Quest Journals
Journal of Research in Applied Mathematics
Volume 3 ~ Issue 6 (2017) pp: 01-04
ISSN(Online) : 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Research Paper

Applications of Numerical Functional Analysis in Atomless Finite Measure Spaces

Yousif Mohammed Modawy Elmahy[1], Montasir Salman ElfadelTyfor[2]
Department of MathematicFaculty ofScience &Arts In Al-mandg Al-BAHA University KS.A[1]
Department of PhysicsFaculty of Science & Arts In Al-mandg Al-BAHA University KS.A[2]
[2]Department of Physics Faculty ofEducationin (Al-Hasahisa), Gezira University Sudan

Received 23Feb, 2017; **A**ccepted 08Mar, 2017 © The author(s) 2017. **P**ublished with open access at **www.questjournals.org**

ABSTRACT:
This paper study the application of numerical Functional analysis in physics as isotope, theorems was derived to applied it on an Atomless finite measure space.
Keyword:*Isotopes, Atomless finite measure space, narrow operator.*

I. INTRODUCTION

This paper attempts to study the application of numerical Functional analysis in many branch of physics we can found it in theoretical physics in Lagrangian formulation of classical mechanics [1] also we found function analysis in quantum mechanics in Hilbert spaces.The basic and historically first class of spaces studied in functional analysis are complete normed vector spaces over the real or complex numbers [8]. Such spaces are calledBanach spaces [4]. An important example is a Hilbert space, where the norm arises from an inner product [9]. These spaces are of fundamental importance in many areas, including the mathematical formulation of quantum mechanics [10].Most spaces considered in functional analysis have finite dimension [5].

II. FUNCTION ANALYSIS IN ATOMLESS FINITE MEASURE SPACE

We start by the applicationon an Atomless finite measure space
We did for rank-one operators.

Lemma (1) [2] Let (Ω, Σ, μ) be an atomless finite measure space, $\varepsilon > 0$, $T \in \mathcal{L}(L_{1+\varepsilon}(\mu))$ a narrow operator, $x_1 \in L_{1+\varepsilon}(\mu)$ a simple function, $T x_1 = x_2$, and$\Omega = D_1 \sqcup \cdots \sqcup D_\ell$ any partition and $\varepsilon > 0$. Then there exists a partition $\Omega = A \sqcup B$ such that

(i)$\|(x_1)_A\|^{1+\varepsilon} = \|(x_1)_B\|^{1+\varepsilon} = 2^{-1}\|x_1\|^{1+\varepsilon}$.

(ii) $\mu(D_j \cap A) = \mu(D_j \cap B) = \frac{1}{2}\mu(D_j)$ for each $j = 1, \dots, \ell$.

(iii) $\|T(x_1)_A - 2^{-1}x_2\| < \varepsilon$ and $\|T(x_1)_B - 2^{-1}x_2\| < \varepsilon$.

Proof. Let $x_1 = \sum_{k=1}^{m} a_k \mathbf{1}_{C_k}$ for some $a_k \in \mathbb{K}$ and $\Omega = C_1 \sqcup \cdots \sqcup C_m$. For each $k = 1, \dots, m$ and $j = 1, \dots, \ell$ define sets$E_{k,j} = C_k \cap D_j$ and, using the definition of narrow operator, choose $u_{k,j} \in L_{1+\varepsilon}(\mu)$ so that

$$u_{k,j}^2 = \mathbf{1}_{E_{k,j}}, \quad \int_\Omega u_{k,j}\, d\mu = 0, \quad \text{and} \quad |a_k|\|Tu_{k,j}\| < \frac{2\varepsilon}{m\ell}.$$

Then set

$$E_{k,j}^+ = \{t \in E_{k,j} : u_{k,j}(t) \ge 0\}, \qquad E_{k,j}^- \setminus E_{k,j}^+$$

Which satisfy$\mu(E_{k,j}^+) = \mu(E_{k,j}^-) = \frac{1}{2}\mu(E_{k,j})$and define

$$A = \bigcup_{k=1}^{m}\bigcup_{j=1}^{\ell} E_{k,j}^+ \text{ and } \quad B = \bigcup_{k=1}^{m}\bigcup_{j=1}^{\ell} E_{k,j}^-.$$

