H}(\mathcal{S}v,v,v)\mathfrak{H}(v,\mathcal{S}v,\mathcal{S}v),\\ \frac{1}{2}\left[\begin{array}{c}\mathfrak{H}\big((\mathcal{S}v,\mathcal{S}v,\mathcal{S}v)\big)\mathfrak{H}(\mathcal{S}v,v,v)+\\ \mathfrak{H}(v,\mathcal{S}v,\mathcal{S}v)\mathfrak{H}(v,v,v)\end{array}\right]\end{array}\right\}$$

$$[1+\hbar\mathfrak{H}(\mathcal{S}v,v,v)]\mathfrak{H}^2(\mathcal{S}v,v,v)\le\hbar\, max\left\{\begin{array}{c}\frac{1}{2}[0+0],\\ 0,\\ 0\end{array}\right\}+\sigma(\mathcal{S}v,v)-\emptyset(\sigma(\mathcal{S}v,v)),$$

$[1+\hbar\mathfrak{H}(\mathcal{S}v,v,v)]\mathfrak{H}^2(\mathcal{S}v,v,v)\le\ \mathfrak{H}^2(\mathcal{S}v,v,v)-\emptyset(\mathfrak{H}^2(\mathcal{S}v,v,v)),$

On simplification, we get $\mathfrak{H}^2(\mathcal{S}v,v,v)=0$. This implies that $\mathcal{S}v=v$. Since $\mathcal{S}(\mathcal{X})\subset\mathcal{B}(\mathcal{X})$, therefore, there exists a point $z\in\mathcal{X}$ such that $\mathcal{S}v=v=\mathcal{B}z$.

Now we show that $v=\mathcal{T}z$. On putting $r=\mathcal{S}r_{2n}$ and $s=z$ in (2.3), we have

$$[1+\hbar\mathfrak{H}(\mathcal{A}\mathcal{S}r_{2n},\mathcal{B}z,\mathcal{B}z)]\mathfrak{H}^2(\mathcal{S}\mathcal{S}r_{2n},\mathcal{T}z,\mathcal{T}z)\le$$

$$\hbar\, max \left\{ \begin{array}{c} \frac{1}{2}\left[\begin{array}{c} \mathfrak{H}^2(\mathcal{AS}r_{2n},\mathcal{SS}r_{2n},\mathcal{SS}r_{2n})\mathfrak{H}(\mathcal{B}z,\mathcal{T}z,\mathcal{T}z) \\ +\mathfrak{H}(\mathcal{AS}r_{2n},\mathcal{SS}r_{2n},\mathcal{SS}r_{2n})\mathfrak{H}^2(\mathcal{B}z,\mathcal{T}z,\mathcal{T}z) \end{array}\right], \\ \mathfrak{H}(\mathcal{AS}r_{2n},\mathcal{SS}r_{2n},\mathcal{SS}r_{2n})\mathfrak{H}(\mathcal{AS}r_{2n},\mathcal{T}z,\mathcal{T}z)\mathfrak{H}(\mathcal{B}z,\mathcal{SS}r_{2n},\mathcal{SS}r_{2n}), \\ \mathfrak{H}(\mathcal{AS}r_{2n},\mathcal{T}z,\mathcal{T}z)\mathfrak{H}(\mathcal{B}z,\mathcal{SS}r_{2n},\mathcal{SS}r_{2n})\mathfrak{H}(\mathcal{B}z,\mathcal{T}z,\mathcal{T}z) \end{array} \right\}$$

$+\sigma(\mathcal{AS}r_{2n},\mathcal{B}z) - \emptyset(\sigma(\mathcal{AS}r_{2n},\mathcal{B}z),$

