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Research Paper

Mathematical Analysis of A Prey-Predator Fishery Model in Three Patch Aquatic Habitat

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ABSTRACT: *This paper examines the analysis of a prey-predator fishery model in a three –patch aquatic habitat. Boundedness of the formulated marine protected system is examined. Continuity, partial differentiability conditions and non-existence of periodic solution of the system is established. The existence of its steady state solutions and their local and global stability are studied using eigen-value analysis. Graphical solutions of the model are provided to complement the analytical findings.*

Keywords: *prey-predator, local and global stability, aquatic habitat, reserve zone*

I. INTRODUCTION

The marine ecosystem needs protection, as well as individual species and habitats (Khamis et.al,2011) .Over-fishing, use of destructive fishing methods and pollution are all taking a toll on marine biodiversity (Kar, 2006). The effectiveness of a protected area, restricted from fishing, depends on a complex set of interactions between biological, economic and institutional factors as it provides protection for critical habitats and cultural heritage sites and some cases conserve biodiversity as a tool to enhance fishery management. Ecological benefits within marine protected areas (MPA'S) are realised as they increase fish abundance in adjacent fishable area (Kribs-zaleta, 2009) which makes up for the lost areas that could have been used for open-access fishery.
Other works done on the effects of harvesting on population growth, from the context of predator-prey interaction include those of Brauer and Soudack(1979a), Dai and Tang(1998), Chaudhuri and Ray (1996), Zhang et.al (2007), Kar (2003), Khamis et.al,2011; Inyama,(2008), Dubey and Patra (2013), Mellachervn et.al (2011), Brauer and Soudack(1979b), Kar and Pahari (2007), Dubey et.al (2003), May (1974) and ece -terra.

Closing off an area that historically contributed a significant catch would prohibits or reduces the total catch at least in the short run, as population levels begin to recover in the MPA (Marine Protected Area) and spill over to the fishable areas increase (Sanchirico et.al, 2002).On the basis of a detailed literature review on our propose fishery model, the issue of dynamics of fisheries proposed in the context of conservation of fishery resources consisting of three zones remain to be an open problem.

II. MATHEMATICAL MODEL

Consider an aquatic ecosystem consisting of reserved zones. Modelling the fishery habitat, we consider that no fishing and other recreational activities are allowed. The fishery habitat under consideration consists of three-patchy zones assumed to be homogeneous and based on a formulation of marine protected area. The model considers two species, the prey's fish and the predator fish species and the growth of the prey's fish population follows the logistic form of the Lotka-Volterra prey –predator equation with the presence of predator in both zones. Prey's migrates between zones randomly at different ratesτ_1, τ_2 and τ_3 for first, second and third prey fish populations respectively.

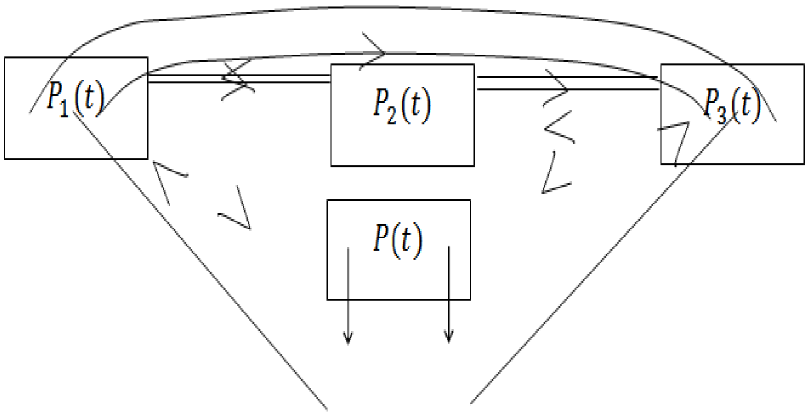

Fig.2.1: Flowchart of the model

Table 2.1:Description of Variables of the model

Variables	Interpretation
$P_1(t)$	Biomass density of the first prey fish at time, t.
$P_2(t)$	Biomass density of the second prey fish at time, t.
$P_3(t)$	Biomass density of the third prey fish at time , t.
$P(t)$	Biomass density of the predator fish at time, t.

Table 2.2: Description of parameters of the model equation

Variables	Interpretation
r_1	Intrinsic growth rate of first prey fish.
r_2	Intrinsic growth rate of second prey fish.
r_3	Intrinsic growth rate of third prey fish.
K_1	Carrying capacity of the first prey fish
K_2	Carrying Capacity of the second prey fish
K_3	Carrying Capacity of the third prey fish
μ_1	Death rate of the predator
μ_2	Intraspecific competition coefficient of the predator
ϕ_1	First prey fish mortality rate due to predation in zone 1
ϕ_2	Second prey fish mortality rate due to predation in zone 2
ϕ_3	Third prey fish mortality rate due to predation in zone 3
τ_1	Rate of migration of the first prey fish.
τ_2	Rate of migration of the second prey fish.
τ_3	Rate of migration of the third prey fish.
ϵ_1	Conversion rate of predator in zone 1
ϵ_2	Conversion rate of predator in zone 2
ϵ_3	Conversion rate of predator in zone 3

2.1 Assumptions of the model

The following assumptions are taken into account in the construction of the model system (2.1):

(1) The prey fish populations are suffering some sort of ecological risk such as predation.
(2) We assume that, since the growth rate of the predator fish population depends on the growth rate of the prey fish populations, it follows that the growth rate of the predator fish can only have negative value, if the birth rate of predator population is less than the death rate of the predator population.
(3) The fish population size is not directly related to age.
(4) We assume that the reproduction of predator after predating the prey fish is instantaneous (no time lag required for gestation of predator).
(5) The variables$P_1(t)$, $P_2(t)$, $P_3(t)$ $and P(t)$ are continuous functions of time.
(6) The predator fish having the prey fish as the only food source decays in the absence of the prey fish.
(7) The prey fish has a logistic growth in the absence of the predator fish.
(8) All the parameters are assumed to be positive.
(9) There are three patches from where the three types of prey fish come from and the patches are homogeneous from ecological point of view,[Khamis et.al, 2011,page 1779].
(10) Prey migrates between zones 1, 2 and 3 randomly.
(11) Predators are assumed to be ubiquitous throughout the whole patches with different migration rates[Khamis et al 2011, page 1780]

(12) We assume $0 < \epsilon_i < 1 (i = 1,\ 2,\ 3)$ as the whole biomass of the prey is not transformed to the biomass of the predator. (Das et.al, 2009).

Applying the assumptions, definition of variables and parameters, and description of terms, the equations governing our fish populations' model in three zones become:

$$\frac{dP_1}{dt} = r_1P_1\left(1 - \frac{P_1}{k_1}\right) - \tau_1P_1 + \tau_2P_2 + \tau_3P_3 - \phi_1P_1P$$
$$\frac{dP_2}{dt} = r_2P_2\left(1 - \frac{P_2}{k_2}\right) + \tau_1P_1 - \tau_2P_2 + \tau_3P_3 - \phi_2P_2P \quad (2.1)$$
$$\frac{dP_3}{dt} = r_3P_3\left(1 - \frac{P_3}{k_3}\right) + \tau_1P_1 + \tau_2P_2 - \tau_3P_3 - \phi_3P_3P$$
$$\frac{dP}{dt} = P(-\mu_1 - \mu_2P + \epsilon_1P_1 + \epsilon_2P_2 + \epsilon_3P_3)$$

These system (2.1) model equations evolve on the basis of initial conditions $P_1(0) > 0,\ P_2(0) > 0,\ P_3(0) > 0\ and\ \ P(0) > 0$. Model (2.1) is biological meaningful if the following conditions are satisfied

$$(i)\ \ (r_1 - \tau_1) > 0,\ \ (ii)(r_2 - \tau_2) > 0, \qquad and\ \ (iii)(r_3 - \tau_3) > 0$$

III. CONTINUITY AND PARTIAL DIFFERENTIABILITY CONDITIONS

3.1 Continuity

Assume an arbitrary steady state solution $(P_{1e}, P_{2e}, P_{3e}, P_e)$ of the model system (2.1)

Let $F_1(P_1, P_2, P_3, P) = r_1P_1\left(1 - \frac{P_1}{k_1}\right) - \tau_1P_1 + \tau_2P_2 + \tau_3P_3 - \phi_1P_1P$

$$\lim_{\substack{P_1\to P_{1e}\\ P_2\to P_{2e}\\ P_3\to P_{3e}\\ P\to P_e}} F_1(P_1, P_2, P_3, P) = F_1(P_{1e}, P_{2e}, P_{3e}, P_e)$$

$$= \lim_{\substack{P_1\to P_{1e}\\ P_2\to P_{2e}\\ P_3\to P_{3e}\\ P\to P_e}} \left[r_1P_1\left(1 - \frac{P_1}{k_1}\right) - \tau_1P_1 + \tau_2P_2 + \tau_3P_3 - \phi_1P_1P\right]$$

$$= r_1P_{1e}\left(1 - \frac{P_{1e}}{k_1}\right) - \tau_1P_{1e} + \tau_2P_{2e} + \tau_3P_{3e} - \phi_1P_{1e}P_e = F_1(P_{1e}, P_{2e}, P_{3e}, P_e) \quad (3.1)$$

Let $F_2(P_1, P_2, P_3, P) = r_2P_2\left(1 - \frac{P_2}{k_2}\right) - \tau_2P_2 + \tau_1P_1 + \tau_3P_3 - \phi_2P_2P$

$$\lim_{\substack{P_1\to P_{1e}\\ P_2\to P_{2e}\\ P_3\to P_{3e}\\ P\to P_e}} F_2(P_1, P_2, P_3, P) = \lim_{\substack{P_1\to P_{1e}\\ P_2\to P_{2e}\\ P_3\to P_{3e}\\ P\to P_e}} \left[r_2P_2\left(1 - \frac{P_2}{k_2}\right) - \tau_2P_2 + \tau_1P_1 + \tau_3P_3 - \phi_2P_2P\right]$$

$$= r_2P_{2e}\left(1 - \frac{P_{2e}}{k_2}\right) - \tau_2P_{2e} + \tau_1P_{1e} + \tau_3P_{3e} - \phi_2P_{2e}P_e = F_2(P_{1e}, P_{2e}, P_{3e}, P_e) \quad (3.2)$$

Let $F_3(P_1, P_2, P_3, P) = r_3P_3\left(1 - \frac{P_3}{k_3}\right) - \tau_3P_3 + \tau_1P_1 + \tau_2P_2 - \phi_3P_3P$

$$\lim_{\substack{P_1\to P_{1e}\\ P_2\to P_{2e}\\ P_3\to P_{3e}\\ P\to P_e}} F_3(P_1, P_2, P_3, P) = \lim_{\substack{P_1\to P_{1e}\\ P_2\to P_{2e}\\ P_3\to P_{3e}\\ P\to P_e}} \left[r_3P_3\left(1 - \frac{P_3}{k_3}\right) - \tau_3P_3 + \tau_1P_1 + \tau_2P_2 - \phi_3P_3P\right]$$

$$= r_3P_{3e}\left(1 - \frac{P_{3e}}{k_3}\right) - \tau_3P_{3e} + \tau_1P_{1e} + \tau_2P_{2e} - \phi_3P_{3e}P_e = F_3(P_{1e}, P_{2e}, P_{3e}, P_e) \quad (3.3)$$

Let $F_4(P_1, P_2, P_3, P) = P(-\mu_1 - \mu_2P + \epsilon_1P_1 + \epsilon_2P_2 + \epsilon_3P_3)$

$$\lim_{\substack{P_1\to P_{1e}\\ P_2\to P_{2e}\\ P_3\to P_{3e}\\ P\to P_e}} F_4(P_1, P_2, P_3, P) = \lim_{\substack{P_1\to P_{1e}\\ P_2\to P_{2e}\\ P_3\to P_{3e}\\ P_1\to P_e}} [P(-\mu_1 - \mu_2P + \epsilon_1P_1 + \epsilon_2P_2 + \epsilon_3P_3)]$$

$$= P_e(-\mu_1 - \mu_2P_e + \epsilon_1P_{1e} + \varepsilon_2P_{2e} + \epsilon_3P_{3e}) = F_4(P_{1e}, P_{2e}, P_{3e}, P_e) \quad (3.4)$$

3.2 Partial differentiability conditions

In this section, we prove the partial differentiability conditions of the model system (2.1)

We prove that, $J_{11} = \frac{\partial F_1}{\partial P_1} = r_1 - \frac{2r_1}{k_1}P_1 - \tau_1 - \phi_1P$

$$\frac{\partial F_1}{\partial P_1} = \lim_{\Delta P_1\to 0} \frac{F_1(P_1 + \Delta P_1, P_2, P_3, P) - F_1(P_1, P_2, P_3, P)}{\Delta P_1} \quad (3.5)$$