Let us show that the partition$\Omega = A \sqcup B$has the desired properties. Indeed, observe that

$$\|(x_1)_A\|^{1+\varepsilon} = \sum_{k=1}^{m}\sum_{j=1}^{\ell}|a_k|^{1+\varepsilon}\,\mu(E_{k,j}^{+}) = \sum_{k=1}^{m}|a_k|^{1+\varepsilon}\sum_{j=1}^{\ell}\frac{1}{2}\,\mu(E_{k,j})$$
$$= \frac{1}{2}\sum_{k=1}^{m}|a_k|^{1+\varepsilon}\,\mu(C_k) = \frac{1}{2}\|x_1\|^{1+\varepsilon}$$

And that one obviously has $\|(x_1)_B\|^{1+\varepsilon} = \|(x_1)_A\|^{1+\varepsilon}$, thus (i) is show.

Since $E_{k,j}^{+} \subseteq E_{k,j} \subseteq D_j$, for each $j_0 \in \{1,\dots,\ell\}$ we have that

$$D_{j_0} \cap A = \bigcup_{k=1}^{m}\bigcup_{j=1}^{\ell}(D_{j_0} \cap E_{k,j}^{+}) = \bigcup_{k=1}^{m} E_{k,j}^{+}$$

And hence

$$\mu(D_{j_0} \cap A) = \sum_{k=1}^{m}\mu(E_{k,j_0}^{+}) = \frac{1}{2}\sum_{k=1}^{m}\mu(E_{k,j_0}) = \frac{1}{2}\sum_{k=1}^{m}\mu(C_k \cap D_{j_0}) = \frac{1}{2}\mu(D_{j_0}).$$

Analogously it is show that $\mu(D_j \cap B) = \frac{1}{2}\mu(D_j)$ for every $j \in \{1,\dots,\ell\}$which finishes (ii). To show (iii) observe that

$$(x_1)_A - (x_1)_B = \sum_{k=1}^{m}\sum_{j=1}^{\ell} a_k\left(\mathbf{1}_{E_{k,j}^{+}} - \mathbf{1}_{E_{k,j}^{-}}\right) = \sum_{k=1}^{m}\sum_{j=1}^{\ell} a_k u_{k,j}$$

And hence,

$$\|T((x_1)_A - (x_1)_B)\| \le \sum_{k=1}^{m}\sum_{j=1}^{\ell}|a_k|\left\|T\,u_{k,j}\right\| < 2\varepsilon.$$

Therefore, one has that

$$\left\|T(x_1)_A - \frac{1}{2}x_2\right\| = \frac{1}{2}\|\,2\,T(x_1)_A - T(x_1)_A - T(x_1)_B\|$$
$$= \frac{1}{2}\|\,T((x_1)_A - (x_1)_B)\| < \varepsilon.$$

Analogously, one obtains $\left\|T(x_1)_B - \frac{1}{2}x_2\right\| < \varepsilon$ finishing the proof of (iii).

Lemma (2) [3] Let (Ω,Σ,μ) be an atomless finite measure space, $\varepsilon > 0$, let $T \in \mathcal{L}(L_{1+\varepsilon}(\mu))$ be a narrow operator, and let$x_1, x_2 \in L_{1+\varepsilon}(\mu)$ be simple functions such that $T\,x_1 = x_2$. Then for each $n \in \mathbb{N}$ and $\varepsilon > 0$ there exists a partition $\Omega = A_1 \sqcup\cdots\sqcup A_{2^n}$ such that for each $k = 1,\dots,2^n$ one has

(a) $\left\|(x_1)_{A_k}\right\|^{1+\varepsilon} = 2^{-n}\|(x_1)\|^{1+\varepsilon}$.

(b)$\left\|(x_2)_{A_k}\right\|^{1+\varepsilon} = 2^{-n}\|(x_2)\|^{1+\varepsilon}$.

