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Research Paper

Block Hybrid Method for the Solution of General Second Order Ordinary Differentials Equations

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ABSTRACT: We consider the construction of block hybrid method for the solution of general second order ODEs. Derivation of the method was based on the use of hermite polynomial as basis function. The main method and its additional equations are obtained from the same continuous formulation via interpolation and collocation procedures. The method is then applied in block form as simultaneous numerical integrator, this approach eliminates requirement for starting values, and it also reduces computational effort. The stability properties of the method is discussed and the stability region shown. Two numerical experiments were given to illustrate the accuracy and efficiency of the new method.

Keywords: Continuous formulation, Block hybrid method, Basis function, Ordinary Differential Equations, Hermite Polynomial.

I. INTRODUCTION

In this paper, efforts are directed towards constructing a uniform order 3 block hybrid method for solution of general second order ordinary differential equation of the form.

$$y'' = f(x, y, y'), y(0) = \alpha, \quad y'(0) = \beta \tag{1}$$

In the past, efforts have been made by eminent scholars to solve higher order initial value problems especially the second order ordinary differential equation. In practice, this class of problem (1) is usually reduced to system of first order differential equation and numerical methods for first order ODEs then employ to solve them, these scholars [4], [11] and [3] showed that reduction of higher order equations to its first order has a serious implication in the results; hence it is necessary to modify existing algorithms to handle directly this class of problem (1). [13] demonstrated a successful application of LMM methods to solve directly a general second order odes of the form (1) though with non-uniform order member block method, this idea is used and now extended to our own uniform order block schemes to solve the type (1) directly. The following scholars also contributed immensely to the development of block hybrid method for the solution of second order ODEs: [2], [10], [1] just to mention a few.

We approximate the exact solution $y(x)$ by seeking the continuous method $\bar{y}(x)$ of the form

$$\bar{y}(x) = \sum_{j=0}^{s-1} \alpha_j(x) y_{n+j} + h^2 \sum_{j=0}^{r-1} \beta_j(x) f_{n+j} \tag{2}$$

Where $x \in [a, b]$ and the following notations are introduced. The positive integer $k \geq 2$ denotes the step number of the method (2), which is applied directly to provide the solution to (1).

II. DERIVATION OF THE METHOD

We propose an approximate solution to (1) in the form:

$$y(x) = \sum_{j=0}^{s+r-1} a_j H_j(x - x_n) \tag{3}$$

where s is the number of interpolation points, r is the number of collocation points and $H_j(x)$ is the Hermite polynomial generated by the formula:

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$$H_n = (-1)^n e^{\left(\frac{x^2}{2}\right)} \frac{d^n}{dx^n} e^{\left(-\frac{x^2}{2}\right)} \tag{4}$$

$$y''(x) = \sum_{j=0}^{s+r-1} a_j H_j''(x - x_n) = f(x, y, y') \tag{5}$$

Now, interpolating (3) at x_n and $x_{n+\frac{3}{2}}$, while collocating (5) at x_n, $x_{n+\frac{3}{2}}$ and at x_{n+2} leads to a system of equations which can be put in matrix form.

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 3 \\ 1 & \frac{3h}{2} & \left(\frac{9h^2}{4} - 1\right) & \left(\frac{27h^3}{8} - \frac{9h}{2}\right) & \left(\frac{81h^4}{16} - \frac{27h^2}{2} + 3\right) \\ 0 & 0 & 2 & 0 & -12 \\ 0 & 0 & 2 & 9h & (27h^2 - 12) \\ 0 & 0 & 2 & 12h & (48h^2 - 12) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} y_n \\ y_{n+\frac{3}{2}} \\ f_n \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{bmatrix} \tag{6}$$