$F_1(P_1+\Delta P_1,P_2,P_3,P)$
$= \mathrm{r}_1(\mathrm{P}_1+\Delta P_1)-\frac{\mathrm{r}_1(\mathrm{P}_1+\Delta P_1)^2}{k_1}-\tau_1(\mathrm{P}_1+\Delta P_1)+\tau_2\mathrm{P}_2+\tau_3\mathrm{P}_3-\phi_1(\mathrm{P}_1+\Delta P_1)P$
$= \mathrm{r}_1\mathrm{P}_1+\mathrm{r}_1\Delta P_1-\frac{\mathrm{r}_1 {P_1}^2}{k_1}-2\frac{\mathrm{r}_1}{\mathrm{k}_1}\mathrm{P}_1 . \Delta P_1 - \frac{\mathrm{r}_1}{\mathrm{k}_1} . (\Delta P_1)^2-\tau_1\mathrm{P}_1-\tau_1\Delta P_1+\tau_2\mathrm{P}_2 + \tau_3\mathrm{P}_3$
$-\phi_1\mathrm{P}_1P+\phi_1\Delta P_1 P$ (3.6)
$F_1(P_1+\Delta P_1,P_2,P_3,P)-F_1(P_1,P_2,P_3,P)$
$= \mathrm{r}_1\Delta P_1-\frac{2\mathrm{r}_1}{\mathrm{k}_1}\mathrm{P}_1\Delta P_1-\frac{\mathrm{r}_1}{\mathrm{k}_1}(\Delta P_1)^2-\tau_1\Delta P_1-\phi_1\Delta P_1 P$
$= \Delta P_1\left[\mathrm{r}_1-\frac{2\mathrm{r}_1}{\mathrm{k}_1}\mathrm{P}_1-\frac{\mathrm{r}_1}{\mathrm{k}_1}(\Delta P_1)-\tau_1-\phi_1 P_1\right]$ (3.7)
$\left\{\frac{F_1(P_1+\Delta P_1,P_2,P_3,P)-F_1(P_1,P_2,P_3,P)}{\Delta P_1}\right\}=\left[\mathrm{r}_1-\frac{2\mathrm{r}_1}{\mathrm{k}_1}\mathrm{P}_1-\frac{\mathrm{r}_1}{\mathrm{k}_1}\Delta P_1-\tau_1-\phi_1 P\right]$
$=\left[\mathrm{r}_1-\frac{2\mathrm{r}_1}{\mathrm{k}_1}\mathrm{P}_1-\frac{\mathrm{r}_1}{\mathrm{k}_1}\Delta P_1-\tau_1-\phi_1 P\right]$
$\lim_{\Delta P_1\to 0}\left[\mathrm{r}_1-\frac{2\mathrm{r}_1}{\mathrm{k}_1}\mathrm{P}_1-\frac{\mathrm{r}_1}{\mathrm{k}_1}\Delta P_1-\tau_1-\phi_1 P\right]==\mathrm{r}_1-\frac{2\mathrm{r}_1}{\mathrm{k}_1}\mathrm{P}_1-\tau_1-\phi_1 P$ (3.8)
Thus $\mathrm{J}_{11}=\frac{\partial \mathrm{F}_1}{\partial P_1}=\mathrm{r}_1-\frac{2\mathrm{r}_1}{k_1}\mathrm{P}_1-\tau_1-\phi_1 P$
$\frac{\partial \mathrm{F}_1}{\partial P_2}=\lim_{\Delta P_1\to 0}\frac{F_1(P_{1,}+P_2+\Delta P_2,P_3,P)-F_1(P_1,P_2,P_3,P)}{\Delta P_2}$ (3.9)
$F_1(P_{1,}+P_2+\Delta P_2,P_3,P)=\mathrm{r}_1-\frac{\mathrm{r}_1}{\mathrm{k}_1}\mathrm{P}_1^2-\tau_1 P_1+\tau_2(P_2+\Delta P_2)+\tau_3 P_3-\phi_1 P_1 P$
$=\mathrm{r}_1-\frac{\mathrm{r}_1}{\mathrm{k}_1}\mathrm{P}_1^2-\tau_1 P_1+\tau_2 P_2+\Delta P_2\tau_2+\tau_3 P_3-\phi_1 P_1 P$
$F_1(P_{1,}+P_2+\Delta P_2,P_3,P)-F_1(P_1,P_2,P_3,P)$
$=\mathrm{r}_1-\frac{\mathrm{r}_1}{\mathrm{k}_1}\mathrm{P}_1^2-\tau_1 P_1+\tau_2 P_2+\tau_2\Delta P_2+\tau_3 P_3-\phi_1 P_1 P-\mathrm{r}_1+\frac{\mathrm{r}_1}{\mathrm{k}_1}\mathrm{P}_1^2+\tau_1 P_1-\tau_2 P_2$
$-\tau_3 P_3+\phi_1 P_1 P \; = \tau_2\Delta P_2$ (3.10)
$\lim_{\Delta P_1\to 0}\frac{\tau_2\Delta P_2}{\Delta P_2}=\lim_{\Delta P_1\to 0}\tau_2 \; =\tau_2$ (3.11)
which proves$\mathrm{J}_{12}=\tau_2=\frac{\partial \mathrm{F}_1}{\partial P_2}$
$\frac{\partial \mathrm{F}_1}{\partial P_3}=\lim_{\Delta P_3\to 0}\frac{F_1(P_{1,}P_2,P_3+\Delta P_3,P)-F_1(P_1,P_2,P_3,P)}{\Delta P_3}$ (3.12)
$F_1(P_{1,}P_2,P_3+\Delta P_3,P)=\mathrm{r}_1-\frac{\mathrm{r}_1}{\mathrm{k}_1}\mathrm{P}_1^2-\tau_1 P_1+\tau_2 P_2+\tau_3(P_3+\Delta P_3)-\phi_1 P_1 P$
$=\mathrm{r}_1-\frac{\mathrm{r}_1}{\mathrm{k}_1}\mathrm{P}_1^2-\tau_1 P_1+\tau_2 P_2+\tau_3 P_3+\tau_3\Delta P_3-\phi_1 P_1 P$ (3.13)
$F_1(P_{1,}P_2,P_3+\Delta P_3,P)-F_1(P_1,P_2,P_3,P)$
$=\mathrm{r}_1-\frac{\mathrm{r}_1}{\mathrm{k}_1}\mathrm{P}_1^2-\tau_1 P_1+\tau_2 P_2+\tau_3 P_3+\tau_3\Delta P_3-\phi_1 P_1 P-\mathrm{r}_1-\frac{\mathrm{r}_1}{\mathrm{k}_1}\mathrm{P}_1^2+\tau_1 P_1-\tau_2 P_2$
$-\tau_3 P_3+\phi_1 P_1 P \quad == \tau_3\Delta P_3$ (3.14)
$\lim_{\Delta P_3\to 0}\frac{\tau_3\Delta P_3}{\Delta P_3}=\lim_{\Delta P_3\to 0}\tau_3=\tau_3$ (3.15)
This confirms $\mathrm{J}_{13}=\tau_3=\frac{\partial \mathrm{F}_1}{\partial P_3}$
$\frac{\partial \mathrm{F}_1}{\partial P}=\lim_{\Delta P\to 0}\frac{F_1(P_{1,}P_2,P_3,P+\Delta P)-F_1(P_1,P_2,P_3,P)}{\Delta P}$ (3.16)
$F_1(P_{1,}P_2,P_3,P+\Delta P)=\mathrm{r}_1-\frac{\mathrm{r}_1}{\mathrm{k}_1}\mathrm{P}_1^2-\tau_1 P_1+\tau_2 P_2+\tau_3 P_3-\phi_1 P_1(P+\Delta P)$
$=\mathrm{r}_1-\frac{\mathrm{r}_1}{\mathrm{k}_1}\mathrm{P}_1^2-\tau_1 P_1+\tau_2 P_2+\tau_3 P_3-\phi_1 P_1 P-\phi_1 P_1\Delta P$ (3.17)
$=\mathrm{r}_1-\frac{\mathrm{r}_1}{\mathrm{k}_1}\mathrm{P}_1^2-\tau_1 P_1+\tau_2 P_2+\tau_3 P_3-\phi_1 P_1 P-\phi_1 P_1\Delta P-\mathrm{r}_1+\frac{\mathrm{r}_1}{\mathrm{k}_1}\mathrm{P}_1^2+\tau_1 P_1-\tau_2 P_2$
$-\tau_3 P_3+\phi_1 P_1 P \quad ==-\phi_1 P_1\Delta$ (3.18)
$\lim_{\Delta P\to 0}\frac{-\phi_1 P_1\Delta P}{\Delta P} \; =-\lim_{\Delta P\to 0}\phi_1 P_1=-\phi_1 P_1$ (3.19)
This confirms $\mathrm{J}_{14}=-\phi_1 P_1=\frac{\partial \mathrm{F}_1}{\partial P}$

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Next, we show that

$$J_{21} = \frac{\partial F_2}{\partial P_1} = \tau_1$$

But, $$\frac{\partial F_2}{\partial P_1} = \lim_{\Delta P_1 \to 0} \frac{F_2(P_1 + \Delta P_1, P_2, P_3, P) - F_2(P_1, P_2, P_3, P)}{\Delta P_1} \quad (3.20)$$

$$F_2(P_1 + \Delta P_1, P_2, P_3, P) = r_2 P_2 - \frac{r_1}{k_1} P_2^2 - \tau_1(P_1 + \Delta P_1) - \tau_2 P_2 + \tau_3 P_3 - \phi_2 P_2 P$$

$$= \tau_2 P_2 - \frac{r_2}{k_2} P_2^2 - \tau_1 P_1 - \Delta P_1 \tau_1 - \tau_2 P_2 + \tau_3 P_3 - \phi_2 P_2 P \quad (3.21)$$

$$F_2(P_1 + \Delta P_1, P_2, P_3, P) - F_2(P_1, P_2, P_3, P)$$

$$= r_2 P_2 - \frac{r_2}{k_2} P_2^2 + \tau_1 P_1 + \Delta P_1 \tau_1 - \tau_2 P_2 + \tau_3 P_3 - \phi_2 P_2 P - r_2 P_2 + \frac{r_2}{k_2} P_2^2 - \tau_1 P_1 + \tau_2 P_2$$

$$-\tau_3 P_3 + \phi_2 P_2 P \quad (3.22)$$

$$\lim_{\Delta P_1 \to 0} \frac{\Delta P_1 \tau_1}{\Delta P_1} = \lim_{\Delta P_1 \to 0} \tau_1 = \tau_1$$

Hence, $$J_{21} = \frac{\partial F_2}{\partial P_1} = \tau_1 \quad (3.23)$$

We prove that

$$J_{22} = \frac{\partial F_2}{\partial P_2} = r_2 - \frac{2r_2}{k_2} P_2 - \tau_2 - \phi_2 P$$

$$\frac{\partial F_2}{\partial P_2} = \lim_{\Delta P_2 \to 0} \frac{F_2(P_1,\ P_2\ + \Delta P_2,\ P_3,\ P) - F_2(P_1,\ P_2,\ P_3,\ P)}{\Delta P_2} \quad (25)$$

$$F_2(P_1,\ P_2 + \Delta P_2, P_3, P) = r_2(P_2 + \Delta P_2) - \frac{r_2}{k_2}(P_2 + \Delta P_2)^2 + \tau_1 P_1 - \tau_2(P_2 + \Delta P_2)$$

$$+\tau_3 P_3 - \phi_2(P_2 + \Delta P_2)P$$

$$= r_2 P_2 + r_2 \Delta P_2 - \frac{r_2}{k_2} P_2^2 - \frac{2r_2}{k_2} P_2 \Delta P_2 - \frac{r_2}{k_2}(\Delta P_2)^2 + \tau_1 P_1 - \tau_2 P_2 - \tau_2 \Delta P_2 + \tau_3 P_3$$

$$+\phi_2 P_2 P - \phi_2 \Delta P_2 P \quad (3.24)$$

$$\Rightarrow F_2(P_1, P_2 + \Delta P_2, P_3, P) - F_2(P_1, P_2, P_3, P)$$

$$= r_2 \Delta P_2 - \frac{2r_2}{k_2} P_2 \Delta P_2 - \frac{r_2}{k_2}(\Delta P_2)^2 - \tau_2 \Delta P_2 - \phi_2 \Delta P_2 P \quad (3.25)$$

$$= \Delta P_2 \left[r_2 - \frac{2r_2}{k_2} P_2 - \frac{r_2}{k_2}(\Delta P_2) - \tau_2 - \phi_2 P \right]$$

$$\Rightarrow \frac{F_2(P_1,\ P_2\ + \Delta P_2,\ P_3,\ P) - F_2(P_1,\ P_2,\ P_3,\ P)}{\Delta P_2}$$

$$\left[r_2 - \frac{2r_2}{k_2} P_2 - \frac{r_2}{k_2} \Delta P_2 - \tau_2 - \phi_2 P \right]$$

$$\lim_{\Delta P_2 \to 0} \left[r_2 - \frac{2r_2}{k_2} P_2 - \frac{r_2}{k_2} \Delta P_2 - \tau_2 - \phi_2 P \right] = r_2 - \frac{2r_2}{k_2} P_2 - \tau_2 - \phi_2 P$$

$$\therefore J_{22} = r_2 - \frac{2r_2}{k_2} P_2 - \tau_2 - \phi_2 P \quad (3.26)$$

We prove that

$$J_{23} = \frac{\partial F_2}{\partial P_3} = \tau_3$$

$$\frac{\partial F_2}{\partial P_3} = \lim_{\Delta P_3 \to 0} \frac{F_2(P_1,\ P_2,\ P_3 + \Delta P_3,\ P) - F_2(P_1,\ P_2,\ P_3,\ P)}{\Delta P_3} \quad (3.27)$$

$$F_2(P_1,\ P_2, P_3 + \Delta P_3, P) = r_2 P_2 - \frac{r_2}{k_2} P_2^2 - \tau_2 P_2 + \tau_3(P_3 + \Delta P_3) + \tau_1 P_1 - \phi_2 P_2 P$$

$$= r_2 P_2 - \frac{r_2}{k_2} P_2^2 - \tau_2 P_2 + \tau_3 P_3 + \Delta P_2 \tau_3 + \tau_1 P_1 - \phi_2 P_2 P \quad (3.28)$$

$$F_2(P_1,\ P_2, P_3 + \Delta P_3, P) - F_2(P_1, P_2, P_3, P)$$

$$= r_2 P_2 - \frac{r_2}{k_2} P_2^2 - \tau_2 P_2 + \tau_3 P_3 - \Delta P_3 \tau_3 + \tau_1 P_1 - \phi_2 P_2 P - r_2 P_2 + \frac{r_2}{k_2} P_2^2 + \tau_2 P_2$$

$$-\tau_3 P_3 - \tau_1 P_1 + \phi_2 P_2 P = \Delta P_3 \tau_3 \quad (3.29)$$

$$\frac{\partial F_2}{\partial P_3} = \lim_{\Delta P_3 \to 0} \frac{\Delta P_3 \tau_3}{\Delta P_3} = \lim_{\Delta P_3 \to 0} \tau_3 = \tau_3 \quad (3.30)$$

We prove that

$J_{24} = \frac{\partial F_2}{\partial P} = -\phi_2 P$

$\frac{\partial F_2}{\partial P} = \lim_{\Delta P \to 0} \frac{F_2(P_1, P_2, P_3, P + \Delta P) - F_2(P_1, P_2, P_3, P)}{\Delta P}$ (33)

$F_2(P_1, P_2, P_3, P + \Delta P) = r_2 P_2 - \frac{r_2}{k_2} P_2 - \tau_2 P_2 + \tau_3 P_3 + \tau_1 P_1 - \phi_2 P_2 (P + \Delta P)$

$= r_2 P_2 - \frac{r_2}{k_2} P_2 - \tau_2 P_2 + \tau_3 P_3 + \tau_1 P_1 - \phi_2 P_2 P - \phi_2 P_2 \Delta P$ (3.31)

$F_2(P_1, P_2, P_3, P + \Delta P) - F_2(P_1, P_2, P_3, P)$

$= r_2 P_2 - \frac{r_2}{k_2} P_2 - \tau_2 P_2 + \tau_3 P_3 + \tau_1 P_1 - \phi_2 P_2 P - \phi_2 P_2 \Delta P_2$

$- r_2 P_2 + \frac{r_2}{k_2} P_2 + \tau_2 P_2 - \tau_3 P_3 - \tau_1 P_1 + \phi_2 P_2 P = -\phi_2 P_2 \Delta P$ (3.32.)

$\lim_{\Delta P \to 0} \frac{F_2(P_1, P_2, P_3, P + \Delta P) - F_2(P_1, P_2, P_3, P)}{\Delta P} = - \lim_{\Delta P \to 0} \frac{\phi_2 P_2 \Delta P}{\Delta P}$

$= - \lim_{\Delta P \to 0} \phi_2 P_2 = -\phi_2 P_2$ (3.33)

We prove that

$J_{31} = \frac{\partial F_3}{\partial P_1} = \tau_1$

$\frac{\partial F_3}{\partial P_1} = \lim_{\Delta P_1 \to 0} \frac{F_3(P_1 + \Delta P_1, P_2, P_3, P) - F_3(P_1, P_2, P_3, P)}{\Delta P_1}$ (3.34)

$F_3(P_1 + \Delta P_1, P_2, P_3, P) = r_3 P_3 - \frac{r_3}{k_3} P_3 - \tau_3 P_3 + \tau_1 (P_1 + \Delta P_1) + \tau_2 P_2 - \phi_3 P_3 P$

$= r_3 P_3 - \frac{r_3}{k_3} P_3 - \tau_3 P_3 + \tau_1 P_1 + \Delta P_1 \tau_1 + \tau_2 P_2 - \phi_3 P_3 P$ (3.35)

$F_3(P_1 + \Delta P_1, P_2, P_3, P) - F_3(P_1, P_2, P_3, P)$

$= r_3 P_3 - \frac{r_3}{k_3} P_3 - \tau_3 P_3 + \tau_1 P_1 + \Delta P_1 \tau_1 + \tau_2 P_2 - \phi_3 P_3 P$

$- r_3 P_3 + \frac{r_3}{k_3} P_3 + \tau_3 P_3 + \tau_1 P_1 - \tau_2 P_2 + \phi_3 P_3 P = \Delta P_1 \tau_1$ (39)

$\lim_{\Delta P_1 \to 0} \frac{F_3(P_1 + \Delta P_1, P_2, P_3, P) - F_3(P_1, P_2, P_3, P)}{\Delta P_1}$

$= \lim_{\Delta P_1 \to 0} \frac{\Delta P_1 \tau_1}{\Delta P_1} = \lim_{\Delta P_1 \to 0} \frac{\tau_1}{\Delta P_1} = \tau_1$ (3.36)

We prove that

$J_{32} = \frac{\partial F_3}{\partial P_2} = \tau_2$

$\frac{\partial F_3}{\partial P_2} = \lim_{\Delta P_2 \to 0} \frac{F_3(P_1, P_2 + \Delta P_2, P_3, P) - F_2(P_1, P_2, P_3, P)}{\Delta P_2}$ (3.37)

$F_3(P_1 + \Delta P_1, P_2, P_3, P) = r_3 P_3 - \frac{r_3}{k_3} {P_3}^2 - \tau_3 P_3 + \tau_1 P_1 + \tau_2 (P_2 + \Delta P_2) - \phi_3 P_3 P$

$= r_3 P_3 - \frac{r_3}{k_3} {P_3}^2 - \tau_3 P_3 + \tau_1 P_1 + \tau_2 P_2 + \tau_2 \Delta P_2 - \phi_3 P_3 P$ (3.38)

$F_3(P_1, P_2 + \Delta P_2, P_3, P) - F_3(P_1, P_2, P_3, P)$

$= r_3 P_3 - \frac{r_3}{k_3} {P_3}^2 - \tau_3 P_3 + \tau_1 P_1 + \tau_2 P_2 + \tau_2 \Delta P_2 - \phi_3 P_3 P - r_3 P_3 + \frac{r_3}{k_3} {P_3}^2 + \tau_3 P_3$

$- \tau_1 P_1 - \tau_2 P + \tau_2 \Delta P_2 + \phi_3 P_3 P = \tau_2 \Delta P_2$ (3.39)

$\lim_{\Delta P_2 \to 0} \frac{F_3(P_1 + \Delta P_1, P_2, P_3, P) - F_3(P_1, P_2, P_3, P)}{\Delta P_2}$

$= \lim_{\Delta P_2 \to 0} \frac{\tau_2 \Delta P_2}{\Delta P_2} = \lim_{\Delta P_2 \to 0} \tau_2 = \tau_2$ (3.40)

We prove that

$J_{33} = \frac{\partial F_3}{\partial P_3} = r_3 - \frac{2 r_3}{k_3} P_3 - \tau_3 - \phi_3 P$

