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Research Paper

On The Existence of Bounded oscillatory Solutions of Impulsive Delay Differential Equations of The Second Order

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ABSTRACT: *In this paper, we consider a certain class of second order delay differential equations with constant impulsive jumps and obtain some sufficient conditions for every bounded solution to be oscillatory. Examples are provided to illustrate the main result.*

Keywords: *Bounded, impulsive, delay, differential equation, second-order, oscillation.*

I. INTRODUCTION

A number of processes in natural evolution feature the fact that at certain moments of time they experience an instantaneous change of state. This has been the main reason for the development of the theory of impulsive ordinary differential equations which has become a new branch of the theory of ordinary differential equations. The first paper on oscillation of impulsive delay differential equations was published in 1989 by Gopalsamy and Zhang [9]. Recently the oscillatory behavior of impulsive delay differential equations has attracted the attention of many researchers. For some contributions in this area, the reader is referred to ([2], [6], [7], [8]). Relatively, it is known that stochastic functional differential equations with state-dependent delay, which are relevant in mathematical models of real phenomena, are an important area of practical application of differential equations with impulses [10].

In this paper, we are concerned with the problem of oscillation of bounded solutions of a class of second order impulsive delay differential equations. In ordinary differential equations, the solutions are continuously differentiable, sometimes at least once, whereas the impulsive differential equations generally possess non-continuous solutions. Since the continuity properties of the solutions play an important role in the analysis of the behaviour, the techniques used to handle the solutions of impulsive differential equations are fundamentally different including the definitions of some of the basic terms. In this section, we examine some of these changes:

Let an evolution process evolve in a period of time J in an open set $\Omega \subset J \times R^n$ and let the function $f : \Omega \to R^n$ be at the least a continuous mapping fulfilling local Lipchitzian condition in $y \in R^n$, $\forall (t,y) \in \Omega$. Let the real numerical sequence $S = \{t_k\}_{k=1}^{\infty} \subset J$ be increasing without finite accumulation point such that $0 \le t_0 < t_1 < \cdots < t_k < \cdots$ with $\lim\limits_{k \to +\infty} t_k = +\infty$. The points t_k are called moments of impulse effect. Then the governing second order impulsive differential equation is of the form

$$\begin{cases} y''(t) = f(t,y,y'),\ t \ne t_k \\ \Delta y'(t_k) = f_k(y,y'),\ t = t_k, \end{cases} \tag{1.1}$$

where $y' = \dfrac{dy}{dt}$, $y'' = \dfrac{d^2y}{dt^2}$, $(t, y(t)) \in \Omega$, $\Delta y^{(i)}(t_k) = y^{(i)}(t_k^+) - y^{(i)}(t_k^-)$, $i = 0,1$ and $y(t_k^-)$, $y(t_k^+)$ represent the left and right limits of $y(t)$ at $t = t_k$, respectively. For the sake of definiteness, we shall suppose that the functions $y(t)$ and $y'(t)$ are continuous from the left at the points t_k such that $y'(t_k^-) = y'(t_k)$, $y(t_k^-) = y(t_k)$.

For the description of the continuous change of such processes ordinary differential equations are used, while the moments and the magnitude of the change by jumps are given
by the jump conditions. Now, in the case of unfixed moments of impulse effects, the impulse points may be time and state dependent, that is, $t_k = t_k(t, y(t))$. When the function t_k depends on the state of the system (1.1), then it is said to have impulses at variable times. This is reflected in the fact that different solutions will tend to undergo impulses at different times.