(c)$\left\|T(x_1)_{A_k} - 2^{-n}x_2\right\| < \varepsilon$.

Proof. Let $x_2 = \sum_{j=1}^{\ell} b_j \mathbf{1}_{D_j}$ for some $b_j \in \mathbb{K}$ and $\Omega = D_1 \sqcup\cdots\sqcup D_\ell$. We proceed by induction on n.Suppose first that $n = 1$ and use Lemma (1) to find a partition $\Omega = A \sqcup B$ satisfying properties (i)–(iii). Then (i) and (iii) mean (a) and (c) for $A_1 = A$, $A_2 = B$. Besides, observe that (b) follows from (ii):

$$\left\|(x_1)_{A_1}\right\|^{1+\varepsilon} = \sum_{j=1}^{\ell}\left|b_j\right|^{1+\varepsilon}\mu(A_1 \cap D_j) = \sum_{j=1}^{\ell}\left|b_j\right|^{1+\varepsilon}\frac{1}{2}\mu(D_j) = \frac{1}{2}\|x_2\|^{1+\varepsilon}$$

And analogously $\left\|(x_2)_{A_2}\right\|^{1+\varepsilon} = 2^{-1}\|x_2\|^{1+\varepsilon}$.

For the induction step suppose that the statement of the lemma is true for $n \in \mathbb{N}$ and find a partition $\Omega = A_1 \sqcup\cdots\sqcup A_{2^n}$ such that for every $k = 1,\dots,2^n$ the following hold:

$$\left\|(x_1)_{A_k}\right\|^{1+\varepsilon} = 2^{-n}\|x_1\|^{1+\varepsilon}, \left\|(x_1)_{A_k}\right\|^{1+\varepsilon} = 2^{-n}\|x_2\|^{1+\varepsilon}, \text{ and} \left\|T(x_1)_{A_k} - 2^{-n}x_2\right\| < \varepsilon. \quad (1)$$

Then, for each $k = 1,\dots,2^n$ use Lemma (1) for $(x_1)_{A_k}$ instead of (x_1), $T(x_1)_{A_k}$ instead of x_2, the decomposition

$$\Omega = \bigcup_{k=1}^{2^{-n}}\bigcup_{j=1}^{\ell}(D_j \cap A_k)$$

Instead of $\Omega = D_1 \sqcup \cdots \sqcup D_\ell$ and $\frac{\varepsilon}{2}$ instead of ε, and find a partition $\Omega = A(k) \sqcup B(k)$ satisfyingproperties (i)–(iii). Namely, for each $k = 1,\ldots,2^n$ we have that:

(i) $\left\|(x_1)_{(A_k \cap A(k))}\right\|^{1+\varepsilon} = \left\|(x_1)_{(A_k \cap B(k))}\right\|^{1+\varepsilon} = 2^{-1}\left\|(x_1)_{A_k}\right\|^{1+\varepsilon}$.

(ii) $\mu\left(D_j \cap A_k \cap A(k)\right) = \mu\left(D_j \cap A_k \cap B(k)\right) = \frac{1}{2}\mu(D_j \cap A_k)$For each $j = 1,\ldots,\ell$.

(iii) $\left\|T(x_1)_{(A_k \cap A(k))} - 2^{-1}T(x_1)_{A_k}\right\| < \frac{\varepsilon}{2}$ and $\left\|T(x_1)_{(A_k \cap B(k))} - 2^{-1}T(x_1)_{A_k}\right\| < \frac{\varepsilon}{2}$.