$\frac{\partial F_3}{\partial P_3} = \lim_{\Delta P_3 \to 0} \frac{F_3(P_1, P_2, P_3 + \Delta P_3, P) - F_3(P_1, P_2, P_3, P)}{\Delta P_3}$ (3.41)

$F_3(P_1, P_2, P_3 + \Delta P_3, P) = r_3 (P_3 + \Delta P_3) - \frac{r_3}{k_3} (P_3 + \Delta P_3)^2 - \tau_3 (P_3 + \Delta P_3)$

$\tau_1 P_1 - \tau_2 P_2 + \phi_3 P (P_3 + \Delta P_3)$

$$= r_3P_3 + \tau_3\Delta P_3 - \frac{r_3}{k_3}P_3^2 - \frac{2r_3}{k_3}P_3\Delta P_3 - \frac{r_3}{k_3}(\Delta P_3)^2 - \tau_3P_3 - \tau_3\Delta P_3 + \tau_1P_1 + \tau_2P_2$$
$$-\phi_3P_3P - \phi_3P\Delta P_3 \quad (3.42)$$
$$F_3(P_1,\ P_2, P_3 + \Delta P_3, P) - F_3(P_1, P_2, P_3, P)$$
$$= r_3P_3 + r_3\Delta P_3 - \frac{r_3}{k_3}P_3^2 - \frac{2r_3}{k_3}P_3\Delta P_3 - \frac{r_3}{k_3}(\Delta P_3)^2 - \tau_3P_3 + \tau_3\Delta P_3 + \tau_1P_1 + \tau_2P_2$$
$$-\phi_3P_3P - \phi_3\Delta P_3 - \tau_3P_3 + \frac{r_3}{k_3}P_3^2 + \tau_3P_3 - \tau_1P_1 - \tau_2P_2$$
$$= r_3\Delta P_3 - \frac{2r_3}{k_3}P_3\Delta P_3 - \tau_3\Delta P_3 - \phi_3P\Delta P_3 (4.49)$$
$$\lim_{\Delta P_3\to 0}\frac{\Delta P_3\left[r_3 - \frac{2r_3}{k_3}P_3 - \tau_3 - \phi_3P - \frac{r_3}{k_3}\Delta P_3\right]}{\Delta P_3}$$
$$= r_3 - \frac{2r_3}{k_3}P_3 - \tau_3 - \phi_3P \quad (3.43)$$

We prove that

$$J_{34} = \frac{\partial F_3}{\partial P} = -\phi_3P_3$$
$$\frac{\partial F_3}{\partial P} = \lim_{\Delta P\to 0}\frac{F_3(P_1,\ P_2,\ P_3,\ P+\Delta P) - F_2(P_1,\ P_2,\ P_3,\ P)}{\Delta P} \quad (3.44)$$
$$F_3(P_1,\ P_2, P_3, P + \Delta P) = r_3P_3 - \frac{r_3}{k_3}P_3^2 - \tau_3P_3 + \tau_1P_1 + \tau_2P_2 - \phi_3P_3(P + \Delta P)$$
$$= r_3P_3 - \frac{r_3}{k_3}P_3^2 - \tau_3P_3 + \tau_1P_1 + \tau_2P_2 - \phi_3P_3P - \phi_3P_3\Delta P \quad (3.45)$$
$$F_3(P_1,\ P_2, P_3, P + \Delta P) - F_2(P_1, P_2, P_3, P)$$
$$= r_3P_3 - \frac{r_3}{k_3}P_3^2 - \tau_3P_3 + \tau_1P_1 + \tau_2P_2 - \phi_3P_3P - \phi_3P_3\Delta P$$
$$- r_3P_3 + \frac{r_3}{k_3}P_3^2 + \tau_3P_3 - \tau_1P_1 + \tau_2P_2 + \phi_3P_3P = -\phi_3P_3\Delta P \quad (50)$$
$$\lim_{\Delta P\to 0}\frac{-\phi_3P_3\Delta P}{\Delta P} = \lim_{\Delta P\to 0} -\phi_3P_3 = -\phi_3P_3 \quad (3.46)$$

We prove that

$$J_{41} = \frac{\partial F_4}{\partial P_1} = \epsilon_1$$
$$\frac{\partial F_4}{\partial P_1} = \lim_{\Delta P\to 0}\frac{F_4(P_1 + \Delta P_1,\ P_2,\ P_3,\ P) - F_4(P_1,\ P_2,\ P_3,\ P)}{\Delta P_1} \quad (3.47)$$
$$F_4(P_1 + \Delta P_1,\ P_2, P_3, P) = -\mu_1P - \mu_2P^2 + \epsilon_1(P_1 + \Delta P_1) + \epsilon_2\ P_2 + \epsilon_3\ P_3$$
$$= -\mu_1P - \mu_2P^2 + \epsilon_1P_1 + \epsilon_1\Delta P_1 + \epsilon_2P_2 + \epsilon_2\Delta P_2 + \epsilon_3\ P_3 \quad (3.48)$$
$$F_4(P_1 + \Delta P_1,\ P_2, P_3, P) - F_4(P_1, P_2, P_3, P)$$
$$= -\mu_1P - \mu_2P^2 + \epsilon_1P_1 + \epsilon_1\Delta P_1 + \epsilon_2P_2 + \epsilon_3\ P_3 + \mu_1P + \mu_2P^2 - \epsilon_1P_1 - \epsilon_2P_2 - \epsilon_3P_3$$
$$= \epsilon_1\Delta P_1 \quad (54)$$
$$\lim_{\Delta P_1\to 0}\frac{\epsilon_1\Delta P_1}{\Delta P_1} = \lim_{\Delta P_1\to 0}\epsilon_1 = \epsilon_1 \quad (3.49)$$

We prove that

$$J_{42} = \frac{\partial F_4}{\partial P_2} = \epsilon_2$$
$$\frac{\partial F_4}{\partial P_2} = \lim_{\Delta P_2\to 0}\frac{F_4(P_1,\ P_2 + \Delta P_2,\ P_3,\ P) - F_4(P_1,\ P_2,\ P_3,\ P)}{\Delta P_2} \quad (3.50)$$
$$F_4(P_1,\ P_2 + \Delta P_2, P_3, P) = -\mu_1P - \mu_2P^2 + \epsilon_1P_1 + \epsilon_2(P_2 + \Delta P_2) + \epsilon_3\ P_3$$
$$= -\mu_1P - \mu_2P^2 + \epsilon_1P_1 + \epsilon_2P_2 + \epsilon_2\Delta P_2 + \epsilon_3\ P_3 \quad (3.51)$$
$$F_4(P_1,\ P_2 + \Delta P_2, P_3, P) - F_4(P_1, P_2, P_3, P)$$
$$= -\mu_1P - \mu_2P^2 + \epsilon_1P_1 + \epsilon_2P_2 + \epsilon_2\Delta P_2 + \epsilon_3P_3 + \mu_1P + \mu_2P^2 - \epsilon_1P_1 - \epsilon_2P_2 - \epsilon_3P_3 = \epsilon_2\Delta P_2$$
$$\lim_{\Delta P_2\to 0}\frac{\epsilon_2\Delta P_2}{\Delta P_2} = \lim_{\Delta P_2\to 0}\epsilon_2 \quad = \epsilon_2 \quad (3.52)$$

We prove that

$$J_{43} = \frac{\partial F_4}{\partial P_3} = \epsilon_3$$

$$\frac{\partial \mathrm{F}_4}{\partial P_3} = \lim_{\Delta P_3 \to 0} \frac{F_4(P_1,\ P_2,\ P_3 + \Delta P_3,\ P) - F_4(P_1,\ P_2,\ P_3,\ P)}{\Delta P_3} \qquad (3.53)$$

$$F_4(P_1,\ P_2, P_3 + \Delta P_3, P) = -\mu_1 P - \mu_2 P^2 + \epsilon_1 P_1 + \epsilon_2 P_2 + \epsilon_3 (P_2 + \Delta P_2)$$
$$= -\mu_1 P - \mu_2 P^2 + \epsilon_1 P_1 + \epsilon_2 P_2 + \epsilon_3\ P_3 + \epsilon_3 \Delta P_3 \qquad (3.54)$$

$$F_4(P_1,\ P_2, P_3 + \Delta P_3, P) - F_4(P_1, P_2, P_3, P)$$
$$= -\mu_1 P - \mu_2 P^2 + \epsilon_1 P_1 + \epsilon_2 P_2 + \epsilon_3 P_3 + \epsilon_3 \Delta P_3 + \mu_1 P + \mu_2 P^2 - \epsilon_1 P_1 - \epsilon_2 P_2 - \epsilon_3 P_3 = \epsilon_3 \Delta P_3 \qquad (61)$$

$$\lim_{\Delta P_3 \to 0} \frac{\epsilon_3 \Delta P_3}{\Delta P_3} = \lim_{\Delta P_3 \to 0} \epsilon_3 \quad = \epsilon_3 \qquad (3.55)$$

We prove that

$$\mathrm{J}_{44} = \frac{\partial \mathrm{F}_4}{\partial P} = -\mu_1 - 2\mu_2 P + \epsilon_1 P_1 + \epsilon_2 P_2 + \epsilon_3 P_3$$

$$\frac{\partial \mathrm{F}_4}{\partial P} = \lim_{\Delta P \to 0} \frac{F_4(P_1,\ P_2,\ P_3,\ P + \Delta P) - F_4(P_1,\ P_2,\ P_3,\ P)}{\Delta P} \qquad (3.56)$$

$$F_4(P_1,\ P_2, P_3, P + \Delta P)$$
$$= -\mu_1 (P + \Delta P) - \mu_2 (P + \Delta P)^2 + \epsilon_1 P_1 (P + \Delta P) + \epsilon_2 P_2 (P + \Delta P) + \epsilon_3 (P + \Delta P)$$
$$= -\mu_1 P - \mu_1 \Delta P - \mu_2 P^2 - 2\mu_2 P \Delta P - \mu_2 (\Delta P)^2 + \ \epsilon_1 P_1 P + \epsilon_1 P_1 \Delta P + \epsilon_2 P_2 P$$
$$+ \epsilon_2 P_2 \Delta P + \epsilon_3\ P_3 P + \epsilon_3 P_3 \Delta P \qquad (3.57)$$

$$F_4(P_1,\ P_2, P_3, P + \Delta P) - F_4(P_1, P_2, P_3, P)$$
$$= -\mu_1 P - \mu_1 \Delta P - \mu_2 P^2 - 2\mu_2 P \Delta P - \mu_2 (\Delta P)^2 + \ \epsilon_1 P_1 P + \epsilon_1 P_1 \Delta P + \epsilon_2 P_2 P$$
$$+ \epsilon_2 P_2 \Delta P + \epsilon_3\ P_3 P + \epsilon_3 P_3 \Delta P + \mu_1 P + \mu_2 P^2 - \epsilon_1 P_1 P - \epsilon_2 P_2 P 8 - \epsilon_3 P_3 P \qquad (3.58)$$
$$= -\mu_1 \Delta P - 2\mu_2 P \Delta P - \mu_2 (\Delta P)^2 + \epsilon_1 P_1 \Delta P + \epsilon_2 P_2 \Delta P + \epsilon_3 P_3 \Delta P$$
$$= \Delta P[-\mu_1 - 2\mu_2 P - \mu_2 \Delta P + \epsilon_1 P_1 + \epsilon_2 P_2 + \epsilon_3 P_3]$$

$$\lim_{\Delta P \to 0} \frac{\Delta P[-\mu_1 - 2\mu_2 P - \mu_2 \Delta P + \epsilon_1 P_1 + \epsilon_2 P_2 + \epsilon_3 P_3]}{\Delta P}$$
$$= -\mu_1 - 2\mu_2 P + \epsilon_1 P_1 + \epsilon_2 P_2 + \epsilon_3 P_3 \qquad (3.59)$$

Hence, the solutions of the model equation (1) with non-negative initial conditions exist and are unique and the interior of R_+^4 is an invariant set for the model equation (2.1)

Lemma 3.1 All the solutions of the model system (2.1) which initiates in $\mathrm{R}_+^4$are uniformly bounded.
Proof: we define the function
$m(t) = P_1(t) + P_2(t) + P_3(t) + P(t) and\ \beta > 0$ be a constant, then we have

$$\frac{dm}{dt} + \beta m = \frac{dP_1}{dt} + \frac{dP_2}{dt} + \frac{dP_3}{dt} + \frac{dP}{dt} + \beta P_1 + \beta P_2 + \beta P_3 + \beta P \qquad (3.60)$$

$$\frac{dm}{dt} + \beta m = r_1 P_1 (1 - \frac{P_1}{k_1}) - \tau_1 P_1 + \tau_2 P_2 + \tau_3 P_3 - \phi_1 P_1 P + \beta P_1 + r_2 P_2 (1 - \frac{P_2}{k_2}) - \tau_2 P_2 + \tau_1 P_1 + \tau_3 P_3 - \phi_2 P_2 P + \beta P_2$$
$$+ r_3 P_3 (1 - \frac{P_3}{k_3}) - \tau_3 P_3 + \tau_1 P_1 + \tau_2 P_2 - \phi_3 P_3 P + \beta P_3 + P(-\mu_1 - \mu_2 P + \varepsilon_1 P_1 + \varepsilon_2 P_2 + \varepsilon_3 P_3) + \beta P \qquad (3$$

$$= r_1 P_1 (1 - \frac{P_1}{k_1}) + \tau_1 P_1 - \phi_1 P_1 P + \beta P_1 + r_2 P_2 (1 - \frac{P_2}{k_2}) + \tau_2 P_{23} - \phi_2 P_2 P + \beta P_2 + r_3 P_3 (1 - \frac{P_3}{k_3}) + \tau_3 P_3 - \phi_3 P_3 P$$
$$+ \beta P_3 - P \mu_1 - \mu_2 P^2 + \varepsilon_1 P_1 P + \varepsilon_2 P_2 P + \varepsilon_3 P_3 P + \beta P \qquad (3.$$

Assuming the whole biomass density of the interacting prey populations is transformed into the biomass of the predator, (3.62) becomes

$$r_1 P_1 - \frac{r_1}{k_1} P_1^2 + \tau_1 P_1 + \beta P_1 + r_2 P_2 - \frac{r_2}{k_2} P_2^2 + \tau_2 P_2 + \beta P_2 + r_3 P_3 - \frac{r_3}{k_3} P_3^2 + \tau_3 P_3 + \beta P_3 - \mu_1 P - \mu_2 P^2 + \beta P \quad (3.63)$$

$$= r_1 P_1 - \frac{r_1}{k_1} P_1^2 + (r_1 + \tau_1 + \beta) P_1 - \frac{r_2}{k_2} P_2^2 + (r_2 + \tau_2 + \beta) P_2 - \frac{r_3}{k_3} P_3^2 + (r_3 + \tau_3 + \beta) P_3 - \mu_2 P^2 + (\beta - \mu_1) P \quad (3.6$$

Completing squares separately for $P_1, P_2, P_3\ and\ P$ we have

$$\frac{dm}{dt}+\beta m=\frac{k_1}{4r_1}(r_1+\tau_1+\beta)^2-\frac{r_1}{k_1}\left\{P_1-\frac{k_1}{2r_1}(r_1+\tau_1+\beta)\right\}^2+\frac{k_2}{4r_2}(r_2+\tau_2+\beta)^2-\frac{r_2}{k_2}\left\{P_2-\frac{k_2}{2r_2}(r_2+\tau_2+\beta)\right\}^2$$

$$\frac{k_3}{4r_3}(r_3+\tau_3+\beta)^2-\frac{r_3}{k_3}\left\{P_3-\frac{k_3}{2r_3}(r_3+\tau_3+\beta)\right\}^2+\frac{1}{4\mu_2}(\beta-\mu_1)^2-\mu_2\left\{P-\frac{1}{\mu_2}(\beta-\mu_1)\right\}^2 \quad (3.65)$$

$$\leq \frac{k_1}{4r_1}(r_1+\tau_1+\beta)^2+\frac{k_2}{4r_2}(r_2+\tau_3+\beta)^2+\frac{k_3}{4r_3}(r_3+\tau_3+\beta)^2+\frac{1}{4\mu_2}(\beta-\mu_1)^2 \quad (3.66)$$

Clearly, we see that the right hand side of **(**3.66**)** is bounded, if we can find a constant $\alpha > 0$ such that $\left|\frac{dm}{dt}+\beta m<\alpha\right.$

By the theory of differential inequality, Birkoff and Rota (1978), we have

$$\frac{dm}{dt}+\beta m = \alpha \quad (3.67)$$

Using integrating factor, $I.F.=e^{\int\beta dt}=e^{\beta t}$

$$m(t)=\frac{1}{e^{\beta t}}\left[\int e^{\beta t}\alpha dt\right]\Rightarrow m=e^{-\beta t}\left[\alpha\frac{e^{\beta t}}{\beta}+c\right]$$

$$\therefore\ m(t)=\frac{\alpha}{\beta}+ce^{-\beta t}$$

Using the initial condition, m(0) = 0, we have,

$$0=\frac{\alpha}{\beta}+ce^0\Rightarrow c=-\frac{\alpha}{\beta}$$

$$\therefore\ m(t)=\frac{\alpha}{\beta}-\frac{\alpha}{\beta}e^{-\beta t}\Rightarrow m(t)=\frac{\alpha}{\beta}(1-e^{-\beta t}) \quad (3.68)$$

We have that,

$$0<m[P_1(t),P_2(t),P_3(t),P(t)]<\frac{\alpha}{\beta}(1-e^{-\beta t})$$

When $t\to\infty,\ 0<m(t)\leq\frac{\alpha}{\beta}$

Note that the quantity $\frac{\alpha}{\beta}$ is the steady state solution of the model system (2.1). Hence all solutions of the model system (2.1) starting in R_+^4 are confined in the region.