where $$\sigma(\mathcal{AS}r_{2n},\mathcal{B}z) = max \left\{ \begin{array}{c} \mathfrak{H}^2(\mathcal{AS}r_{2n},\mathcal{B}z,\mathcal{B}z), \\ \mathfrak{H}(\mathcal{AS}r_{2n},\mathcal{SS}r_{2n},\mathcal{SS}r_{2n})\mathfrak{H}(\mathcal{B}z,\mathcal{T}z,\mathcal{T}z), \\ \mathfrak{H}(\mathcal{AS}r_{2n},\mathcal{T}z,\mathcal{T}z)\mathfrak{H}(\mathcal{B}z,\mathcal{SS}r_{2n},\mathcal{SS}r_{2n}), \\ \frac{1}{2}\left[\begin{array}{c} \mathfrak{H}(\mathcal{AS}r_{2n},\mathcal{SS}r_{2n},\mathcal{SS}r_{2n})\mathfrak{H}(\mathcal{AS}r_{2n},\mathcal{T}z,\mathcal{T}z) + \\ \mathfrak{H}(\mathcal{B}z,\mathcal{SS}r_{2n},\mathcal{SS}r_{2n})\mathfrak{H}(\mathcal{B}z,\mathcal{T}z,\mathcal{T}z) \end{array}\right] \end{array} \right\}$$

Proceeding limit as $n \to \infty$, we get

$$[1+\hbar\mathfrak{H}(v,v,v)]\mathfrak{H}^2(v,\mathcal{T}z,\mathcal{T}z) \le$$

$$\hbar\, max \left\{ \begin{array}{c} \frac{1}{2}\left[\begin{array}{c} \mathfrak{H}^2(v,v,v)\mathfrak{H}(v,\mathcal{T}z,\mathcal{T}z) \\ +\mathfrak{H}((v,v,v))\mathfrak{H}^2(v,\mathcal{T}z,\mathcal{T}z) \end{array}\right], \\ \mathfrak{H}((v,v,v))\mathfrak{H}(v,\mathcal{T}z,\mathcal{T}z)\mathfrak{H}(v,v,v), \\ \mathfrak{H}(v,\mathcal{T}z,\mathcal{T}z)\mathfrak{H}(v,v,v)\mathfrak{H}(v,\mathcal{T}z,\mathcal{T}z) \end{array} \right\} + \sigma(v,v) - \emptyset(\sigma(v,v)$$

where $$\sigma(v,v) = max \left\{ \begin{array}{c} \mathfrak{H}^2(v,v,v), \\ \mathfrak{H}((v,v,\mathcal{S}v))\mathfrak{H}(v,v,v), \\ \mathfrak{H}(v,\mathcal{T}z,\mathcal{T}z)\mathfrak{H}(v,v,v), \\ \frac{1}{2}\left[\begin{array}{c} \mathfrak{H}((v,v,v))\mathfrak{H}(v,\mathcal{T}z,\mathcal{T}z) + \\ \mathfrak{H}(v,v,v)\mathfrak{H}(v,\mathcal{T}z,\mathcal{T}z) \end{array}\right] \end{array} \right\} = 0.$$

$$[1+\hbar\mathfrak{H}(v,v,v)]\mathfrak{H}^2(v,\mathcal{T}z,\mathcal{T}z) \le \hbar\, max \left\{ \begin{array}{c} \frac{1}{2}[0+0], \\ 0, \\ 0 \end{array} \right\} + 0 - \emptyset(0),$$

On simplification, we get $\mathfrak{H}^2(v,\mathcal{T}z,\mathcal{T}z) = 0$. This implies that $\mathcal{T}z = v$.
Since the pair $(\mathcal{B},\mathcal{T})$ is weak commutative, therefore, we have

$$\mathfrak{H}(\mathcal{TB}z,\mathcal{BT}z,\mathcal{BT}z) \le \mathfrak{H}(\mathcal{T}z,\mathcal{B}z,\mathcal{B}z) = \mathfrak{H}(z,z,z) = 0.$$