Where the coefficients a_j, $j = 0(1)4$ are obtained as

$$a_0 = y_n + \frac{1}{2} f_n + \frac{1}{12h^2}\left(3 f_{n+2} - 4 f_{n+\frac{3}{2}} + f_n\right)$$

$$a_1 = -\frac{2}{3h} y_n + \frac{2}{3h} y_{n+\frac{3}{2}} - \frac{1}{96h}\left(56 f_n - 128 f_{n+\frac{3}{2}} + 72 f_{n+2}\right) - \frac{h}{96}\left(39 f_n + 60 f_{n+\frac{3}{2}} - 27 f_{n+2}\right)$$

$$a_2 = \frac{1}{6h^2}\left(f_n - 4 f_{n+\frac{3}{2}} + 3 f_{n+2}\right) + \frac{1}{2} f_n$$

$$a_3 = \frac{1}{36h}\left(-7 f_n + 16 f_{n+\frac{3}{2}} - 9 f_{n+2}\right)$$

$$a_4 = \frac{1}{36h^2}\left(f_n - 4 f_{n+\frac{3}{2}} + 3 f_{n+2}\right) \tag{7}$$

Putting (7) in (3) and evaluating at some points yield the following discrete schemes:

$$y_{n+1} - \frac{2}{3} y_{n+\frac{3}{2}} - \frac{1}{3} y_n = h^2\left(\frac{-7}{96} f_n - \frac{7}{24} f_{n+\frac{3}{2}} + \frac{11}{96} f_{n+2}\right)$$

$$y_{n+2} - \frac{4}{3} y_{n+\frac{3}{2}} + \frac{1}{3} y_n = h^2\left(\frac{11}{144} f_n + \frac{19}{36} f_{n+\frac{3}{2}} - \frac{5}{48} f_{n+2}\right)$$

$$hy'_n - \frac{2}{3} y_{n+\frac{3}{2}} + \frac{2}{3} y_n = \frac{h^2}{32}\left(-13 f_n - 20 f_{n+\frac{3}{2}} + 9 f_{n+2}\right)$$

$$hy'_{n+1} + \frac{2}{3} y_n - \frac{2}{3} y_{n+\frac{3}{2}} = h^2\left(\frac{35}{288} f_n + \frac{19}{72} f_{n+\frac{3}{2}} - \frac{13}{96} f_{n+2}\right)$$

$$hy'_{n+\frac{3}{2}} - \frac{2}{3} y_{n+\frac{3}{2}} + \frac{2}{3} y_n = \frac{h^2}{32}\left(5 f_n + 28 f_{n+\frac{3}{2}} - 9 f_{n+2}\right)$$

$$hy'_{n+2} - \frac{2}{3} y_{n+\frac{3}{2}} + \frac{2}{3} y_n = \frac{h^2}{288}\left(43 f_n + 332 f_{n+\frac{3}{2}} - 15 f_{n+2}\right) \tag{8}$$

III. ANALYSIS OF THE METHOD

3.1 Order and error constant

Following [5, 6, 7] and [11, 12], we define the local truncation error associated with the conventional form of (2) to be the linear difference operator.

$$L[y(x);h]=\sum_{j=0}^{k}[\alpha_j(x)y(x+jh)-h^2\beta_j(x)y''(x+jh)]=0 \quad (9)$$

The Taylor's series expansion about the point x gives

$$L[y(x);h]=c_0y(x)+c_1hy'(x)+c_2h^2y''(x)+...+c_ph^py^p(x) \quad (10)$$

According to [8], we say that the method (2) has order p if the difference operator L is of order p and if $c_0=c_1=c_2=...=c_p=c_{p+1}=0,$ and $c_{p+2}\neq 0$ where $c_{p+2}\neq 0$ is called the error constant. We report that the method (7) have uniform order $p=3$ and error constants $c_{p+2}=\left(-\frac{13}{576}\ \ \frac{7}{288}\ \ -\frac{21}{320}\ \ \frac{91}{2880}\ \ \frac{33}{640}\ \ \frac{131}{2880}\right)^T$