In this paper, we shall restrict ourselves to the investigation of conditions for bounded oscillatory solutions of impulsive differential equations for which the impulse effects take place at fixed moments of time $\{t_k\}$. Our equation under consideration is of the form

$$\begin{cases} \left[r(t)y'(t)\right]' = p(t)y(g(t)), t \geq t_0, t \notin S \\ \Delta\left[r(t_k)y'(t_k)\right] = p_k y(g(t_k)), t_k \geq t_0, \forall t_k \in S \end{cases} \tag{1.2}$$

where $t, t_k \geq 0$. Without further mention throughout this paper, we will assume that every solution $y(t)$ of equation (1.2) that is under consideration here, is continuous from the left and is nontrivial. That is, $y(t)$ is defined on some half-line $[T_y, \infty)$ and $sup\left\{|y(t)| : t \geq T\right\} > 0$ for all $T \geq T_y$. Such a solution is called a regular solution of equation (1.2).

We say that a real valued function $y(t)$ defined on an interval $[a, +\infty)$ fulfills some property *finally* if there exists $T \geq a$ such that $y(t)$ has this property on the interval $[T, +\infty)$.

Definition 1.1 The solution $y(t)$ of the impulsive differential equation (1.2) is said to be

i) finally positive (finally negative) if there exist $T \geq 0$ such that $y(t)$ is defined and is strictly positive (negative) for $t \geq T$ ([4]);
ii) non-oscillatory, if it is either finally positive or finally negative; and
iii) oscillatory, if it is neither finally positive nor finally negative ([1], [5]).

Remark 1.1: All functional inequalities considered in this paper are assumed to hold finally, that is, they are satisfied for all t large enough.

II. STATEMENT OF THE PROBLEM

At this point, may we recall that the problem under consideration is the second order linear impulsive differential equation with delay of the form

$$\begin{cases} \left[r(t)y'(t)\right]' = p(t)y(g(t)), t \geq t_0, t \notin S \\ \Delta\left[r(t_k)y'(t_k)\right] = p_k y(g(t_k)), t_k \geq t_0, \forall t_k \in S \end{cases} \tag{2.1}$$

where $\Delta\left(r(t_k)y'(t_k)\right) = r\left(t_k^+\right)y'\left(t_k^+\right) - r(t_k)y'(t_k)$.

We introduce the following conditions:

C2.1: $g(t) \in C([t_0, \infty), R), g(t)$ is a non-decreasing function in $R_+, g(t) \geq t$ for $t \in R_+$ and $\lim_{t\to\infty} g(t) = +\infty;$

C2.2: $r \in PC^1\left([t_0, \infty), R_+\right)$ and $r(t) > 0, r\left(t_k^+\right) > 0,$ for $t, t_k \in R_+$;

C2.3: $p \in PC\left([t_0, \infty), R_+\right)$ and $p_k \geq 0, k \in N;$

At the initial point $t_0 \geq 0$ the following initial conditions are imposed on the solution of equation (2.1):

$y(t) = \phi(t)$ for $t \in E_{t_0}$, $y'(t_0^+) = y'_0$, where $E_{t_0} = \{t_0\} \cup \left\{g(t) : g(t) < t_0,\ t > t_0\right\}$; $\phi \in C(E_{t_0}, R)$. Our aim is to establish some sufficient conditions for every bounded solution of equation (2.1) to be oscillatory. Here, we demonstrate how well-known mathematical techniques and methods, after suitable modification, is extended in proving an oscillation theorem for impulsive delay differential equations.

III. MAIN RESULTS

The following theorem is the impulsive extension of Theorem 4.3.1 of the monograph by Ladde *et al* [3] while the salient techniques for the proof are obtained from studies by Bainov and Simeonov [1].

Theorem 3.1: Assume that

i) conditions C2.1—C2.3 hold

ii) $p(t)>0$ for $t\in R_+$;

iii) $r(t)$ is a non-decreasing function in R_+ and $\lim_{t\to\infty}\int_{t_0}^{t}\frac{ds}{r(s)}=\infty$

iv) $$\lim_{t\to\infty}\sup\frac{1}{r(t)}\left[\int_{g(t)}^{t}\big(u-g(t)\big)p(u)du+\sum_{g(t)\le t_k<t}\big(t_k-g(t_k)\big)p_k\right]>1 \qquad (3.1)$$

Then every bounded solution $y(t)$ of equation (2.1) is oscillatory.