Theorem 3.1: The model system of equation (2.1) cannot have limit cycle in the interior of the positive quadrant.

Proof:

$$Let\ W(P_1,P_2,P_3,P)=\frac{1}{P_1P_2P_3\,P} \quad (3.69)$$

be a suitable Dulac multiplier and the interacting functions given as follows

$$F_1(P_1,P_2,P_3,P)=r_1P_1-\frac{r_1}{k_1}P_1^2-\tau_1P_1-\tau_2P_2+\tau_3P_3-\phi_1P_1P$$

$$F_2(P_1,P_2,P_3,P)=r_2P_2-\frac{r_2}{k_2}P_2^2-\tau_2P_2+\tau_1P_1+\tau_3P_3-\phi_2P_2P$$

$$F_3(P_1,P_2,P_3,P)=r_3P_3-\frac{r_3}{k_3}P_3^2-\tau_3P_3+\tau_1P_1+\tau_2P_2-\phi_3P_3P$$

$$F_4(P_1,P_2,P_3,P)=P\left(-\mu_1-\mu_2P+\epsilon_1P_1+\epsilon_2P_2+\epsilon_3P_3\right) \quad (3.70)$$

Clearly, we see that $W(P_1,P_2,P_3,P)>0$ in the interior of the positive quadrant of $P_1,\ P_2,\ P_3,\ and\ P$ plane, then, we have

$$\Delta(P_1,P_2,P_3,P)=\frac{\partial(WF_1)}{\partial P_1}+\frac{\partial(WF_2)}{\partial P_2}+\frac{\partial(WF_3)}{\partial P_3}+\frac{\partial(WF_4)}{\partial P} \quad (3.71)$$

$$WF_1=\frac{1}{P_1P_2P_2\ P}\left[r_1P_1-\frac{r_1}{k_1}P_1^2-\tau_1P_1+\tau_2P_2+\tau_3P_3-\phi_1P_1P\right]$$

$$\frac{\partial(W\mathrm{F}_1)}{\partial P_1} = -\frac{\mathrm{r}_1}{k_1 P_2 P_3\, P} - \frac{\tau_2}{\mathrm{P}_1^2 P_3\, P} - \frac{\tau_3}{\mathrm{P}_1^2 P_2\, P}$$
$$= -\frac{1}{P}\left[\frac{\mathrm{r}_1}{k_1 P_2 P_3} + \frac{\tau_2}{\mathrm{P}_1^2 P_3} + \frac{\tau_3}{\mathrm{P}_1^2 P_2}\right] \quad (3.72)$$
$$W\mathrm{F}_2 = \frac{1}{P_1 P_2 P_3\, P}\left[r_2 P_2 - \frac{\mathrm{r}_2}{\mathrm{k}_2}\mathrm{P}_2^2 - \tau_2 P_2 + \tau_1 P_1 + \tau_3 P_3 - \phi_2 P_2 P\right]$$
$$\frac{\partial(W\mathrm{F}_2)}{\partial P_2} = -\frac{1}{P}\left[\frac{\mathrm{r}_2}{k_2 P_1 P_3} + \frac{\tau_1}{\mathrm{P}_2^2 P_3} + \frac{\tau_3}{P_1 \mathrm{P}_2^2}\right] \quad (3.73)$$
$$W\mathrm{F}_3 = \frac{1}{P_1 P_2 P_3\, P}\left[r_3 P_3 - \frac{\mathrm{r}_2}{\mathrm{k}_2}\mathrm{P}_3^2 - \tau_3 P_3 + \tau_1 P_1 + \tau_2 P_2 - \phi_3 P_3 P\right]$$
$$\frac{\partial(W\mathrm{F}_3)}{\partial P_3} = -\frac{1}{P}\left[\frac{\mathrm{r}_3}{k_3 P_1 P_2} + \frac{\tau_1}{P_2 \mathrm{P}_3^2} + \frac{\tau_2}{P_1 \mathrm{P}_3^2}\right] \quad (3.74)$$
$$W\mathrm{F}_4 = \frac{1}{P_1 P_2 P_3\, P}\left[-\mu_1 P - \mu_2 P^2 + \epsilon_1 P_1 P + \epsilon_2 P_2 P + \epsilon_3 P_3 P\right]$$
$$\frac{\partial(W\mathrm{F}_4)}{\partial P} = -\left[\frac{\mu_2}{P_1 P_2 P_3}\right]$$
Combining, we have
$$\Delta(P_1 P_2 P_3\, P) = -\left\{\frac{1}{P}\left(\frac{\mathrm{r}_1}{k_1 P_2 P_3} + \frac{\tau_2}{\mathrm{P}_1^2 P_3} + \frac{\tau_3}{\mathrm{P}_1^2 P_2}\right) + \frac{1}{P}\left(\frac{\mathrm{r}_2}{k_2 P_1 P_3} + \frac{\tau_1}{\mathrm{P}_2^2 P_3} + \frac{\tau_3}{P_1 \mathrm{P}_2^2}\right) + \frac{1}{P}\left(\frac{\mathrm{r}_3}{k_3 P_1 P_2} + \frac{\tau_1}{P_2 \mathrm{P}_3^2} + \frac{\tau_2}{P_1 \mathrm{P}_3^2}\right) + \left[\frac{\mu_2}{P_1 P_2 P_3}\right]\right\} < 0 \quad (3.75)$$
The above result shows that $\Delta(P_1 P_2 P_3\, P)$ is not identically zero in the positive quadrant of P_1, P_2, P_3 *and* P plane and does not change sign (Dubey, et. al. 2003). Thus, by Bendixson -Dulac criteria, it follows that the model system (2.1) has no closed trajectory and no existence of periodic solutions in the interior of the positive quadrant.

This completes the proof ■

4.0 Steady state solution

In this section we shall consider the steady state solutions of the model system (2.1) in two scenarios: absence and presence of predation

4.1 Steady state solution in the absence of predation

In the absence of predation, the model system (2.1) becomes
$$\frac{\mathrm{dP}_1}{\mathrm{dt}} = \mathrm{r}_1\mathrm{P}_1 - \frac{\mathrm{r}_1}{\mathrm{k}_1}\mathrm{P}_1^2 - \tau_1\mathrm{P}_1 + \tau_2\mathrm{P}_2 + \tau_3\mathrm{P}_3$$
$$\frac{\mathrm{dP}_2}{\mathrm{dt}} = \mathrm{r}_2\mathrm{P}_2 - \frac{\mathrm{r}_2}{\mathrm{k}_2}\mathrm{P}_2^2 - \tau_2\mathrm{P}_2 + \tau_1\mathrm{P}_1 + \tau_3\mathrm{P}_3 \quad (3.76)$$
$$\frac{\mathrm{dP}_3}{\mathrm{dt}} = \mathrm{r}_3\mathrm{P}_3 - \frac{\mathrm{r}_3}{\mathrm{k}_3}\mathrm{P}_3^2 - \tau_3\mathrm{P}_3 + \tau_1\mathrm{P}_1 + \tau_2\mathrm{P}_2$$
At Steady State solutions, we have
$$\frac{\mathrm{dP}_1}{\mathrm{dt}} = \frac{\mathrm{dP}_2}{\mathrm{dt}} = \frac{\mathrm{dP}_3}{\mathrm{dt}} = 0$$
This implies that
$$r_1 P_1 - \frac{r_1}{k_1} P_1^2 - \tau_1 P_1 + \tau_2 P_2 + \tau_3 P_3 = 0$$
$$r_2 P_2 - \frac{r_2}{k_2} P_2^2 - \tau_2 P_2 + \tau_1 P_1 + \tau_3 P_3 = 0 \quad (3.77)$$
$$r_3 P_3 - \frac{r_3}{k_3} P_3^2 - \tau_3 P_3 + \tau_1 P_1 + \tau_2 P_2 = 0$$

The model system (2.1) is four dimensional; the analytical study of the system is difficult to tractable (Mellachervu et.al. 2011. To determine the co-existence steady state solution $E_1(P_1, P_2, P_3)$ from equation (1) in terms of the parameter values analytically is a daunting task due to the presence of non-linearity term in the equation. To overcome this analytical barrier, we proposed to use Newton – Raphson numerical scheme to calculate the approximate steady state solution. We shall omit the numerical solution here and proceed to investigate its local and global stability.

4.2Local Stability of the Model System (2.1) in the absence of Predation

Theorem 4.1: The steady state of system (2.1) in the absence of the Predator, $E_1(P_1, P_2, P_3)$, is locally Asymptotically stable.

Proof: Let

$$F_1(P_1, P_2, P_3) = r_1P_1 - \frac{r_1}{k_1}P_1^2 - \tau_1P_1 - \tau_2P_2 + \tau_3P_3$$

$$F_2(P_1, P_2, P_3) = r_2P_2 - \frac{r_2}{k_2}P_2^2 - \tau_2P_2 + \tau_1P_1 + \tau_3P_3 \qquad (4.1)$$

$$F_3(P_1, P_2, P_3) = r_3P_3 - \frac{r_3}{k_3}P_3^2 - \tau_3P_3 + \tau_1P_1 + \tau_2P_2$$

Let the system (4.1) be the interaction functions and

The Jacobian matrix for system (4.1) at $E_1(P_1, P_2, P_3)$ is given by

$$J = \begin{pmatrix} (\tau_1 - r_1) - \frac{2r_1P_1}{k_1} & \tau_2 & \tau_3 \\ \tau_1 & (\tau_2 - r_2) - \frac{2r_2P_2}{k_2} & \tau_3 \\ \tau_1 & \tau_2 & (\tau_3 - r_3) - \frac{2r_3P_3}{k_3} \end{pmatrix} \qquad (4.2)$$

From (4.1), we have

$$P_1\left[r_1 - \frac{r_1}{k_1}P_1 - \tau_1 + \frac{\tau_2P_2}{P_1} + \frac{\tau_3P_3}{P_1}\right] = 0 \qquad (4.3)$$

$$P_2\left[r_2 - \frac{r_2}{k_2}P_2 - \tau_2 + \frac{\tau_1P_1}{P_1} + \frac{\tau_3P_3}{P_1}\right] = 0 \qquad (4.4)$$

$$P_3\left[r_3 - \frac{r_3}{k_3}P_3 - \tau_3 + \frac{\tau_1P_1}{P_3} + \frac{\tau_2P_2}{P_3}\right] = 0 \qquad (4.5)$$

From (4.3), we have

$$r_1 - \tau_1 - \frac{r_1}{k_1}P_1 = -\frac{\tau_2P_2}{P_1} - \frac{\tau_3P_3}{P_1}$$

$$r_1 - \tau_1 = \frac{r_1}{k_1}P_1 - \frac{\tau_2P_2}{P_1} - \frac{\tau_3P_3}{P_1}$$

$$r_1 - \tau_1 - \frac{2r_1}{k_1}P_1 = -\frac{r_1P_1}{k_1} - \frac{\tau_2P_2}{P_1} - \frac{\tau_3P_3}{P_1}$$

$$= -\left(\frac{r_1P_1}{k_1} + \frac{\tau_2P_2}{P_1} + \frac{\tau_3P_3}{P_1}\right) \qquad (4.6)$$

Similarly,

From (4.4), we have

$$r_2 - \tau_2 = +\frac{r_2P_2}{k_2} - \frac{\tau_1P_1}{P_2} - \frac{\tau_3P_3}{P_2}$$

$$r_2 - \tau_2 - \frac{2r_2}{k_2}P_2 = -\left(\frac{r_2P_2}{k_2} + \frac{\tau_1P_1}{P_2} + \frac{\tau_3P_3}{P_2}\right) \qquad (4.7)$$

From (4.5), we have

$$r_3 - \tau_3 = +\frac{r_3P_3}{k_3} - \frac{\tau_1P_1}{P_3} - \frac{\tau_2P_2}{P_3}$$

$$r_3 - \tau_3 - \frac{2r_3}{k_3}P_3 = -\left(\frac{r_3P_3}{k_3} + \frac{\tau_1P_1}{P_3} + \frac{\tau_2P_2}{P_3}\right) \qquad (4.8)$$

Substituting (4.6), (4.7) and (4.8) into the linearized matrix (4.2) gives

$$J = \begin{pmatrix} -\left(\frac{r_1P_1}{k_1} + \frac{\tau_1P_1}{P_3} + \frac{\tau_3P_3}{P_3}\right) & \tau_2 & \tau_3 \\ \tau_1 & -\left(\frac{r_2P_2}{k_2} + \frac{\tau_1P_1}{P_2} + \frac{\tau_3P_3}{P_2}\right) & \tau_3 \\ \tau_1 & \tau_2 & -\left(\frac{r_3P_3}{k_3} + \frac{\tau_1P_1}{P_3} + \frac{\tau_2P_2}{P_3}\right) \end{pmatrix} \qquad (4.9)$$

$$|J - \lambda I| = \begin{vmatrix} -\left(\frac{r_1P_1}{k_1} + \frac{\tau_1P_1}{P_3} + \frac{\tau_2P_2}{P_3}\right) - \lambda & \tau_2 & \tau_3 \\ \tau_1 & -\left(\frac{r_2P_2}{k_2} + \frac{\tau_1P_1}{P_2} + \frac{\tau_3P_3}{P_2}\right) - \lambda & \tau_3 \\ \tau_1 & \tau_2 & -\left(\frac{r_3P_3}{k_3} + \frac{\tau_1P_1}{P_3} + \frac{\tau_2P_2}{P_3}\right) - \lambda \end{vmatrix} = 0$$

$$\Rightarrow -\left(\frac{r_1 P_1}{k_1}+\frac{\tau_1 P_1}{P_3}+\frac{\tau_2 P_2}{P_3}\right)-\lambda \begin{vmatrix} -\left(\frac{r_2 P_2}{k_2}+\frac{\tau_1 P_1}{P_2}+\frac{\tau_3 P_3}{P_2}\right)-\lambda & \tau_3 \\ \tau_2 & -\left(\frac{r_3 P_3}{k_3}+\frac{\tau_1 P_1}{P_3}+\frac{\tau_2 P_2}{P_3}\right)-\lambda \end{vmatrix}$$

$$-\tau_2 \begin{vmatrix} \tau_1 & \tau_3 \\ \tau_1 & -\left(\frac{r_3 P_3}{k_3}+\frac{\tau_1 P_1}{P_3}+\frac{\tau_2 P_2}{P_3}\right)-\lambda \end{vmatrix} + \tau_3 \begin{vmatrix} \tau_1 & -\left(\frac{r_2 P_2}{k_2}+\frac{\tau_1 P_1}{P_2}+\frac{\tau_3 P_3}{P_2}\right)-\lambda \\ \tau_1 & \tau_2 \end{vmatrix} \quad (4.10)$$

$$\Rightarrow \left[-\left(\frac{r_3 P_3}{k_3}+\frac{\tau_1 P_1}{P_3}+\frac{\tau_2 P_2}{P_3}\right)-\lambda\right]\left[\left(\frac{r_2 P_2}{k_2}+\frac{\tau_1 P_1}{P_2}+\frac{\tau_3 P_3}{P_2}\right)\left(\frac{r_3 P_3}{k_3}+\frac{\tau_1 P_1}{P_3}+\frac{\tau_2 P_2}{P_3}\right)\right.$$
$$\left.+\left(\frac{r_2 P_2}{k_2}+\frac{\tau_1 P_1}{P_2}+\frac{\tau_3 P_3}{P_2}\right)\lambda+\left(\frac{r_3 P_3}{k_3}+\frac{\tau_1 P_1}{P_3}+\frac{\tau_2 P_2}{P_3}\right)\lambda+\lambda^2-\tau_2\tau_2\right]$$
$$-\tau_2\left[-\left(\frac{r_3 P_3}{k_3}+\frac{\tau_1 P_1}{P_3}+\frac{\tau_2 P_2}{P_3}\right)\tau_1-\lambda\tau_1-\tau_1\tau_3\right]$$
$$+\tau_3\left[\tau_1\tau_2+\tau_1\left(\frac{r_2 P_2}{k_2}+\frac{\tau_1 P_1}{P_2}+\frac{\tau_3 P_3}{P_2}\right)+\lambda\tau_1\right]=0 \quad (4.11)$$