Let us show that the partition

$$\Omega = \left(A_1 \cap A(1)\right) \sqcup \cdots \sqcup \left(A_{2^n} \cap A(2^n)\right) \sqcup \left(A_1 \cap B(1)\right) \sqcup \cdots \sqcup \left(A_{2^n} \cap B(2^n)\right)$$

Has the desired properties for $n+1$:

Property (a): using (i) and (1), one obtains

$$\left\|(x_1)_{(A_k \cap A(k))}\right\|^{1+\varepsilon} = \left\|(x_1)_{(A_k \cap B(k))}\right\|^{1+\varepsilon} = 2^{-1}\left\|(x_1)_{A_k}\right\|^{1+\varepsilon} = 2^{-(n+1)}\|x_1\|^{1+\varepsilon}.$$

Property (b): for each $k = 1,\ldots,2^n$ use (ii) and (1) to obtain

$$\left\|(x_2)_{(A_k \cap A(k))}\right\|^{1+\varepsilon} = \sum_{j=1}^{\ell}\left|b_j\right|^{1+\varepsilon}\mu\left(D_j \cap A_k \cap A(k)\right)$$

$$= \frac{1}{2}\sum_{j=1}^{\ell}\left|b_j\right|^{1+\varepsilon}\frac{1}{2}\mu\left(D_j \cap A_k\right)$$

$$= \frac{1}{2}\left\|(x_2)_{A_k}\right\|^{1+\varepsilon} = 2^{-(n+1)}\|x_2\|^{1+\varepsilon}$$

And analogously $\left\|(x_2)_{(A_k \cap B)}\right\|^{1+\varepsilon} = 2^{-(n+1)}\|x_2\|^{1+\varepsilon}$.

Property (c): for each $k = 1,\ldots,2^n$ use (iii) and (1) to write

$$\left\|T(x_1)_{(A_k \cap A(k))} - 2^{-(n+1)}x_2\right\| \le \left\|T(x_1)_{(A_k \cap A(k))} - 2^{-1}T(x_1)_{A_k}\right\| + \frac{1}{2}\left\|T(x_1)_{A_k} - 2^{-n}x_2\right\|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

And analogously$\left\|T(x_1)_{(A_k \cap B(k))} - 2^{-(n+1)}x_2\right\| < \varepsilon$, which completes the proof.

III .APPLICATIONS OF NUMERICAL FUNCTIONAL IN ATOMLESS FINITE MEASURE SPACE

Lemma (3) [6] Let (Ω, Σ, μ) be an atomless finite measure space, $\varepsilon > 0$, let $T \in \mathcal{L}(L_{1+\varepsilon}(\mu))$ be a narrow operator, and let x_1, $x_2 \in L_{1+\varepsilon}(\mu)$ be simple functions such that $Tx_1 = x_2$. Then for each $n \in \mathbb{N}$, each number $\lambda$of the form$\lambda = \frac{j}{2^n}$ where $j \in \{1,\ldots,2^n - 1\}$and each $\varepsilon > 0$there exists a partition $\Omega = A \sqcup B$ such that:

(A) $\|(x_1)_A\|^{1+\varepsilon} = \lambda\|x_1\|^{1+\varepsilon}$.

(B) $\|(x_2)_B\|^{1+\varepsilon} = (1-\lambda)\|x_2\|^{1+\varepsilon}$.

(C) $\|T(x_1)_A - \lambda x_1\| < \varepsilon$.

Proof. Use Lemma (2) to choose a partition

$\Omega = A_1 \sqcup \cdots \sqcup A_{2^n}$ Satisfying properties (a)–(c) with ε/j instead of ε. Then, setting

$$A = \bigcup_{k=1}^{j} A_k \; and B = \bigcup_{k=j+1}^{2^n} A_k,$$

One obtains

$$\|(x_1)_A\|^{1+\varepsilon} = \sum_{k=1}^{j}\left\|(x_1)_{A_k}\right\|^{1+\varepsilon} = \sum_{k=1}^{j} 2^{-n}\|x_1\|^{1+\varepsilon} = \lambda\|x_1\|^{1+\varepsilon},$$

$$\|(x_2)_B\|^{1+\varepsilon} = \sum_{k=j+1}^{2^n}\left\|(x_2)_{A_k}\right\|^{1+\varepsilon} = \sum_{k=j+1}^{2^n} 2^{-n}\|x_2\|^{1+\varepsilon} = (1+\lambda)\|x_1\|^{1+\varepsilon},$$

and

$$\|T(x_1)_A - \lambda x_2\| = \left\|\sum_{k=1}^{j} T(x_1)_{A_k} - \sum_{k=1}^{j} 2^{-n}x_2\right\| \le \sum_{k=1}^{j}\left\|T(x_1)_{A_k} - 2^{-n}x_2\right\| < j\frac{\varepsilon}{j} = \varepsilon$$

As desired.