Thus $\mathcal{B}v = \mathcal{T}v$.
Now we show that $v = \mathcal{T}v$. On putting $r = r_{2n}$ and $s = v$ in (2.3), we have

$$[1+\hbar\mathfrak{H}(\mathcal{A}r_{2n},\mathcal{B}v,\mathcal{B}v)]\mathfrak{H}^2(\mathcal{S}r_{2n},\mathcal{T}v,\mathcal{T}v) \le$$

$$\hbar\, max \left\{ \begin{array}{c} \frac{1}{2}\left[\begin{array}{c} \mathfrak{H}^2(\mathcal{A}r_{2n},\mathcal{S}r_{2n},\mathcal{S}r_{2n})\mathfrak{H}(\mathcal{B}v,\mathcal{T}v,\mathcal{T}v) \\ +\mathfrak{H}(\mathcal{A}r_{2n},\mathcal{S}r_{2n},\mathcal{S}r_{2n})\mathfrak{H}^2(\mathcal{B}v,\mathcal{T}v,\mathcal{T}v) \end{array}\right], \\ \mathfrak{H}(\mathcal{A}r_{2n},\mathcal{S}r_{2n},\mathcal{S}r_{2n})\mathfrak{H}(\mathcal{A}r_{2n},\mathcal{T}v,\mathcal{T}v)\mathfrak{H}(\mathcal{B}v,\mathcal{S}r_{2n},\mathcal{S}r_{2n}), \\ \mathfrak{H}(\mathcal{A}r_{2n},\mathcal{T}v,\mathcal{T}v)\mathfrak{H}(\mathcal{B}v,\mathcal{S}r_{2n},\mathcal{S}r_{2n})\mathfrak{H}(\mathcal{B}v,\mathcal{T}v,\mathcal{T}v) \end{array} \right\}$$

$+\sigma(\mathcal{A}r_{2n},\mathcal{B}v) - \emptyset(\sigma(\mathcal{A}r_{2n},\mathcal{B}v),$

where $$\sigma(\mathcal{A}r_{2n},\mathcal{B}v) = max \left\{ \begin{array}{c} \mathfrak{H}^2(\mathcal{A}r_{2n},\mathcal{B}v,\mathcal{B}v), \\ \mathfrak{H}(\mathcal{A}r_{2n},\mathcal{S}r_{2n},\mathcal{S}r_{2n})\mathfrak{H}(\mathcal{B}v,\mathcal{T}v,\mathcal{T}v), \\ \mathfrak{H}(\mathcal{A}r_{2n},\mathcal{T}v,\mathcal{T}v)\mathfrak{H}(\mathcal{B}v,\mathcal{S}r_{2n},\mathcal{S}r_{2n}), \\ \frac{1}{2}\left[\begin{array}{c} \mathfrak{H}(\mathcal{A}r_{2n},\mathcal{S}r_{2n},\mathcal{S}r_{2n})\mathfrak{H}(\mathcal{A}r_{2n},\mathcal{T}v,\mathcal{T}v) + \\ \mathfrak{H}(\mathcal{B}v,\mathcal{S}r_{2n},\mathcal{S}r_{2n})\mathfrak{H}(\mathcal{B}v,\mathcal{T}v,\mathcal{T}v) \end{array}\right] \end{array} \right\}$$

Proceeding limit as $n \to \infty$, we get

$$[1+\hbar\mathfrak{H}(v,\mathcal{T}v,\mathcal{T}v)]\mathfrak{H}^2(v,\mathcal{T}v,\mathcal{T}v) \le$$

$$\hbar\, max \left\{ \begin{array}{c} \frac{1}{2}\left[\begin{array}{c} \mathfrak{H}^2(v,v,v)\mathfrak{H}(\mathcal{B}v,\mathcal{T}v,\mathcal{T}v) \\ +\mathfrak{H}((v,v,v))\mathfrak{H}^2(\mathcal{B}v,\mathcal{T}v,\mathcal{T}v) \end{array}\right], \\ \mathfrak{H}((v,v,v))\mathfrak{H}(v,\mathcal{T}v,\mathcal{T}v)\mathfrak{H}(\mathcal{T}v,v,v), \\ \mathfrak{H}(v,\mathcal{T}v,\mathcal{T}v)\mathfrak{H}(v,v,v)\mathfrak{H}(\mathcal{B}v,\mathcal{T}v,\mathcal{T}v) \end{array} \right\} + \sigma(v,\mathcal{T}v) - \emptyset(\sigma(v,\mathcal{T}v)$$