$$\Rightarrow -\left(\frac{r_1 P_1}{k_1}+\frac{\tau_1 P_1}{P_3}+\frac{\tau_2 P_2}{P_3}\right)\left(\frac{r_2 P_2}{k_2}+\frac{\tau_1 P_1}{P_2}+\frac{\tau_3 P_3}{P_2}\right)\left(\frac{r_3 P_3}{k_3}+\frac{\tau_1 P_1}{P_3}+\frac{\tau_2 P_2}{P_3}\right)$$
$$-\left(\frac{r_1 P_1}{k_1}+\frac{\tau_1 P_1}{P_3}+\frac{\tau_2 P_2}{P_3}\right)\left(\frac{r_2 P_2}{k_2}+\frac{\tau_1 P_1}{P_2}+\frac{\tau_3 P_3}{P_2}+\frac{r_3 P_3}{k_3}+\frac{\tau_1 P_1}{P_3}+\frac{\tau_2 P_2}{P_3}\right)\lambda$$
$$-\left(\frac{r_1 P_1}{k_1}+\frac{\tau_1 P_1}{P_3}+\frac{\tau_2 P_2}{P_3}\right)\lambda^2+\left(\frac{r_1 P_1}{k_1}+\frac{\tau_1 P_1}{P_3}+\frac{\tau_2 P_2}{P_3}\right)\tau_2\tau_3$$
$$-\left(\frac{r_2 P_2}{k_2}+\frac{\tau_1 P_1}{P_2}+\frac{\tau_3 P_3}{P_2}\right)\left(\frac{r_3 P_3}{k_3}+\frac{\tau_1 P_1}{P_3}+\frac{\tau_2 P_2}{P_3}\right)\lambda$$
$$-\left(\frac{r_2 P_2}{k_2}+\frac{\tau_1 P_1}{P_2}+\frac{\tau_3 P_3}{P_2}+\frac{r_3 P_3}{k_3}+\frac{\tau_1 P_1}{P_3}+\frac{\tau_2 P_2}{P_3}\right)\lambda^2-\lambda^3-\tau_2\tau_3\lambda$$
$$+\tau_1\tau_1\left(\frac{r_3 P_3}{k_3}+\frac{\tau_1 P_1}{P_3}+\frac{\tau_2 P_2}{P_3}\right)+\tau_1\tau_2\lambda+\tau_1\tau_2\tau_3+\tau_1\tau_2\tau_3$$
$$+\tau_2\tau_3\left(\frac{r_2 P_2}{k_2}+\frac{\tau_1 P_1}{P_2}+\frac{\tau_3 P_3}{P_2}\right)+\tau_1\tau_3\lambda=0 \quad (4.12)$$

Let the characteristic equation of the model system (4.10) be given as

$\lambda^3+c_1\lambda^2+c_2\lambda+c_3=0$

Where, $c_i, i = 1,2,3$ are given by

Re-arranging (3.89) we have

$$c_1=\left[\frac{r_1 P_1}{k_1}+\frac{\tau_1 P_1}{P_3}+\frac{\tau_2 P_2}{P_3}+\frac{r_2 P_2}{k_2}+\frac{\tau_1 P_1}{P_2}+\frac{\tau_3 P_3}{P_2}+\frac{r_3 P_3}{k_3}+\frac{\tau_1 P_1}{P_3}+\frac{\tau_2 P_2}{P_3}\right]$$
$$c_2=\left[\left(\frac{r_1 P_1}{k_1}+\frac{\tau_1 P_1}{P_3}+\frac{\tau_2 P_2}{P_3}\right)\left(\frac{r_2 P_2}{k_2}+\frac{\tau_1 P_1}{P_2}+\frac{\tau_3 P_3}{P_2}+\frac{r_3 P_3}{k_3}+\frac{\tau_1 P_1}{P_3}+\frac{\tau_2 P_2}{P_3}\right)\right.$$
$$\left.+\left(\frac{r_2 P_2}{k_2}+\frac{\tau_1 P_1}{P_2}+\frac{\tau_3 P_3}{P_2}\right)\left(\frac{r_3 P_3}{k_3}+\frac{\tau_1 P_1}{P_3}+\frac{\tau_2 P_2}{P_3}\right)+\tau_2\tau_3-\tau_1\tau_2-\tau_1\tau_3\right]$$
$$c_3=\left[\left(\frac{r_1 P_1}{k_1}+\frac{\tau_1 P_1}{P_3}+\frac{\tau_2 P_2}{P_3}\right)\left(\frac{r_2 P_2}{k_2}+\frac{\tau_1 P_1}{P_2}+\frac{\tau_3 P_3}{P_2}\right)\left(\frac{r_3 P_3}{k_3}+\frac{\tau_1 P_1}{P_3}+\frac{\tau_2 P_2}{P_3}\right)\right. \quad (4.13)$$
$$-\left(\frac{r_1 P_1}{k_1}+\frac{\tau_1 P_1}{P_3}+\frac{\tau_2 P_2}{P_3}\right)\tau_2\tau_3-\left(\frac{r_3 P_3}{k_3}+\frac{\tau_1 P_1}{P_3}+\frac{\tau_2 P_2}{P_3}\right)\tau_1\tau_2$$
$$\left.-\left(\frac{r_2 P_2}{k_2}+\frac{\tau_1 P_1}{P_2}+\frac{\tau_3 P_3}{P_2}\right)\tau_1\tau_3-2\tau_1\tau_2\tau_3\right]$$

Applying the Routh-Hurwitz criteria

$E_1(P_1, P_2, P_3)$ is LAS (locally asymptotically stability)

if $c_1>0, c_3>0$ *and* $c_1c_2-c_3>0$

Clearly we see that

$c_1>0,\ c_2>0$ *and* $c_3>0$,

provided $(\tau_2\tau_3)>(\tau_1\tau_2+\tau_1\tau_3)$ and

$$\left[\left(\frac{r_1 P_1}{k_1}+\frac{\tau_1 P_1}{P_3}+\frac{\tau_2 P_2}{P_3}\right)\left(\frac{r_2 P_2}{k_2}+\frac{\tau_1 P_1}{P_2}+\frac{\tau_3 P_3}{P_2}\right)\left(\frac{r_3 P_3}{k_3}+\frac{\tau_1 P_1}{P_3}+\frac{\tau_2 P_2}{P_3}\right)\right]> \quad (4.14)$$
$$\left[\left(\frac{r_1 P_1}{k_1}+\frac{\tau_1 P_1}{P_3}+\frac{\tau_2 P_2}{P_3}\right)\tau_2\tau_3+\left(\frac{r_3 P_3}{k_3}+\frac{\tau_1 P_1}{P_3}+\frac{\tau_2 P_2}{P_3}\right)\tau_1\tau_2+\left(\frac{r_2 P_2}{k_2}+\frac{\tau_1 P_1}{P_2}+\frac{\tau_3 P_3}{P_2}\right)\tau_1\tau_3+2\tau_1\tau_2\tau_3\right]$$

The stability characterization of $E_1(P_1, P_2, P_3)$ is determined by the sign of $c_1c_2-c_3$. By direct algebraic calculation, we have

$$\left[\frac{r_1P_1}{k_1}+\frac{\tau_1P_1}{P_3}+\frac{\tau_2P_2}{P_3}+\frac{r_2P_2}{k_2}+\frac{\tau_1P_1}{P_2}+\frac{\tau_3P_3}{P_2}+\frac{r_3P_3}{k_3}+\frac{\tau_1P_1}{P_3}+\frac{\tau_2P_2}{P_3}\right]*$$
$$\left[\left(\frac{r_1P_1}{k_1}+\frac{\tau_1P_1}{P_3}+\frac{\tau_2P_2}{P_3}\right)\left(\frac{r_2P_2}{k_2}+\frac{\tau_1P_1}{P_2}+\frac{\tau_3P_3}{P_2}+\frac{r_3P_3}{k_3}+\frac{\tau_1P_1}{P_3}+\frac{\tau_2P_2}{P_3}\right)\right.$$
$$\left.+\left(\frac{r_2P_2}{k_2}+\frac{\tau_1P_1}{P_2}+\frac{\tau_3P_3}{P_2}\right)\left(\frac{r_3P_3}{k_3}+\frac{\tau_1P_1}{P_3}+\frac{\tau_2P_2}{P_3}\right)+\tau_2\tau_3-\tau_1\tau_2-\tau_1\tau_3\right]-$$
$$\left[\left(\frac{r_1P_1}{k_1}+\frac{\tau_1P_1}{P_3}+\frac{\tau_2P_2}{P_3}\right)\left(\frac{r_2P_2}{k_2}+\frac{\tau_1P_1}{P_2}+\frac{\tau_3P_3}{P_2}\right)\left(\frac{r_3P_3}{k_3}+\frac{\tau_1P_1}{P_3}+\frac{\tau_2P_2}{P_3}\right)\right.$$
$$-\left(\frac{r_1P_1}{k_1}+\frac{\tau_1P_1}{P_3}+\frac{\tau_2P_2}{P_3}\right)\tau_2\tau_3-\left(\frac{r_3P_3}{k_3}+\frac{\tau_1P_1}{P_3}+\frac{\tau_2P_2}{P_3}\right)\tau_1\tau_2$$
$$\left.-\left(\frac{r_2P_2}{k_2}+\frac{\tau_1P_1}{P_2}+\frac{\tau_3P_3}{P_2}\right)\tau_1\tau_3-2\tau_1\tau_2\tau_3\right]>0 \qquad (4.15)$$

Clearly we see that $(c_1 .\ c_2)-c_3>0$, provided the condition (4.14) hold.
Thus, all eigenvalues are negative and therefore, the steady state solution $E_1(P_1, P_2, P_3)$ is locally asymptotically stability (LAS). ■

4.3 Global Stability of the Model System (2.1)

In this section, we discuss the global stability of the steady state solution $E_1(P_1, P_2, P_3)$ using the standard Lyapunov function

Theorem 4.2: The $steady\ state, E_1(P_1, P_2, P_3)$, of system (2.1) in the absence of the Predator is globally Asymptotically stable.

Proof:

$$V(P_1,P_2,P_3)=\left(P_1-P_1^+-P_1^+\log\frac{P_1}{P_1^+}\right)+n_1\left(P_2-P_2^+-P_2^+\log\frac{P_2}{P_2^+}\right)$$
$$+n_2\left(P_3-P_3^+-P_3^+\log\frac{P_3}{P_3^+}\right) \qquad (4.16)$$

where $P_1^+, P_2^+\ and\ P_3^+$ are time independent parameters and $\eta_1\ and\ \eta_2$ are positive constants to be chosen later on.

Differentiating $V(P_1,P_2,P_3)$ with respect to time, t, along the solution of **(**2.1**)** we have,

$$\frac{dv}{dt}=\left[\frac{dP_1}{dt}-\frac{dP_1^+}{dt}-P_1^+\left(\frac{d}{dt}\log\frac{P_1}{P_1^+}\right)\right]+n_1\left[\frac{dP_2}{dt}-\frac{dP_2^+}{dt}-P_2^+\left(\frac{d}{dt}\log\frac{P_2}{P_2^+}\right)\right]$$
$$+n_2\left[\frac{dP_3}{dt}-\frac{dP_3^+}{dt}-P_3^+\left(\frac{d}{dt}\log\frac{P_3}{P_3^+}\right)\right] \qquad (4.17)$$

$$Recall\ that\ \frac{d}{dt}(\log\alpha)=\frac{1}{\alpha}.\frac{d}{dt}(\alpha)$$

$$\frac{dv}{dt}=\frac{dP_1}{dt}-0-P_1^+\left[\frac{1}{\frac{P_1}{P_1^+}}\ .\ \frac{d}{dt}\left(\frac{P_1}{P_1^+}\right)\right]+n_1\left[\frac{dP_2}{dt}-0-P_2^+\left(\frac{1}{\frac{P_2}{P_2^+}}\frac{d}{dt}\left(\frac{P_2}{P_2^+}\right)\right)\right]$$
$$+n_2\left[\frac{dP_3}{dt}-0-P_3^+\left(\frac{1}{\frac{P_3}{P_3^+}}\frac{d}{dt}\left(\frac{P_3}{P_3^+}\right)\right)\right] \qquad (4.18)$$
$$=\left[\frac{dP_1}{dt}-\frac{P_1^+}{P_1}\frac{dP_1}{dt}\right]+n_1\left[\frac{dP_2}{dt}-\frac{P_2^+}{P_2}\frac{dP_2}{dt}\right]+n_2\left[\frac{dP_3}{dt}-\frac{P_3^+}{P_3}\frac{dP_3}{dt}\right] \qquad (4.19)$$
$$=\left(1-\frac{P_1^+}{P_1}\right)\frac{dP_1}{dt}+n_1\left(1-\frac{P_2^+}{P_2}\right)\frac{dP_2}{dt}+n_2\left(1-\frac{P_3^+}{P_3}\right)\frac{dP_3}{dt} \qquad (4.20)$$
$$=\frac{(P_1-P_1^+)}{P_1}\frac{dP_1}{dt}+n_1\frac{(P_2-P_2^+)}{P_2}\frac{dP_2}{dt}+n_2\frac{(P_3-P_3^+)}{P_3}\frac{dP_3}{dt} \qquad (4.21)$$
$$For\ \ \frac{(P_1-P_1^+)}{P_1}\frac{dP_1}{dt}=\frac{(P_1-P_1^+)}{P_1}\left[(r_1-\tau_1)P_1-\frac{r_1P_1^2}{k_1}+\tau_2P_2+\tau_3P_3\right]$$
$$=\frac{(P_1-P_1^+)}{P_1}\left[(r_1-\tau_1)P_1-\frac{r_1P_1^2}{k_1}+\tau_2P_2+\tau_3P_3\right] \qquad (4.22)$$

From the first equation of (84), we have

$$P_1\left[r_1-\tau_1-\frac{r_1}{k_1}P_1+\frac{\tau_2P_2}{P_1}+\frac{\tau_3P_3}{P_1}\right]=0$$
$$\Rightarrow r_1-\tau_1=\frac{r_1}{k_1}P_1-\frac{\tau_2P_2}{P_1}-\frac{\tau_3P_3}{P_1} \qquad (4.23)$$

The model system (1) under investigation is biologically relevance if

$r_1 - \tau_1 > 0, \qquad r_2 - \tau_2 > 0 \; and \; r_3 - \tau_3 > 0$

From the point of view of biological relevance of the model, the right hand side of equation (4.23) must be express in positive sense and also as parameter value

$$r_1 - \tau_1 = \frac{r_1 P_1^+}{k_1} - \frac{\tau_2 P_2}{P_1^+} - \frac{\tau_3 P_3}{P_1^+} \qquad (4.24)$$

In positive sense, the right hand side of (4.24) becomes

$$r_1 - \tau_1 > 0 \Rightarrow r_1 - \tau_1 = \frac{r_1 P_1^+}{k_1} + \frac{\tau_2 P_2}{P_1^+} + \frac{\tau_3 P_3}{P_1^+} \qquad (4.25)$$

Substituting $(r_1 - \tau_1)$ into (107), we have

$$\frac{(P_1 - P_1^+)}{P_1}\frac{dP_1}{dt} = \frac{(P_1 - P_1^+)}{P_1}\left[\left(\frac{r_1 P_1^+}{k_1} + \frac{\tau_2 P_2}{P_1^+} + \frac{\tau_3 P_3}{P_1^+}\right)P_1 - \frac{r_1 P_1^2}{k_1} + \tau_2 P_2 + \tau_3 P_3\right] \qquad (4.26)$$

Opening bracket of (4.26) and collecting like terms, we have

$$\frac{(P_1 - P_1^+)}{P_1}\frac{dP_1}{dt} = -\frac{r_1}{k_1}(P_1 - P_1^+)^2 \qquad (4.27)$$

Similarly,

$$\frac{(P_2 - P_2^+)}{P_2}\frac{dP_2}{dt} = \frac{(P_2 - P_2^+)}{P_2}\left[\left(\frac{r_2 P_2^+}{k_2} + \frac{\tau_2 P_1}{P_2^+} + \frac{\tau_3 P_3}{P_2^+}\right)P_2 - \frac{r_2 P_2^2}{k_2} + \tau_1 P_1 + \tau_3 P_3\right]$$

$$= -\frac{r_2}{k_2}(P_2 - P_2^+)^2 \qquad (4.28)$$

and

$$\frac{(P_3 - P_3^+)}{P_3}\frac{dP_3}{dt} = \frac{(P_3 - P_3^+)}{P_3}\left[\left(\frac{r_3 P_3^+}{k_3} + \frac{\tau_1 P_1}{P_3^+} + \frac{\tau_2 P_2}{P_3^+}\right)P_3 - \frac{r_3 P_3^2}{k_3} + \tau_1 P_1 + \tau_2 P_2\right]$$

$$= -\frac{r_3}{k_3}(P_3 - P_3^+)^2 \qquad (4.29)$$

Combining, we have

$$\frac{dv}{dt} = -\frac{r_1}{k_1}(P_1 - P_1^+)^2 - \eta_1\frac{r_2}{k_2}(P_2 - P_2^+)^2 - \eta_2\frac{r_3}{k_3}(P_3 - P_3^+)^2 \qquad (4.30)$$

$$Choosing \;\; \eta_1 = \left(\frac{P_2^+}{P_1^+}\right)\left(\frac{\tau_2}{\tau_1}\right) \; and \; \eta_2 = \left(\frac{P_3^+}{P_1^+}\right)\left(\frac{\tau_3}{\tau_1}\right) \qquad (4.31)$$

$$\frac{dv}{dt} = -\left[\frac{r_1}{k_1}(P_1 - P_1^+)^2 + \frac{P_2^+\tau_2 r_2}{P_1^+\tau_1 k_2}(P_2 - P_2^+)^2 + \frac{P_3^+\tau_3 r_3}{P_1^+\tau_1 k_3}(P_3 - P_3^+)^2\right] \qquad (4.32)$$

Thus, $\frac{dV}{dt}$ is negative definite.