Proof: Assume by contradiction that $y(t)$ is a bounded finally positive solution of equation (2.1). That is, there exist constants $T_1>0$ and $L>0$ such that $0<y(t)\le L$ for $t\ge T_1$.

Hence, $(r(t)y'(t))'\ge 0$, $\Delta(r(t_k)y'(t_k))\ge 0$ for $t\ge T_2\ge T_1$ and $\forall k: t_k\ge T_2\ge T_1$, where T_2 is sufficiently large. This means that $r(t)y'(t)$ is a non-decreasing function for $t\ge T_2$.

Here, we observe that there exist two possibilities:

a) If $r(t)y'(t)>c>0$ and $t\ge T_3\ge T_2$, then $y'(t)>\frac{c}{r(t)}$, $t\ne t_k$ and $\Delta y(t_k)=0$ for $t\ge T_3$. Integrating this and taking into account condition (iii) results in the fact that $y(t)$ is an unbounded function which contradicts our earlier assumption.

b) If $r(t)y'(t)\le 0$ for $t\ge T_2$, then $y'(t)\le 0$ for $t\ge T_2$. That is $y(t)$ is a non-increasing function for $y(t)\ge T_2$.

Integrating equation (2.1) from s to t, we have

$$r(t)y'(t)-r(s)y'(s)=\int_{s}^{t}p(u)y\big(g(u)\big)du+\sum_{s\le t_k\le t}p_k y\big(g(t_k)\big). \qquad (3.3)$$

Again, integrating equation (3.3) in s from $g(t)$ to t, we obtain

$$r(t)y'(t)\big(t-g(t)\big)=\int_{g(t)}^{t}r(s)y'(s)ds+\int_{g(t)}^{t}\left(\int_{s}^{t}p(u)y(g(u))du+\sum_{s\le t_k<t}p_k y(g(t_k))\right)ds \qquad (3.4)$$

We now change the order of integration in equation (3.4), rearrange and obtain

$$0\ge r(t)y(t)-r(g(t))y(g(t))-y(g(t))(r(t)-r(g(t)))+\int_{g(t)}^{t}(u-g(t))p(u)y(g(u))du+$$
$$+\sum_{g(t)\le t_k<t}(t_k-g(t))p_k y(g(t_k))$$
$$\ge r(t)(y(t)-y(g(t)))+\int_{g(t)}^{t}(u-g(t))p(u)y(g(u))du+\sum_{g(t)\le t_k\le t}(t_k-g(t))p_k y(g(t_k))$$

or

$$0 \geq y(t) - y(g(t)) + \frac{1}{r(t)}\left[\int_{g(t)}^{t}(u-g(t))p(u)y(g(u))du + \sum_{g(t)\leq t_k \leq t}(t_k - g(t))p_k y(g(t_k))\right]. \tag{3.5}$$

Dividing inequality (3.5) by $y(g(t))$ and using the monotonicity of $y(t)$, we get

$$0 \geq \frac{y(t)}{y(g(t))} + \left[\frac{1}{r(t)}\left\{\int_{g(t)}^{t}(u-g(t))p(u)du + \sum_{g(t)\leq t_k < t}(t_k - g(t))p_k\right\} - 1\right]$$

which is a contradiction to equation (3.1). This, therefore, completes the proof.

Corollary 3.1: If $\tau \geq 0$, $p(t) \geq 0$ is continuous, $p_k \geq 0$ and $\tau^2 p(t) \geq 2$ for $t \geq 0$, then bounded solutions of the equation

$$\begin{cases} y''(t) - p(t)y(t-\tau) = 0, \; t \notin S \\ \Delta y'(t_k) - p_k y(t_k - \tau) = 0, \; \forall \, t_k \in S \end{cases}$$

are oscillatory.