3.2 Convergence

The block methods shown in (7) can be represented by a matrix finite difference equation in the form:

$$\varphi Y_{w+1}=\eta y_{w-1}+h^2[\beta_1F_{w+1}+\beta_0f_{w-1}] \quad (11)$$

where

$$Y_{w+1}=(y_{n+1},...,y_{n+3},y'_{n+1},...,y'_{n+3})^T,\ y_{w-1}=(y_{n-3},...,y_{n-\frac{3}{2}},y'_{n-1},...,y'_n)^T$$

$$F_{w+1}=(f_{n-2},...,f_{n-1},f_{n+1},...,f_{n+2})^T,\ f_{w-1}=(f_{n-4},...,f_{n-2},f_{n-\frac{3}{2}},...,f_n)^T$$

where

$$\varphi=\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix},\ \eta=\begin{bmatrix}0&0&0&0&1&0\\0&0&0&0&1&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\end{bmatrix},\ B_0=\begin{bmatrix}0&0&0&0&\frac{1}{3}h^2&-\frac{1}{6}h^2\\0&0&0&0&\frac{15}{16}h^2&-\frac{27}{64}h^2\\0&0&0&0&\frac{16}{9}h^2&-\frac{2}{3}h^2\\0&0&0&0&\frac{8}{9}h^2&-\frac{5}{12}h^2\\0&0&0&0&\frac{3}{2}h^2&-\frac{9}{16}h^2\\0&0&0&0&\frac{16}{9}h^2&-\frac{1}{3}h^2\end{bmatrix},$$

$$,\ B_1=\begin{bmatrix}0&0&0&0&0&-\frac{1}{3}h^2\\0&0&0&0&0&\frac{39}{64}h^2\\0&0&0&0&0&\frac{8}{9}h^2\\0&0&0&0&0&\frac{19}{36}h\\0&0&0&0&0&\frac{9}{16}h\\0&0&0&0&0&\frac{5}{9}h\end{bmatrix}$$

It is worth noting that zero-stability is concerned with the stability of the difference system in the limit as h tends to zero. Thus, as h → 0, the method (11) tends to the difference system.

$$\varphi Y_{w+1} - \eta y_{w-1} = 0 \quad (12)$$

Whose first characteristic polynomial $\rho(\varphi)$ is given by

$$\rho(\varphi) = \det(\varphi I - \eta) = 0$$

$$\rho(\varphi) = \varphi^5(\varphi - 1) \quad (13)$$

Following [7], the block method (7) is zero-stable, since from (13), $\rho(\varphi) = 0$ satisfy $|\varphi_j| \leq 1,\ j = 1, \ldots, k$ and for those roots with $|\varphi_j| = 1$, the multiplicity does not exceed 2. The block method (11) is consistent as it has order P > 1. Accordingly, from [8], we assert the convergence of the block hybrid method (7).

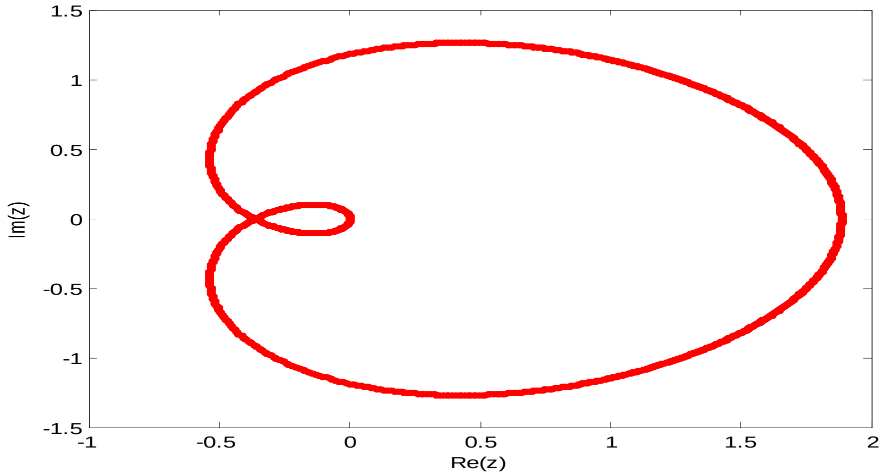

Figure 1: Absolute Stability Region of the Block hybrid method

IV. NUMERICAL EXAMPLE

In this section, we report two numerical examples taken from relevant literature to compare the performance of our method. The absolute errors obtained from computed results at selected mesh points are also reported in the tables shown.