Therefore, $E_1(P_1, P_2, P_3)$ is globally asymptotically stable (GAS) ■

Lemma 4.1 The steady state solution $E_1(P_1, P_2, P_3)$ is a hyperbolic saddle point if and only if $(\tau_i - r_i) < 2\frac{r_i P_i}{k_i}, \; i = 1, 2, 3$

Proof: The Jacobian matrix of (4.1) evaluated at $E_1(P_1, P_2, P_3)$ is

$$J(P_1, P_2, P_3) = \begin{pmatrix} (\tau_1 - r_1) - \frac{2r_1P_1}{k_1} & \tau_2 & \tau_3 \\ \tau_1 & (\tau_2 - r_2) - \frac{2r_2P_2}{k_2} & \tau_3 \\ \tau_1 & \tau_2 & (\tau_3 - r_3) - \frac{2r_3P_3}{k_3} \end{pmatrix}$$

Let $\beta_1 = (\tau_1 - r_1) - \frac{2r_1P_1}{k_1}$, $\beta_2 = (\tau_2 - r_2) - \frac{2r_2P_2}{k_2}$ and $\beta_3 = (\tau_3 - r_3) - \frac{2r_3P_3}{k_3}$

So

$det\;[J(P_1, P_2, P_3)] = \beta_1(\beta_2\beta_3 - \tau_2\tau_3) - \tau_2(\tau_1\beta_3 - \tau_1\tau_3) + \tau_3(\tau_1\tau_2 - \tau_1\beta_2)$

$det\;[J(P_1, P_2, P_3)] < 0$ provided $\beta_2\beta_3 < \tau_2\tau_3, \; \tau_1\beta_3 < \tau_1\tau_3 \; and \; \tau_1\tau_2 < \tau_1\beta_2$

and $tr[J(P_1, P_2, P_3)] = (\tau_1 - r_1) - \frac{2r_1P_1}{k_1} + (\tau_2 - r_2) - \frac{2r_2P_2}{k_2} + (\tau_3 - r_3) - \frac{2r_3P_3}{k_3} < 0$

Provided $(\tau_i - r_i) < 2\frac{r_i P_i}{k_i}, \; i = 1, 2, 3$

We have that $det[J(P_1, P_2, P_3)] < 0$ and $tr[J(P_1, P_2, P_3)] < 0$, then the steady state solution $E_1(P_1, P_2, P_3)$ is a hyperbolic saddle. ■

4.4 Steady state solution in the presence of Predation

At steady state solution in the presence of predation, we have

$$\frac{dP_1}{dt} = \frac{dP_2}{dt} = \frac{dP_3}{dt} = \frac{dP}{dt} = 0 \qquad (4.33)$$

From the model system (2.1) it becomes

$$r_1P_1\left(1 - \frac{P_1}{k_1}\right) - \tau_1P_1 + \tau_2P_2 + \tau_3P_3 - \phi_1P_1P = 0$$
$$r_2P_2\left(1 - \frac{P_2}{k_2}\right) + \tau_1P_1 - \tau_2P_2 + \tau_3P_3 - \phi_2P_2P = 0 \qquad (4.34)$$
$$r_3P_3\left(1 - \frac{P_3}{k_3}\right) + \tau_1P_1 + \tau_2P_2 - \tau_3P_3 - \phi_3P_3P = 0$$
$$P(-\mu_1 - \mu_2P + \varepsilon_1P_1 + \varepsilon_2P_2 + \varepsilon_3P_3) = 0$$

The possible steady state solutions of the model equations (2.1) are

$$E_0(0,0,0,0), E_1(P_1,P_2,P_3,0), E_2(P_1,P_2,P_3,P), E_3(P_1,P_2,0,P), E_4(0,P_2,P_3,P) \, and \, E_5(P_1,0,P_3,P) \quad (4.35)$$

For ecological relevance, we will consider the realistic steady state solutions:

$$E_0(0,0,0,0), E_1(P_1,P_2,P_3,0), and \, E_2(P_1,P_2,P_3,P) \qquad (4.36)$$

4.5 Stability Characterization of the Steady State $E_0(0,0,0,0)$

Theorem 4.3: The trivial steady state of (2.1) in the presence of the Predator, $E_0(0,0,0,0)$ *is locally asymptotically stable*

Proof:

Let the interaction functions of the model system (2.1) be given as follows

$$F_1(P_1,P_2,P_3,P) = r_1P_1 - \frac{r_1}{k_1}P_1^2 - \tau_1P_1 - \tau_2P_2 + \tau_3P_3 - \phi_1P_1P$$
$$F_2(P_1,P_2,P_3,P) = r_2P_2 - \frac{r_2}{k_2}P_2^2 - \tau_2P_2 + \tau_1P_1 + \tau_3P_3 - \phi_2P_2P$$
$$F_3(P_1,P_2,P_3,P) = r_3P_3 - \frac{r_3}{k_3}P_3^2 - \tau_3P_3 + \tau_1P_1 + \tau_2P_2 - \phi_3P_3P$$
$$F_4(P_1,P_2,P_3,P) = -\mu_1P - \mu_2P^2 + \epsilon_1P_1 + \epsilon_2P_2 + \epsilon_3P_3. \qquad (4.37)$$

.

The Jacobian matrix at $E_0(0,0,0,0)$ become

$$J = \begin{pmatrix} r_1 - \tau_1 & \tau_2 & \tau_3 & 0 \\ \tau_1 & r_2 - \tau_2 & \tau_3 & 0 \\ \tau_1 & \tau_2 & r_3 - \tau_3 & 0 \\ 0 & 0 & 0 & -\mu_1 \end{pmatrix} \quad (123)$$

We have

$$|J - \lambda I| = \begin{vmatrix} r_1 - \tau_1 - \lambda & \tau_2 & \tau_3 & 0 \\ \tau_1 & r_2 - \tau_2 - \lambda & \tau_3 & 0 \\ \tau_1 & \tau_2 & r_3 - \tau_3 - \lambda & 0 \\ 0 & 0 & 0 & -\mu_1 - \lambda \end{vmatrix} = 0 \qquad (4.38)$$

$$\Rightarrow \left(-\mu_1 - \lambda\right)\begin{vmatrix} -\lambda & \tau_2 & \tau_3 \\ \tau_1 r_1 - \tau_1 & r_2 - \tau_2 - \lambda & \tau_3 \\ \tau_1 & \tau_2 & r_3 - \tau_3 - \lambda \end{vmatrix} = 0$$

$\Rightarrow \left(-\mu_1 - \lambda\right)\{(r_1 - \tau_1 - \lambda)[(r_2 - \tau_2 - \lambda)(r_3 - \tau_3 - \lambda) - \tau_2\tau_3] - \tau_2[r_3\tau_1 - \tau_1\tau_3 - \tau_1\lambda - \tau_1\tau_3] +$
τ3τ1τ2−τ1□2−τ1τ2−τ1□=0 (4.39)

Simplifying gives the characteristic equation

$\left(-\mu_1 - \lambda\right)\{-\lambda^3 - (\tau_1 + \tau_2 + \tau_3 - r_1 - r_2 - r_3)\lambda^2 - (r_1r_2 + 2r_2r_3 + 2\tau_1\tau_3 - r_1\tau_2 - r_1\tau_3 - r_3\tau_3 -$
□3τ2−□3τ1−□2τ1□−□1□2τ3+□1□3τ2+□2□3τ1+□2τ1τ3−□1□2□3−□2τ1τ3−2τ1τ2τ3=0 (4.40)

$\left(-\mu_1 - \lambda\right) = 0$ *or*

$\lambda^3 + (\tau_1 + \tau_2 + \tau_3 - r_1 - r_2 - r_3)\lambda^2 + (r_1r_2 + 2r_2r_3 + 2\tau_1\tau_3 - r_1\tau_2 - r_1\tau_3 - r_3\tau_3 - r_3\tau_2 - r_3\tau_1 -$
□2τ1□+□1□2τ3+□1□3τ2+□2□3τ1+□2τ1τ3−□1□2□3−□2τ1τ3−2τ1τ2τ3=0 (4.41)

Let $L_1 = [(\tau_1 + \tau_2 + \tau_3) - (r_1 + r_2 + r_3)]$

$L_2 = [(r_1r_2 + 2r_2r_3 + 2\tau_1\tau_3) - (r_1\tau_2 + r_1\tau_3 + r_3\tau_3 + r_3\tau_2 + r_3\tau_1 + r_2\tau_1)]$

(4.42)

$L_3 = [(r_1r_2\tau_3 + r_1r_3\tau_2 + r_2r_3\tau_1 + r_2\tau_1\tau_3) - (r_1r_2r_3 + r_2\tau_1\tau_3 + 2\tau_1\tau_2\tau_3)]$

The cubic characteristic equation becomes
$\lambda^3 + L_1\lambda^2 + L_2\lambda + L_3 = 0$
(4.43)
Applying the Routh-Hurwitz criteria, we have
$-\mu_1 - \lambda = 0 \implies \lambda = -\mu_1 < 0$ and the eigen values of the cubic equation must have negative real parts
if $L_1 > 0,\ L_2 > 0,\ L_3 > 0$ and $L_1L_2 - L_3 > 0,$ (4.44)
the stability characterization of $E_0(0,0,0,0)$ is determined by the sign of
$L_1L_2 - L_3 > 0.$ (4.45)
Following the biological relevance of the model equation (1) the growth rate of the interacting prey populations must be greater than its migration rate.
On the basis of the above statement,
$L_1 = [(\tau_1 + \tau_2 + \tau_3) - (r_1 + r_2 + r_3)] < 0$
(4.46)
Therefore, the eigenvalues of the cubic characteristics equation has opposite signs because $L_1L_2 - L_3 < 0.$
Thus the trivial steady state solution $E_0(0,0,0,0)$ is unstable, ■

4.6 Co-existence Steady state solution $E_2(P_1, P_2, P_3, P)$

The model system (2.1) is four dimensional; the analytical study of the system is difficult to tractable (Mellachervu et.al. 2011). To determine the co-existence steady state solution $E_2(P_1, P_2, P_3, P)$ from equation (1) in terms of the parameter values analytically is a daunting task due to the presence of non-linearity term in the equation. To overcome this analytical barrier, we proposed to use Newton – Raphson numerical scheme to calculate the approximate steady state solution. We shall omit the numerical solution here and proceed to investigate its local and global stability.

4.7 Local Stability of Model System (2.1)

Theorem 4.4: The steady state of system (2.1), $E_2(P_1, P_2, P_3, P)$ is locally asymptotically stable
Proof:
The Jacobian matrix of model system (1) at $E_2(P_1,\ P_2,\ P_3,\ P)$ is given by

$$J(P_1,P_2,P_3)=\begin{pmatrix} r_1-\frac{2r_1P_1^*}{k_1}-\tau_1-\phi_1P^* & \tau_2 & \tau_3 & -\phi_1P_1^* \\ \tau_1 & r_2-\frac{2r_2P_2^*}{k_2}-\tau_2-\phi_2P^* & \tau_3 & -\phi_2P_2^* \\ \tau_1 & \tau_2 & r_2-\frac{2r_2P_2^*}{k_2}-\tau_2-\phi_2P^* & -\phi_3P_3^* \\ \varepsilon_1P^* & \varepsilon_2P^* & \varepsilon_2P^* & -\mu_1-2\mu_2P^*+\varepsilon_1P_1^*+\varepsilon_2P_2^*+\varepsilon_3P_3^* \end{pmatrix}$$

(4.47)

The system has steady state at $\frac{dP_1}{dt} = \frac{dP_2}{dt} = \frac{dP_3}{dt} = \frac{dP}{dt} = 0$, which implies that

$$P_1\left[r_1 - \frac{r_1P_1}{k_1} - \tau_1 + \frac{\tau_2P_2}{P_1} + \frac{\tau_3P_3}{P_1} - \phi_1P\right] = 0 \quad (4.48)$$
$$P_2\left[r_2 - \frac{r_2P_2}{k_2} - \tau_2 + \frac{\tau_1P_1}{P_2} + \frac{\tau_3P_3}{P_2} - \phi_2P\right] = 0 \quad (4.49)$$
$$P_3\left[r_3 - \frac{r_3P_3}{k_3} - \tau_3 + \frac{\tau_1P_1}{P_3} + \frac{\tau_2P_2}{P_3} - \phi_3P\right] = 0 \quad (4.50)$$
$$P\left[-\mu_1 - \mu_2P + \epsilon_1P_1 + \epsilon_2P_2 + \epsilon_3P_3\right] = 0 \quad (4.51)$$

From (4.48), we have
$$r_1 - \frac{r_1P_1}{k_1} - \tau_1 + \frac{\tau_2P_2}{P_1} + \frac{\tau_3P_3}{P_1} - \phi_1P = 0$$
$$r_1 - \tau_1 - \frac{r_1P_1}{k_1} = \phi_1P - \frac{\tau_2P_2}{P_1} - \frac{\tau_3P_3}{P_1}$$

$$\mathrm{r}_1 - \tau_1 - \frac{\mathrm{r}_1\mathrm{P}_1}{\mathrm{k}_1} = \phi_1 P - \frac{\tau_2\mathrm{P}_2}{\mathrm{P}_1} - \frac{\tau_3\mathrm{P}_3}{\mathrm{P}_1}$$

$$\therefore \mathrm{r}_1 - \tau_1 - \frac{2\mathrm{r}_1\mathrm{P}_1}{\mathrm{k}_1} - \phi_1 P = \mathrm{r}_1 - \tau_1 - \frac{\mathrm{r}_1\mathrm{P}_1}{\mathrm{k}_1} - \frac{\mathrm{r}_1\mathrm{P}_1}{\mathrm{k}_1} - \phi_1 P$$

$$= \phi_1 P - \frac{\tau_2\mathrm{P}_2}{\mathrm{P}_1} - \frac{\tau_3\mathrm{P}_3}{\mathrm{P}_1} - \frac{\mathrm{r}_1\mathrm{P}_1}{\mathrm{k}_1} - \phi_1 P$$

$$= -\left(\frac{\mathrm{r}_1\mathrm{P}_1}{\mathrm{k}_1} + \frac{\tau_2\mathrm{P}_2}{\mathrm{P}_1} + \frac{\tau_3\mathrm{P}_3}{\mathrm{P}_1}\right) \qquad (4.52)$$

Similarly

From (4.49), we have

$$\mathrm{r}_2 - \tau_2 - \frac{2\mathrm{r}_2\mathrm{P}_2}{\mathrm{k}_2} - \phi_2 = -\left(\frac{\tau_2\mathrm{P}_2}{\mathrm{k}_2} + \frac{\mathrm{r}_1\mathrm{P}_1}{\mathrm{P}_2} + \frac{\tau_3\mathrm{P}_3}{\mathrm{P}_2}\right) \qquad (4.53)$$

From (4.50), we have

$$\mathrm{r}_3 - \tau_2 - \frac{2\mathrm{r}_3\mathrm{P}_3}{\mathrm{k}_3} - \phi_3 = -\left(\frac{\tau_3\mathrm{P}_3}{\mathrm{k}_3} + \frac{\mathrm{r}_1\mathrm{P}_1}{\mathrm{P}_3} + \frac{\tau_2\mathrm{P}_2}{\mathrm{P}_3}\right) \qquad (4.54)$$