Corollary 3.2: If $\ell > 0$, $p(t) \geq 0$ is piece-wise continuous, $p_k \geq 0$ and

$$\begin{cases} p(t) \geq \dfrac{2\ell^2}{((1-\ell)t)^2} \\ p_k \geq \dfrac{2\ell^2}{((1-\ell)t_k)^2} \end{cases}$$

for large t the bounded solutions of

$$\begin{cases} y''(t) - p(t)y(\frac{t}{\ell}) = 0, \; t \notin S \\ \Delta y'(t_k) - p_k y(\frac{t_k}{\ell}) = 0, \; \forall \, t_k \in S \end{cases}$$

are oscillatory.

Example 3.1: The equation

$$\begin{cases} \left(\frac{1}{t}y'\right)' - 4ty(t-\pi) = 0, t \geq 2, \; t \notin S \\ \Delta\left(\frac{1}{t_k}y'\right) - 4t_k y(t_k - \pi) = 0, t_k \geq 2, \; \forall \, t_k \in S \end{cases}$$

satisfies the conditions of Theorem 3.1. Therefore, all bounded solutions are oscillatory. In particular, $y(t) = \cos t^2$ is one such solution.

Example 3.2: The equation

$$\begin{cases} y''(t) - y(t-\tau) = 0, \; t \notin S \\ \Delta y'(t_k) - y(t_k - \tau) = 0, \; \forall \, t_k \in S, \end{cases}$$

where $0 \leq \tau \leq 2e^{-1}$, does not satisfy the conditions of Theorem 3.1 as expected. This equation has a bounded non-oscillatory solution. $y(t) = e^{\lambda t}$ is one such solution.

Remark 3.1: If we do not require that $\int^{\infty} \frac{ds}{r(s)} = \infty$, but $r(t)$ is non-decreasing and condition (3.1) is satisfied, then the conclusion of Theorem 3.1 remains valid.

REFERENCES

[1]. D. D. Bainov and P. S. Simeonov, Oscillation Theory of Impulsive Differential Equations, International Publications Orlando, Florida, 1998.

[2]. L. Berezansky and E. Braverman, Oscillation of a linear delay impulsive differential equation, Comm. Appl. Nonlinear Anal., 3(1):61–77, 1996.

[3]. G. S. Ladde, V. Lakshmikantham and B. G. Zhang, Oscillation Theory of Differential Equations with Deviating Arguments, Marcel Dekker, New York, 1987.

[4]. I. O. Isaac and Z. Lipcsey and U. J. Ibok, Nonoscillatory and Oscillatory Criteria for First Order Nonlinear Neutral Impulsive Differential Equations, Journal of Mathematics Research, Vol. 3 Issue 2, (2011), 52-65.

[5]. I. O. Isaac and Z. Lipcsey, Oscillations of Scalar Neutral Impulsive Differential Equations of the First Order with variable Coefficients, Dynamic Systems and Applications, 19, (2010), 45-62.

[6]. Jurang Yan, Oscillation properties of a second-order impulsive delay differential equation, Comput. Math. Appl., 47(2-3):253–258, 2004.

[7]. Guo Liang Huang and Jian Hua Shen, Oscillation of a second-order linear ODE with impulses, J. Nat. Sci. Hunan Norm. Univ., 26(4):8–10, 2003.

[8]. Chaolong Zhang and Xiaojian Hu, Oscillations of higher order nonlinear ordinary differential equations with impulses, J. Zhongkai Univ., 18(1):45–51, 2005.

[9]. K. Gopalsamy and B. G. Zhang, On delay differential equations with impulses, Journal of Mathematical Analysis and Applications, 139 (1989), pp. 110–122.

[10]. U. A. Abasiekwere and I. U. Moffat, Oscillation Theorems for Linear Neutral Impulsive Differential Equations of the Second Order with Variable Coefficients and Constant Retarded Arguments, Applied Mathematics, Vol. 7 No. 3, 2017, pp. 39-43. doi: 10.5923/j.am.20170703.01.

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