where $$\sigma(v,\mathcal{T}v) = max \left\{ \begin{array}{c} \mathfrak{H}^2(v,\mathcal{T}v,\mathcal{T}v), \\ \mathfrak{H}((v,v,v))\mathfrak{H}(\mathcal{T}v,\mathcal{T}v,\mathcal{T}v), \\ \mathfrak{H}(v,\mathcal{T}v,\mathcal{T}v)\mathfrak{H}(\mathcal{T}v,v,v), \\ \frac{1}{2}\left[\begin{array}{c} \mathfrak{H}((v,v,v))\mathfrak{H}(v,\mathcal{T}v,\mathcal{T}v) + \\ \mathfrak{H}(\mathcal{T}v,v,v)\mathfrak{H}(\mathcal{T}v,\mathcal{T}v,\mathcal{T}v) \end{array}\right] \end{array} \right\}.$$

$$[1+\hbar\mathfrak{H}(v,\mathcal{T}v,\mathcal{T}v)]\mathfrak{H}^2(v,\mathcal{T}v,\mathcal{T}v)\le \hbar\, max\begin{Bmatrix}\frac{1}{2}[0+0],\\ 0,\\ 0\end{Bmatrix}$$

$+\mathfrak{H}^2(v,\mathcal{T}v,\mathcal{T}v)-\emptyset(\mathfrak{H}^2(v,\mathcal{T}v,\mathcal{T}v))$,

On simplification, we get $\mathfrak{H}^2(v,\mathcal{T}v,\mathcal{T}v)=0$. This implies that $\mathcal{T}v=v$.
Since $\mathcal{T}(\mathcal{X})\subset\mathcal{A}(\mathcal{X})$, therefore, there exists a point $w\in\mathcal{X}$ such that $\mathcal{T}v=v=\mathcal{A}w$.
We claim that $v=\mathcal{S}w$.
For this, we put $r=w$ and $s=v$ in (2.3) we have

$$[1+\hbar\mathfrak{H}(\mathcal{A}w,\mathcal{B}v,\mathcal{B}v)]\mathfrak{H}^2(\mathcal{S}w,\mathcal{T}v,\mathcal{T}v)\le$$

$$\hbar\, max\begin{Bmatrix}\frac{1}{2}\begin{bmatrix}\mathfrak{H}^2(\mathcal{A}w,\mathcal{S}w,\mathcal{S}w)\mathfrak{H}(\mathcal{B}v,\mathcal{T}v,\mathcal{T}v)\\+\mathfrak{H}(\mathcal{A}w,\mathcal{S}w,\mathcal{S}w)\mathfrak{H}^2((\mathcal{B}v,\mathcal{T}v,\mathcal{T}v))\end{bmatrix},\\ \mathfrak{H}(\mathcal{A}w,\mathcal{S}w,\mathcal{S}w)\mathfrak{H}(\mathcal{A}w,\mathcal{T}v,\mathcal{T}v)\mathfrak{H}(\mathcal{B}v,\mathcal{S}w,\mathcal{S}w),\\ \mathfrak{H}(\mathcal{A}w,\mathcal{T}v,\mathcal{T}v)\mathfrak{H}(\mathcal{B}v,\mathcal{S}w,\mathcal{S}w)\mathfrak{H}((\mathcal{B}v,\mathcal{T}v,\mathcal{T}v))\end{Bmatrix}+\sigma(\mathcal{A}w,\mathcal{B}v)-\emptyset(\sigma(\mathcal{A}w,\mathcal{B}v),$$

where $$\sigma(\mathcal{A}w,\mathcal{B}v)=max\begin{Bmatrix}\mathfrak{H}^2(\mathcal{A}w,\mathcal{B}v,\mathcal{B}v),\\ \mathfrak{H}(\mathcal{A}w,\mathcal{S}w,\mathcal{S}w)\mathfrak{H}((\mathcal{B}v,\mathcal{T}v,\mathcal{T}v)),\\ \mathfrak{H}(\mathcal{A}w,\mathcal{T}v,\mathcal{T}v)\mathfrak{H}(\mathcal{B}v,\mathcal{S}w,\mathcal{S}w),\\ \frac{1}{2}\begin{bmatrix}\mathfrak{H}(\mathcal{A}w,\mathcal{S}w,\mathcal{S}w)\mathfrak{H}(\mathcal{A}w,\mathcal{T}v,\mathcal{T}v)+\\ \mathfrak{H}(\mathcal{B}v,\mathcal{S}w,\mathcal{S}w)\mathfrak{H}((\mathcal{B}v,\mathcal{T}v,\mathcal{T}v))\end{bmatrix}\end{Bmatrix}.$$