Problem 1:

$$y'' = y',\ y(0) = 0,\ y'(0) = -1,\ h = 0.1$$

Analytical solution: $y(x) = 1 - \exp(x)$

Source: Yahaya and Badmus (2009)

Table 1: Comparism of absolute errors of the new method with other authors for problem 1

x	Analytical Solution	BHyMS	Errors in [13]	Errors in BHyMS
0.1	-0.10517091807565	-0.10517041901188	8.79316 X 10^{-05}	4.99063768 X 10^{-7}
0.2	-0.22140275816017	-0.22140087554722	3.26718 X 10^{-04}	1.88261295 X 10^{-6}
0.3	-0.34985880757600	-0.34985480195761	2.21556 X 10^{-03}	4.00561839 X 10^{-6}
0.4	-0.49182469764127	-0.49181732954579	4.85709 X 10^{-03}	7.36809500 X 10^{-6}
0.5	-0.64872127070013	-0.64870946105267	9.09773 X 10^{-03}	1.18096475 X 10^{-5}
0.6	-0.82211880039051	-0.82210084090240	1.43914 X 10^{-02}	1.79594881 X 10^{-5}
0.7	-1.01375270747048	-1.01372706531323	2.14379 X 10^{-02}	2.56421573 X 10^{-5}
0.8	-1.22554092849247	-1.22550527972633	2.98987 X 10^{-02}	3.56487661 X 10^{-5}
0.9	-1.45960311115695	-1.45955532117372	4.03007 X 10^{-02}	4.77899832 X 10^{-5}
1.0	-1.71828182845905	-1.71821876879647	5.25521 X 10^{-02}	6.30596627 X 10^{-5}

Problem 2:

$$y'' + 1001y' + 1000y = 0,\ y(0) = 1,\ y'(0) = -1,\ h = 0.05$$

Analytical solution: $y(x) = \exp(-x)$

Source: Jator (2007)

Table 2: Comparism of absolute errors of the new method with other authors for problem 2

x	Analytical Solution	BHyMS	Errors in [9]	Errors in BHyMS
0.1	0.90483741803596	0.90483742302973	6.98677 X 10^{-12}	1.62 X 10^{-12}
0.2	0.81873075307798	0.81873076660624	1.00275 X 10^{-12}	2.94 X 10^{-12}
0.3	0.74081822068172	0.74081824239740	7.85878 X 10^{-12}	4.00 X 10^{-12}
0.4	0.67032004603564	0.67032007464740	1.04778 X 10^{-11}	4.86 X 10^{-12}
0.5	0.60653065971263	0.60653069382600	6.32212 X 10^{-11}	5.50 X 10^{-12}
0.6	0.54881163609403	0.54881167443656	1.00508 X 10^{-11}	5.95 X 10^{-12}
0.7	0.49658530379141	0.49658534525922	9.36336 X 10^{-12}	6.31 X 10^{-12}
0.8	0.44932896411722	0.44932900777118	2.64766 X 10^{-12}	6.51 X 10^{-12}
0.9	0.40656965974060	0.40656970478997	1.06793 X 10^{-11}	6.62 X 10^{-12}
1.0	0.36787944117144	0.36787948695556	2.32731 X 10^{-11}	6.66 X 10^{-12}

V. CONCLUSION

We developed a third order block hybrid block methods for the solution of general second order Ordinary differential equations (BhyMS). The method (7) was derived using hermite polynomial as basis function. Graph of its absolute stability region suggest that it is $A(\alpha)-stable$ (figure 1). The method proved to be very efficient when tested on two numerical problems and when compared with results obtained from relevant authors (tables 1 and table 2).

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