By using anther way (in general case), about above lemma:

Lemma (4) [6]Let (Ω,Σ,μ) be an Atomless finite measure space, $\varepsilon > 0$ let $T \in \mathcal{L}(L_{1+\varepsilon}(\mu))$ be a narrow operator, and let $x_n,\ x_{n+1} \in L_{1+\varepsilon}(\mu)$ be of simple functions such that $Tx_n = x_{n+1}$. Then for each $n \in \mathbb{N}$, each number $\lambda$of the form $\lambda = \frac{j}{2^n}$ where $j \in \{1,\ldots,2^n - 1\}$and each $\varepsilon > 0$ there exists a partition $\Omega = A \sqcup B$ such that:

(i) $\|(x_n)_A\|^{1+\varepsilon} = \lambda\|x_n\|^{1+\varepsilon}$.
(ii) $\|(x_{n+1})_B\|^{1+\varepsilon} = (1-\lambda)\|x_{n+1}\|^{1+\varepsilon}$.
(iii) $\|T(x_n)_A - \lambda x_{n+1}\| < \varepsilon$.

Proof. Use Lemma (2) to choose a partition $\Omega = A_1 \sqcup \cdots \sqcup A_{2^n}$ satisfying properties (i)–(iii) with ε/j instead of ε. Then, setting

$$A = \bigcup_{k=1}^{j} A_k \ and B = \bigcup_{k=j+1}^{2^n} A_k,$$

We have

$$\|(x_n)_A\|^{1+\varepsilon} = \sum_{k=1}^{j} \left\|(x_n)_{A_k}\right\|^{1+\varepsilon} = \sum_{k=1}^{j} 2^{-n}\|x_n\|^{1+\varepsilon} = \lambda\|x_n\|^{1+\varepsilon},$$

$$\|(x_{n+1})_B\|^{1+\varepsilon} = \sum_{k=j+1}^{2^n} \left\|(x_{n+1})_{A_k}\right\|^{1+\varepsilon} = \sum_{k=j+1}^{2^n} 2^{-n}\|x_{n+1}\|^{1+\varepsilon} = (1+\lambda)\|x_n\|^{1+\varepsilon},$$

and

$$\|T(x_n)_A - \lambda x_{n+1}\| = \left\|\sum_{k=1}^{j} T(x_n)_{A_k} - \sum_{k=1}^{j} 2^{-n}x_n\right\|$$

$$\leq \sum_{k=1}^{j} \left\|T(x_n)_{A_k} - 2^{-n}x_{n+1}\right\| < j\frac{\varepsilon}{j} = \varepsilon.$$

as desired.

REFERENCE

[1]https://www.quora.com/What-are-common-uses-of-functional-analysis-in-theoretical-physics
[2] Miguel Martín a,∗, Javier Mería, Mikhail Popov , On the numerical radius of operators in Lebesgue spaces
[3] M. Martín, J. Merí, M. Popov, on the numerical radius of operators in Lebesgue spaces^☆, J. of functional analysis, 261 (2011) 149-168.
[4] Hall, B.C. (2013), Quantum Theory for Mathematicians, Springer, p. 147
[5] Rudin, Walter (1991). Functional analysis. McGraw-Hill Science/Engineering/Math. ISBN 978-0-07-054236-5.
[6] M. Martín, J. Merí, M. Popov, on the numerical radius of operators in Lebesgue spaces^☆, J. of functional analysis, 261 (2011) 149-168.
[7] M. Martín, J. Merí, M. Popov, on the numerical radius of operators in Lebesgue spaces^☆, J. of functional analysis, 261 (2011) 149-168.
[8] https://en.wikipedia.org/wiki/Functional_analysis.
[9]https://en.wikipedia.org/wiki/Functional_analysis.
[10] Same as reference [8]