Now we have

$$[1+\hbar\mathfrak{H}(v,v,v)]\mathfrak{H}^2(\mathcal{S}w,v,v)\le$$

$$\hbar\, max\begin{Bmatrix}\frac{1}{2}\begin{bmatrix}\mathfrak{H}^2(v,\mathcal{S}w,\mathcal{S}w)\mathfrak{H}(v,v,v)\\+\mathfrak{H}((v,\mathcal{S}w,\mathcal{S}w))\mathfrak{H}^2(v,v,v)\end{bmatrix},\\ \mathfrak{H}((v,\mathcal{S}w,\mathcal{S}w))\mathfrak{H}(v,v,v)\mathfrak{H}(v,\mathcal{S}w,\mathcal{S}w),\\ \mathfrak{H}(v,v,v)\mathfrak{H}(v,\mathcal{S}w,\mathcal{S}w)\mathfrak{H}(v,v,v)\end{Bmatrix}+\sigma(v,v)-\emptyset(\sigma(v,v)$$

where $$\sigma(v,v)=max\begin{Bmatrix}\mathfrak{H}^2(v,v,v),\\ \mathfrak{H}((v,\mathcal{S}w,\mathcal{S}w))\mathfrak{H}(v,v,v),\\ \mathfrak{H}(v,v,v)\mathfrak{H}(v,\mathcal{S}w,\mathcal{S}w),\\ \frac{1}{2}\begin{bmatrix}\mathfrak{H}((v,\mathcal{S}w,\mathcal{S}w))\mathfrak{H}(v,v,v)+\\ \mathfrak{H}(v,\mathcal{S}w,\mathcal{S}w)\mathfrak{H}(v,v,v)\end{bmatrix}\end{Bmatrix}=0.$$

Therefore, we get

$$[1+\hbar\mathfrak{H}(v,v,v)]\mathfrak{H}^2(\mathcal{S}w,v,v)\le\hbar\, max\begin{Bmatrix}\frac{1}{2}[0+0],\\ 0,\\ 0\end{Bmatrix}+0-\emptyset(0),$$

$(\mathfrak{H}^2(\mathcal{S}w,v,v))$. This implies that $v=\mathcal{S}w$.
Since the pair $(\mathcal{S},\mathcal{A})$ is weakly commuting on $\mathcal{X}$, therefore,
$\mathfrak{H}(\mathcal{A}v,\mathcal{S}v,\mathcal{S}v)=\mathfrak{H}(\mathcal{A}\mathcal{S}w,\mathcal{S}\mathcal{A}w,\mathcal{S}\mathcal{A}w)\le\mathfrak{H}(\mathcal{S}w,\mathcal{A}w,\mathcal{A}w)=\mathfrak{H}(v,v,v)=0$,, therefore, $\mathcal{A}v=\mathcal{S}v$.
Hence $v=\mathcal{A}v=\mathcal{S}v=\mathcal{B}v=\mathcal{T}v$.

Case 4. Suppose that $\mathcal{T}$ is continuous, we can obtain a similar result by way of case 3. **Uniqueness**: Suppose $v\ne w$ be two common fixed points of $\mathcal{A}$,$\mathcal{B}$,$\mathcal{S}$ and $\mathcal{T}$.
On putting $r=v$ and $s=w$ in (2.3), we have

$$[1+\hbar\mathfrak{H}(\mathcal{A}v,\mathcal{B}w,\mathcal{B}w)]\mathfrak{H}^2(\mathcal{S}v,\mathcal{T}w,\mathcal{T}w)\le$$