From (4.51), we have

$$-\mu_1 - \mu_2 P = -\epsilon_1 P_1 - \epsilon_2 P_2 - \epsilon_3 P_3$$

$$-\mu_1 - \mu_2 P - \mu_2 P + \epsilon_1 P_1 + \epsilon_2 P_2 + \epsilon_3 = -\mu_2 P \qquad (4.55)$$

Substituting (4.52), (4.53), (4.54) and (4.55) into the Jacobian matrix (4.47), we have

$$J = \begin{pmatrix} -\left(\frac{\mathrm{r}_1\mathrm{P}_1}{\mathrm{k}_1} + \frac{\tau_2\mathrm{P}_2}{\mathrm{P}_1} + \frac{\tau_3\mathrm{P}_3}{\mathrm{P}_1}\right) & \tau_2 & \tau_3 & -\phi_1 P_1 \\ \tau_1 & -\left(\frac{\tau_2\mathrm{P}_2}{\mathrm{k}_2} + \frac{\mathrm{r}_1\mathrm{P}_1}{\mathrm{P}_2} + \frac{\tau_3\mathrm{P}_3}{\mathrm{P}_2}\right) & \tau_3 & -\phi_2 P_2 \\ \tau_1 & \tau_2 & -\left(\frac{\tau_3\mathrm{P}_3}{\mathrm{k}_3} + \frac{\mathrm{r}_1\mathrm{P}_1}{\mathrm{P}_3} + \frac{\tau_2\mathrm{P}_2}{\mathrm{P}_3}\right) & -\phi_3 P_3 \\ \epsilon_1 P & \epsilon_2 P & \epsilon_2 P & -\mu_2 P \end{pmatrix} \qquad (4.56)$$

$$|J - \lambda I| = \begin{vmatrix} -\left(\frac{\mathrm{r}_1\mathrm{P}_1}{\mathrm{k}_1} + \frac{\tau_2\mathrm{P}_2}{\mathrm{P}_1} + \frac{\tau_3\mathrm{P}_3}{\mathrm{P}_1}\right) - \lambda & \tau_2 & \tau_3 & -\phi_1 P_1 \\ \tau_1 & -\left(\frac{\tau_2\mathrm{P}_2}{\mathrm{k}_2} + \frac{\mathrm{r}_1\mathrm{P}_1}{\mathrm{P}_2} + \frac{\tau_3\mathrm{P}_3}{\mathrm{P}_2}\right) - \lambda & \tau_3 & -\phi_2 P_2 \\ \tau_1 & \tau_2 & -\left(\frac{\tau_3\mathrm{P}_3}{\mathrm{k}_3} + \frac{\mathrm{r}_1\mathrm{P}_1}{\mathrm{P}_3} + \frac{\tau_2\mathrm{P}_2}{\mathrm{P}_3}\right) - \lambda & -\phi_3 P_3 \\ \epsilon_1 P & \epsilon_2 P & \epsilon_2 P & -\mu_2 P - \lambda \end{vmatrix} = 0 \qquad (4.57)$$

$$Let \quad \alpha_1 = \left(\frac{\mathrm{r}_1\mathrm{P}_1}{\mathrm{k}_1} + \frac{\tau_2\mathrm{P}_2}{\mathrm{P}_1} + \frac{\tau_3\mathrm{P}_3}{\mathrm{P}_1}\right), \quad \alpha_2 = \left(\frac{\tau_2\mathrm{P}_2}{\mathrm{k}_2} + \frac{\mathrm{r}_1\mathrm{P}_1}{\mathrm{P}_2} + \frac{\tau_3\mathrm{P}_3}{\mathrm{P}_2}\right) \quad and \quad \alpha_3 = \left(\frac{\tau_3\mathrm{P}_3}{\mathrm{k}_3} + \frac{\mathrm{r}_1\mathrm{P}_1}{\mathrm{P}_3} + \frac{\tau_2\mathrm{P}_2}{\mathrm{P}_3}\right) \qquad (4.58)$$

$$\Rightarrow (-\alpha_1 - \lambda)\begin{vmatrix} -\alpha_2 - \lambda & \tau_3 & -\phi_2 P_2 \\ \tau_2 & -\alpha_3 - \lambda & -\phi_3 P_3 \\ \epsilon_2 P & \epsilon_3 P & -\mu_2 P - \lambda \end{vmatrix} - \tau_2 \begin{vmatrix} \tau_1 & \tau_3 & -\phi_2 P_2 \\ \tau_2 & -\alpha_3 - \lambda & -\phi_3 P_3 \\ \epsilon_1 P & \epsilon_3 P & -\mu_2 P - \lambda \end{vmatrix}$$

$$\tau_2 \begin{vmatrix} \tau_1 & -\alpha_2 - \lambda & -\phi_2 P_2 \\ \tau_2 & \tau_2 & -\phi_3 P_3 \\ \epsilon_1 P & \epsilon_2 P & -\mu_2 P - \lambda \end{vmatrix} - (-\phi_1 P_1)\begin{vmatrix} \tau_1 & -\alpha_2 - \lambda & \tau_3 \\ \tau_2 & \tau_2 & -\alpha_2 - \lambda \\ \epsilon_1 P & \epsilon_2 P & \epsilon_3 P \end{vmatrix} = 0 \qquad (4.59)$$

$$\Rightarrow (-\alpha_1 - \lambda)\left\{(-\alpha_2 - \lambda)\begin{vmatrix} -\alpha_3 - \lambda & -\phi_3 P_3 \\ \epsilon_2 P & -\mu_2 P - \lambda \end{vmatrix} - \tau_3 \begin{vmatrix} \tau_2 & -\phi_3 P_3 \\ \epsilon_2 P & -\mu_2 P - \lambda \end{vmatrix} - \phi_2 P_2 \begin{vmatrix} \tau_2 & -\alpha_2 - \lambda \\ \epsilon_2 P & \epsilon_3 P \end{vmatrix}\right\} = 0 \qquad (4.60)$$

$(-\alpha_1 - \lambda)\{(-\alpha_2 - \lambda)[\alpha_3\mu_2 P + \alpha_3\lambda + \mu_2 P\lambda + \lambda] - \tau_3[-\tau_2\mu_2 P - \tau_2\lambda + \epsilon_2 P\phi_3 P_3 + \epsilon_2 P\phi_3 P_3] -$
□2□2−τ2□3□+□3□2□+□2□□=0 (4.61)

$(-\alpha_1 - \lambda)\{-\alpha_2\alpha_3\mu_2 P - \alpha_2\alpha_3\lambda - \alpha_2\mu_2 P\lambda - \alpha_2\lambda^2 - \alpha_3\mu_2 P - \alpha_3\lambda^2 - \mu_2 P\lambda^2 - \lambda^3 + \tau_2\tau_3\mu_2 P +$
τ2τ3□−τ3□3□□3□3−□2□3□□3□3−□2□□3□3□−□2□2τ2□3□−□2□2□3□3□−□2□2□3□□=0
(4.62)

$(-\alpha_1 - \lambda)\{\lambda^3 - (\alpha_2 + \alpha_3 + \mu_2 P)\lambda^2 + (\tau_2\tau_3 - \alpha_2\alpha_3 - \alpha_2\mu_2 P - \alpha_3 P\phi_3 P_3 - \phi_2 P_2\epsilon_2 P)\lambda -$
□2□3□2□+τ2τ3□2□+τ2τ3□□3□3+□3□2□□3□3+□2□2τ3□3□+□2□2□3□3□=0
(4.63)

Expanding (4.63) and Collecting like terms, we have

$\lambda^4 + (\alpha_1 + \alpha_2 + \alpha_3 + \mu_2 P)\lambda^3 + [\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_1\mu_2 P + \tau_2\tau_3 - \alpha_2\alpha_3 - \alpha_2\mu_2 P - \alpha_3\mu_2 P -$
□2□□3□3−□2□2□2□−τ1τ2−□1□1□1□□2+□1τ2τ3−□1□2□3−□1□2□2□−□1□3□2□−□1□2□□3□3−
□1□2□2□3□+□2□3□2□+□3□3□2□+□2□3□□3□3+□2□2□□3□3+□2□2τ2□3□+□2□2□3□3□−τ1τ2
□3−τ1τ2□2□−τ1τ2τ3+□2□2□1□τ2+τ2τ3□2□−τ1τ2τ3−□1τ2τ3−□1□□3□3τ3+□1□1□2τ1□+□1□1□3□1
□−□1□1τ1□3□−□3□1□□1□1□□1□2□3□2□+□1τ2τ3□2□+□1τ3□3□□3□3+□1□2□3□□3□3+□2□2τ
2□3□□1+□2□2□2□3□3□−τ1τ2□3□2□−τ1τ2□3□□3□3−τ1τ2τ3□3□2□+τ2τ3□3□1□3□3+τ1τ2□2□2ϵ3
□+□2□2τ2□1□□3+□3τ3□1□□3□3−τ1τ2τ3□2□−□2τ2τ3□2□−□2□2τ2τ3□3□+□2□2τ2τ3□1□+τ1τ2□3
□□1□+□3□3τ1□□1□1+□1□1□2□3□1□+□1□1τ1τ3□3□−□1□1τ2τ3□1□=0 (4.64)

Let the characteristic equation be given as

$$\lambda^4 + A_1\lambda^3 + A_2\lambda^2 + A_3\lambda + A_4 = 0 \quad (4.65)$$

where

$$A_1 = (\alpha_1 + \alpha_2 + \alpha_3 + \mu_2 P)$$

$$A_2 = [(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_1\mu_2 P + \tau_2\tau_3) - (\alpha_2\alpha_3 + \alpha_2\mu_2 P + \alpha_3\mu_2 P + \epsilon_2 P\phi_3 P_3 + \phi_2 P_2\epsilon_2 P + \tau_1\tau_2 + \phi_1 P_1\epsilon_1 P)]$$

$$A_3 = [(\alpha_1\tau_2\tau_3 + \alpha_2\alpha_3\mu_2 P + \tau_3\tau_2\mu_2 P + \alpha_2\epsilon_3 P\phi_3 P_3 + \alpha_2\epsilon_2 P\phi_3 P_3 + \phi_2 P_2\tau_2\epsilon_3 P + \phi_2 P_2\alpha_3\epsilon_3 P + \phi_2 P_2\epsilon_1 P\tau_2 + \tau_2\tau_3\mu_2 P + \phi_1 P_1\epsilon_2\tau_1 P + \phi_1 P_1\alpha_3\epsilon_1 P) - (\alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\mu_2 P + \alpha_1\alpha_3\mu_2 P + \alpha_1\epsilon_2 P\phi_3 P_3 + \alpha_1\phi_2 P_2\epsilon_2 P + \tau_1\tau_2\alpha_3 + \tau_1\tau_2\mu_2 P + \tau_1\tau_2\tau_3 + \tau_1\tau_2\tau_3 + \alpha_1\tau_2\tau_3 + \epsilon_1 P\phi_3 P_3\tau_3 + \phi_1 P_1\tau_1\epsilon_3 P + \alpha_3\epsilon_1 P\phi_1 P_1)]$$

$A_4 =$
$[(\alpha_1\alpha_2\alpha_3\mu_2 P + \alpha_1\tau_2\tau_3\mu_2 P + \alpha_1\tau_3\epsilon_3 P\phi_3 P_3 + \alpha_1\alpha_2\epsilon_2 P\phi_3 P_3 + \phi_2 P_2\tau_2\epsilon_3 P\alpha_1 + \alpha_1\alpha_2\phi_2 P_2\epsilon_3 P +$
τ2τ3□1□1□3□3+τ1τ2□2□2□3□+□2□2τ2□1□τ3+□3τ3□1□□3□3+□2□2τ2τ3□1□+τ1τ2□3□□1□++□3
□3τ1□□1□1+□1□1□2□3□1□+□1□1τ1τ3□3□−τ1τ2□3□2□−τ1τ2□3□□3□3−τ1τ2τ3□3□2□+τ1τ2τ3□2
□+□2τ2τ3□2□+□2□2τ2τ3□3□+□1□1τ2τ3□1□=0
(4.66)

Applying Routh – Hurwitz criteria $E_2(P_1,\ P_2,\ P_3,\ P)$ is locally asymptotically stable (LAS), if $A_1 > 0,\ A_2 > 0,\ A_3 > 0,\ A_4 > 0,\ and\ \ A_1(A_2A_3 - A_4) > 0.$
Clearly, we see that $A_1 > 0,\ A_2A_3 - A_4 > 0.$ (4.67)
The stability characterization of $E_2(P_1,\ P_2,\ P_3,\ P)$ is determined by the sign of $A_2A_3 - A_4 > 0$, by direct algebraic calculate we have
$[(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_1\mu_2 P + \tau_2\tau_3) - (\alpha_2\alpha_3 + \alpha_2\mu_2 P + \alpha_3\mu_2 P + \epsilon_2 P\phi_3 P_3 + \phi_2 P_2\epsilon_2 P + \tau_1\tau_2 +$
□1□1□1□ .
$= [(\alpha_1\tau_2\tau_3 + \alpha_2\alpha_3\mu_2 P + \tau_3\tau_2\mu_2 P + \alpha_2\epsilon_3 P\phi_3 P_3 + \alpha_2\epsilon_2 P\phi_3 P_3 + \phi_2 P_2\tau_2\epsilon_3 P + \phi_2 P_2\alpha_3\epsilon_3 P +$
□2□2□1□τ2+τ2τ3□2□+□1□1□2τ1□+□1□1□3□1□−□1□2□3+□1□2□2□+□1□3□2□+□1□2□□3□3+
□1□2□2□2□+τ1τ2□3+τ1τ2□2□+τ1τ2τ3+τ1τ2τ3+□1τ2τ3+□1□□3□3τ3+□1□1τ1□3P+□3□1□□1□1 −
$[(\alpha_1\alpha_2\alpha_3\mu_2 P + \alpha_1\tau_2\tau_3\mu_2 P + \alpha_1\tau_3\epsilon_3 P\phi_3 P_3 + \alpha_1\alpha_2\epsilon_2 P\phi_3 P_3 + \phi_2 P_2\tau_2\epsilon_3 P\alpha_1 + \alpha_1\alpha_2\phi_2 P_2\epsilon_3 P +$
τ2τ3□1□1□3□3+τ1τ2□2□2□3□+□2□2τ2□1□τ3+□3τ3□1□□3□3+□2□2τ2τ3□1□+τ1τ2□3□□1□++□3
□3τ1□□1□1+□1□1□2□3□1□+□1□1τ1τ3□3□−τ1τ2□3□2□−τ1τ2□3□□3□3−τ1τ2τ3□3□2□+τ1τ2τ3□2
□+□2τ2τ3□2□+□2□2τ2τ3□3□+□1□1τ2τ3□1□>0
(4.68)
Clearly, we see that $A_2A_3 - A_4 > 0$. Thus, all eigenvalues are negative and therefore, the steady state solution $E_2(P_1,\ P_2,\ P_3,\ P)$ i s locally asymptotically stable, (LAS). ■

Theorem 4.5: The co-existence steady state solution $(P_1,\ P_2,\ P_3,\ P)$ of system (2.1) is globally asymptotically stable (GAS)

Proof: We will adopt the idea of Eduardo and Jaime (2011) and Kar (2009) to prove Theorem 4.3
Choosing a suitable Dulacs multiplier function $\mathcal{W}(P_1,\ P_2,\ P_3,\ P) = \frac{1}{P_1P_2P_3P}$ in Theorem 4.1, we established the non-existence of periodic orbit within the interior of the first quadrant around$E_2(P_1,\ P_2,\ P_3,\ P)$. So, local stability of the steady state $E_2(P_1,\ P_2,\ P_3,\ P)$ implies the global asymptotic stability (Eduardo and Jaime (2011) and Kar and Chattopadhyay (2009).