$$\hbar\, max\begin{Bmatrix}\frac{1}{2}\begin{bmatrix}\mathfrak{H}^2(\mathcal{A}v,\mathcal{S}v,\mathcal{S}v)\mathfrak{H}(\mathcal{B}w,\mathcal{T}w,\mathcal{T}w)\\+\mathfrak{H}(\mathcal{A}v,\mathcal{S}v,\mathcal{S}v)\mathfrak{H}^2\mathcal{B}w,\mathcal{T}w,\mathcal{T}w\end{bmatrix},\\ \mathfrak{H}(\mathcal{A}v,\mathcal{S}v,\mathcal{S}v)\mathfrak{H}(\mathcal{A}v,\mathcal{T}w,\mathcal{T}w)\mathfrak{H}(\mathcal{B}w,\mathcal{S}v,\mathcal{S}v),\\ \mathfrak{H}(\mathcal{A}v,\mathcal{T}s,\mathcal{T}s)\mathfrak{H}(\mathcal{B}s,\mathcal{S}v,\mathcal{S}v)\mathfrak{H}(\mathcal{B}s,\mathcal{T}s,\mathcal{T}s)\end{Bmatrix}+\sigma(\mathcal{A}v,\mathcal{B}w)-\emptyset(\sigma(\mathcal{A}v,\mathcal{B}w),$$

where $$\sigma(\mathcal{A}v,\mathcal{B}w)=max\begin{Bmatrix}\mathfrak{H}^2(\mathcal{A}v,\mathcal{B}w,\mathcal{B}w),\\ \mathfrak{H}(\mathcal{A}v,\mathcal{S}v,\mathcal{S}v)\mathfrak{H}(\mathcal{B}w,\mathcal{T}w,\mathcal{T}w),\\ \mathfrak{H}(\mathcal{A}v,\mathcal{T}w,\mathcal{T}w)\mathfrak{H}(\mathcal{B}w,\mathcal{S}v,\mathcal{S}v),\\ \frac{1}{2}\begin{bmatrix}\mathfrak{H}(\mathcal{A}v,\mathcal{S}v,\mathcal{S}v)\mathfrak{H}(\mathcal{A}v,\mathcal{T}w,\mathcal{T}w)+\\ \mathfrak{H}(\mathcal{B}w,\mathcal{S}v,\mathcal{S}v)\mathfrak{H}(\mathcal{B}w,\mathcal{T}w,\mathcal{T}w)\end{bmatrix}\end{Bmatrix}$$

$[1+\hbar\mathfrak{H}(\mathcal{A}v,\mathcal{B}w,\mathcal{B}w)]\mathfrak{H}^2(\mathcal{S}v,\mathcal{T}w,\mathcal{T}w)\le\ \hbar\, max\{0,0,0\}+\sigma(\mathcal{A}v,\mathcal{B}w)-\emptyset(\sigma(\mathcal{A}v,\mathcal{B}w))$.
On solving we have $\mathfrak{H}^2(v,w,w)=0$. This implies $v=w$.
This completes the proof.

Example 2.1 Let $\mathcal{X} = [2,20]$ and let $(\mathcal{X}, \mathfrak{H})$ be a $\mathfrak{H}$ −metric space defined by $\mathfrak{H}(r,s,t) = |r-s| + |s-t| + |t-r|$ for all $r,s,t \in \mathcal{X}$. Let $\mathcal{A}, \mathcal{B}, \mathcal{S}, \mathcal{T}$ are self mappings defined by

$$\mathcal{A}r = \begin{cases} 12 & if\ 2 < r \leq 5 \\ r-3 & if\ r > 5 \\ 2 & if\ r = 2. \end{cases} \qquad \mathcal{S}r = \begin{cases} 6 & if\ 2 < r \leq 5 \\ 2 & if\ r > 5 \\ r & if\ r = 2. \end{cases};$$

$$\mathcal{B}r = \begin{cases} 2 & if\ r = 2 \\ 6 & if\ r > 2 \end{cases} \qquad \mathcal{T}r = \begin{cases} r & if\ r = 2 \\ 3 & if\ r > 2 \end{cases}$$

Let us consider a sequence $\{x_n\}$ with $x_n = 2$, and define $\emptyset: [0,\infty) \to [0,\infty)$ by $\emptyset(t) = \frac{t}{3}$. For any values of $\hbar > 0$, then it is easy to verify the inequality (2.3) holds. Hence the Theorem 2.1 holds well.

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