5.0 Numerical simulation

For the numerical simulation of the model systems, we choose a hypothetical data and set of parameters value as specified by Khamis et.al (2011) and Kar(2006) as follows:

r_1 =3, r_2=1.5, r_3 =1.2, k_1=110, k_2 =100, k_3=90, τ_1 = 0.5, τ_2=0.4 , τ_3=0.3, ϵ_1= 0.03, ϵ_2=0.02, ϵ_3=0.01, ϕ_1=0.3, ϕ_2=0.2, ϕ_3=0.1, μ_1=0.6 and μ_2=0.05 in appropriate units.
Using these parameter values, we determine the solutions for the steady state solution $E_1(P_1, P_2, P_3, 0)$ *and* $E_1(P_1, P_2, P_3, P)$with the help of Newton-Raphson Numerical scheme with MATLAB , and is given by $P_1 = 135.4092,\ P_2 = 217.0719$ *and* $P_3 = 249.0389$ for $E_1(P_1, P_2, P_3, 0)$ $P_1 = 70.8828,\ P_2 = 128.3600,\ P_3 = 185.6401$ and $P = 6.9229$ for $E_1(P_1, P_2, P_3, P)$
Now we plot the dynamics of the system for the set of parameter values given above with the help of MATLAB 7.5. The behaviour of P_1, P_2 and P_3 with respect to time in the absence of predation is plotted in Fig.5.1 and Fig.5.2, the behaviour of P_1 with respect to time , the behaviour of P_2 with respect to time and the behaviour of P_3 with respect to time is plotted in Fig.5.3, Fig.5.4 and Fig.5.5 respectively. Plot of prey fish populations versus time in the absence of predation is plotted in Fig.5.6. Figures 5.7-5.9 show the plot of fish population in the presence of predation. Fig. 5.10 is the graph showing global stability of E_2 with the colorbar representing the predator population

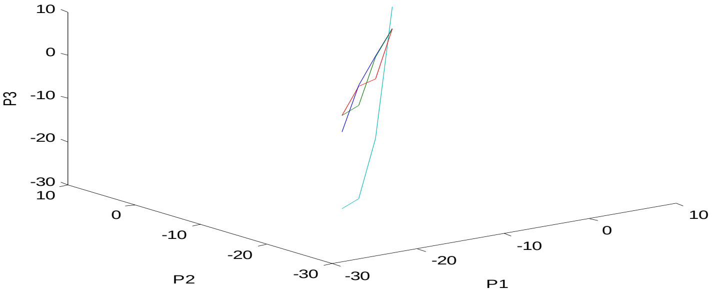

Fig5.1: Global stability of E_1

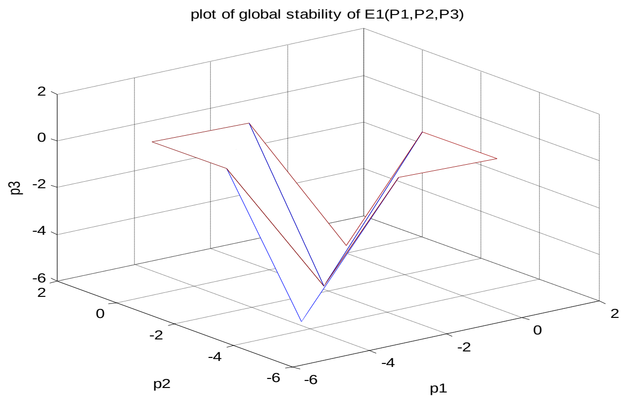

Fig 5.2: Global stability of E_1

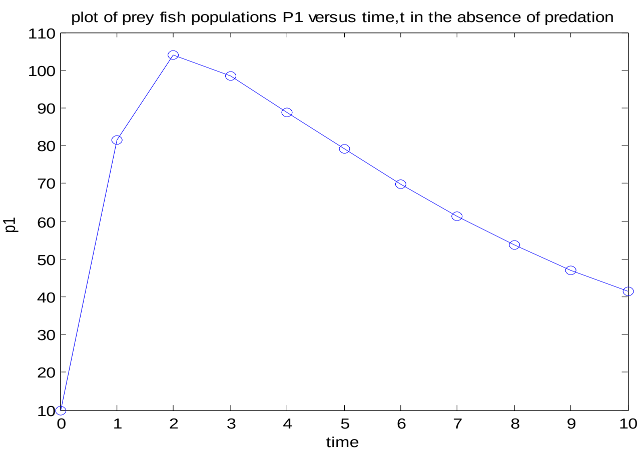

Fig 5.3: Plot of prey fish population P_1 versus time

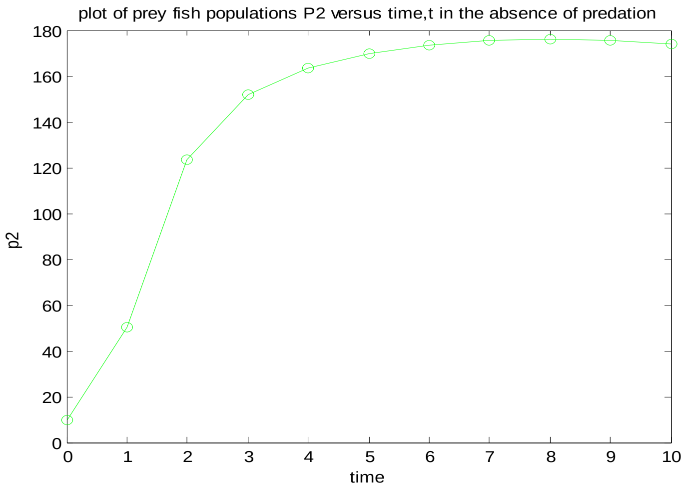

Fig 5.4: Plot of prey fish population P_2 versus time

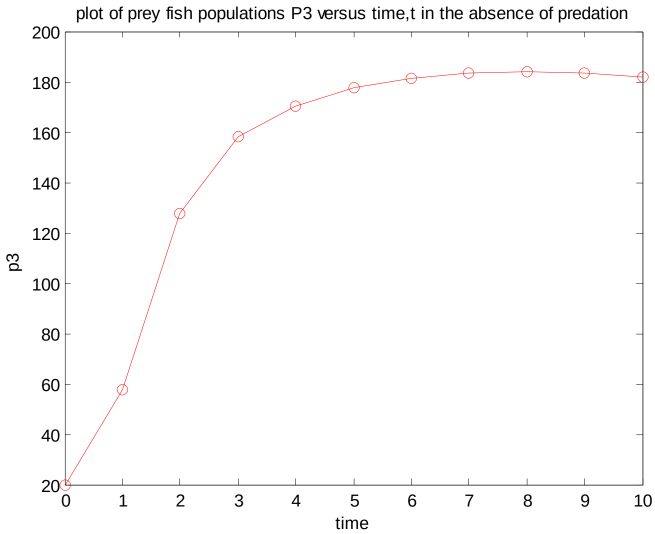

Fig 5.5: Plot of prey fish population P_3 versus time

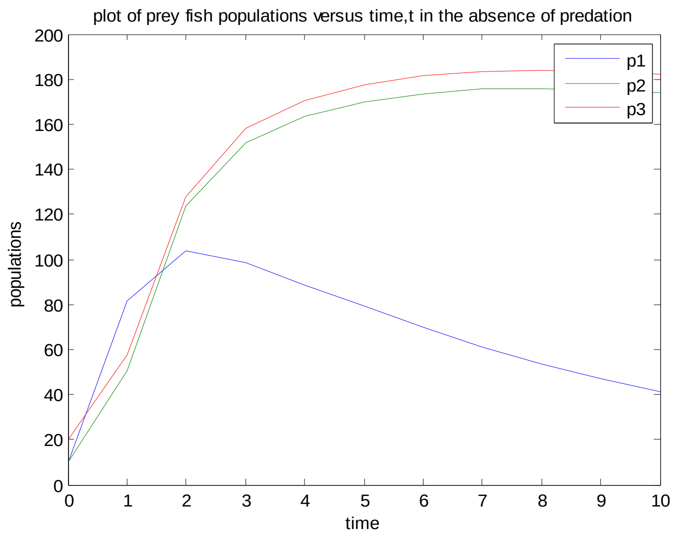

Fig. 5.6: Plot of prey fish populations versus time in the absence of predation

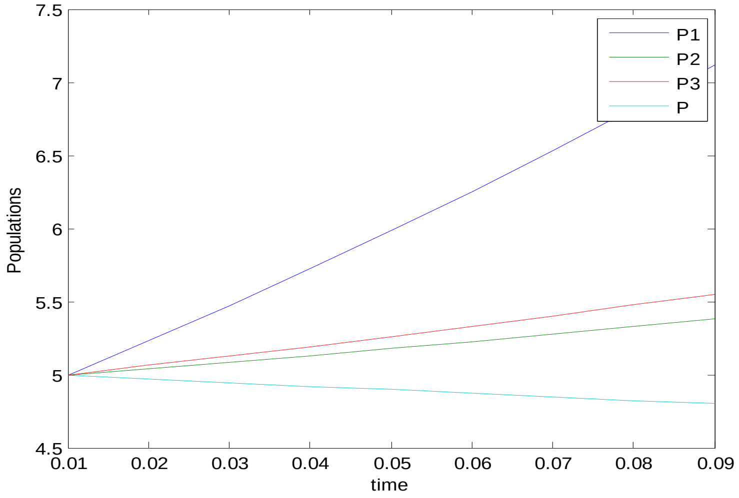

Fig.5.7: Plot of fish populations versus time in the presence of predation for T-final=0.09

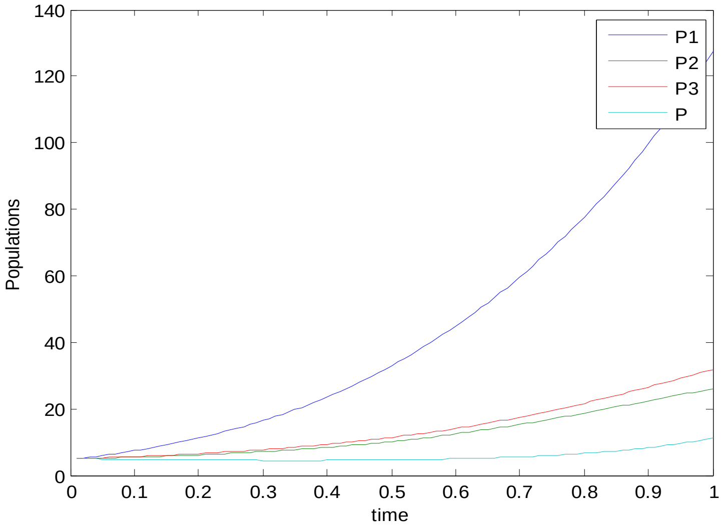

Fig.5.8: Plot of fish populations versus time in the presence of predation for T-final=1

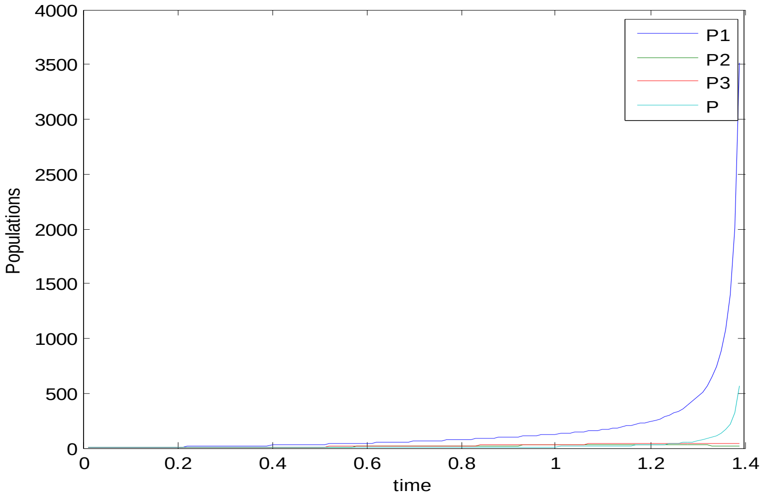

Fig.5.9: Plot of fish populations versus time in the presence of predation for T-final=1.4

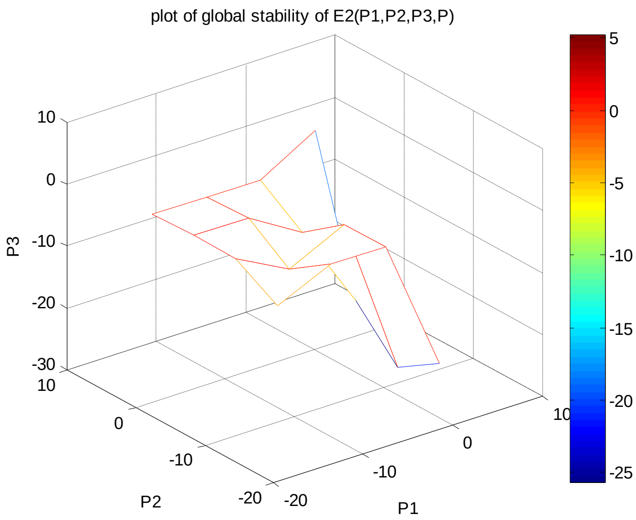

Fig. 5.10: Showing global stability of E_2 with the colorbar representing the predator population

IV. CONCLUSION

We have formulated and analysed a prey-predator fishery model with prey migration in a three –patch aquatic habitat for marine reserve. The existence of the possible steady states with their local stability is discussed and also conditions for global stability of the system is derived using Lyapunov function .We found that the first fish population is to some extent resilient to predation as seen in the graphical solution. Our overall observation is that whether in the presence or in the absence of predation, the fish population of the reserve ecosystem is sustained at an appropriate steady state level with indication of biodiversity loss as a result of predation. Having obtained a co-existence steady state solution that is globally asymptotically stable, it will be necessary to plan harvesting of the prey-predator fishery for sustainable development of the ecosystem and this form the basis of our further work.

REFERENCES

[1]. Birkoff, G and Rota, G (1978) Ordinary Differential Equation; John Wiley and Sons, New York.
[2]. Brauer, F. and Sondack, A. C. (1979) Stability Regions in Predator-Prey systems with constant-Rate Prey Harvesting; Journal of Mathematical Biology, Vol. 8, N0. 1, pp 55-71
[3]. Chattopadhyay (2009)
[4]. Chaudhuri, K. S. and Ray, S. S. (1996) On the Combined Harvesting of a Predator Prey System. Journal of Biological systems, Vol. 4, No. 3, pp 373-389
[5]. Dai, G. and Tang, M (1998) Coexistence Region and Global Dynamics of a Harvested Predator Prey System; SIAM Journal on Applied Mathematics, Vol.58, No. 1, pp 193-210.
[6]. Das Tapasi, Mukherjee and Chaudhuri (2009) Bionomic Harvesting of a Prey-Predator Fishery. Journal of Biological Dynamics, Vol.3, No.5, PP 447-462.
[7]. Dubey, Balram and Patra, Atasi (2013) Optimal Management of a Renewable Resource Utilized by a Poulation with Taxation as a Control Variable. Nonlinear Analysis: Modelling and Control, vol. 18, No. 1, pp 37-52.
[8]. Dubey, B., Chandra, P. And Sinha, P. (2003) A Model for Fishery Resource with Reserve Area; Non-linear Analysis: Real World Applications, 4, Pp. 625-637
[9]. Eduardo González-Olivares and Jaime Huincahue-Arcos (2011) A Two-patch Model for the
[10]. Optimal Management of a Fishing Resource Considering a Marine Protected Area; Nonlinear
[11]. Analysis: Real World Applications, Vol.12, PP2489-2499
[12]. Inyama S. C (2008). Mathematical Model of Fish Harvesting in a common access fishery. Journal of
[13]. Mathematical Sciences, Vol.19, No.1, PP 43-47
[14]. Kar, T. K. (2003) Selective Harvesting in a Prey-Predator Fishery with Time Delay. Mathematical and Computer Modelling, Vol. 38, pp 449-458
[15]. Kar, T. K. (2006) A Model for Fisheries Resources with Reserve Area and Facing Prey-Predator Interaction, J. Can, Appl. Math. Q. 14(4), pp. 385 -399.
[16]. Kar, T. K. and Chattopadhyay, S. K. (2009) Bio-economic Modelling: An Application to the North-
[17]. East- Atlantic cod Fishery, Journal of Mathematics Research, Vol.1, No.2, pp. 164-178.
[18]. Kar, T. K. and Pahari, U. K (2007) A Model for Prey-Predator Fishery with Marine Reserve, J. Fish;

[19]. Aquat, Sci 2(3), pp. 195-205.
[20]. Khamis, Sara A., Tchuenche, Jean M., Lukka, Markka and Heilio, Matti (2011) 'Dynamics of Fisheries with Prey Reserve and Harvesting, International Journal of Computer Mathematics, 88:8, 1775 – 1802
[21]. Kribs-zaleta, C. M. (2009) Sharpness of Saturation in Harvesting and Predator, Math. Bioscience
[22]. Eng. 6(4), pp. 719-742.
[23]. May, R. M. (1974) Stability and Complexity in Model Ecosystem, Princeton University Press, Princeton New Jessy, USA
[24]. Mellachervu, Naga S, Mantripragada Ananata, S. S., Kalyan Das and Nural Huda Gazi (2011) Prey Predator Fishery Model with Stage Structure in two-patchy marine Aquatic Environment. Applied Mathematics Vol. 2, pp 1405-1416
[25]. Sanchirico, J. N.; Cochran, K. A. and Emerson, P. M. (2002) Marine Protected
[26]. Areas: Economic and Social Implications Resources for the Future, pp. 1-24.
[27]. Zhang, R, Sun, J. F. and Yang, H. X. (2007) Analysis of a Prey-Predator Fishery Model with Prey Reserve; Applied Mathematical Science, Vol. 50, No. 1, pp 2481-2492.

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