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**Research Paper**

# Combined Properties of Finite Sums & Finite Products near Zero

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***ABSTRACT:*** *It was proved that wheneveris partitioned into finitely manycells, one cell must contain arbitrary length geo-arithmetic progressions. It was also proved that arithmetic and geometric progressions can be nicely in-entwined in one cell of partition, whenever ℕ is partitioned into finitely many cells. In this article we shall prove that similar types of results also hold near zero in some suitable dense sub semigroup S of ((0, ∞) ,+), using the Stone-Čech compactification βS.*
**Key words and phrases.**Ramsey theory, Central sets near zero, Finite Sums, Image partition near zero, IP set near zero.

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## I. INTRODUCTION

One of the famous Ramsey theoretic results is van der Waerden's Theorem [11], which says that whenever the set N of natural numbers is divided into finitely many classes, one of these classes contains arbitrarily long arithmetic progressions. The analogous statement about geometric progressions is easily seen to be equivalent via the homomorphisms p: (ℕ, +) ⟶ (ℕ, ·) such that p (x) = $2^x$, andq: (ℕ \ {1}, ·) ⟶ (ℕ, +),where q (x) is the length of the prime factorization of x.

It has been shown in [1, Theorem 3.11] that any set which is multiplicatively large, that is a piecewise syndetic IP set in (ℕ, · ) must contain substantial combined additive and multiplicative structure; in particular it must contain arbitrarily large geo-arithmetic progressions, that is, sets of the form$r^j(a + id)$ :i, j ∈{1, 2, . . . , k}.

A well-known extension of van der Waerden's Theorem allows one to get the additive increment of the arithmetic progression in the same cell as the arithmetic progression. Similarly, for any finite partition of N there exist some cell A and b, r ∈ℕsuch that$\{r, b, br, \ldots, br^k\}$ ⊆ A. It is proved in [1, Theorem 1.5] that these two facts can be intertwined.

**Theorem 1.1.** Let r, k∈ℕ and ℕ=$\cup_{i=1}^{r} A_i$. Then there existss∈{1,2, . . ., r}and a, b, d ∈ $A_s$, such that
$\{b(a + id)^j : i, j \in \{0, 1, \ldots, k\}\}$□$\{bdj : j \in \{0, 1, \ldots, k\}\}$
□{a + id : i∈ {0, 1, . . . , k}}⊆$A_s$

We know that if A⊆ℕ belongs to every idempotent in βℕ, then it is calledan IP* set. Given a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in ℕ, we let FP$\langle x_n \rangle_{n=1}^{\infty}$ be the product analogue of Finite Sum. Given a sequence$\langle x_n \rangle_{n=1}^{\infty}$in ℕ, we say that $\langle y_n \rangle_{n=1}^{\infty}$ is a sum subsystem of $\langle x_n \rangle_{n=1}^{\infty}$ provided there is a sequence $\langle H_n \rangle_{n=1}^{\infty}$ of nonempty finite subsets of ℕ such that max $H_n$< min $H_{n+1}$, and $y_n = \sum_{t \in H_n} x_t$ for each n ∈ ℕ.

**Theorem 1.2.** Let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence in ℕ and A be an IP* set in (ℕ,+). Then there exists a sum subsystem $\langle y_n \rangle_{n=1}^{\infty}$ of $\langle x_n \rangle_{n=1}^{\infty}$ such thatFS$\langle y_n \rangle_{n=1}^{\infty}$∪FP$\langle y_n \rangle_{n=1}^{\infty}$⊆ A.

*Proof.* [2, Theorem 2.6] or see [9, Corollary 16.21].

The algebraic structure of the smallest ideal of βS has played a significant role in Ramsey Theory. It is known that any central subset of (ℕ, +) is guaranteed to have substantial additive structure. But Theorem 16.27 of [9] shows that central sets in (ℕ, +) need not have any multiplicative structure at all. On the other hand, in [2] we see that sets which belong to every minimal idempotent of ℕ, called central* sets, must have significant multiplicative structure.
In case of central* sets a similar result has been proved in [4] for a restricted class of sequences called minimal sequences, where a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in N is said to be a minimal sequence if

$$\cap_{m=1}^{\infty} \overline{FS(\langle x_n \rangle_{n=1}^{\infty})} \cap K(\beta N) \neq \emptyset.$$

**Theorem 1.3.** Let $\langle y_n \rangle_{n=1}^{\infty}$ be a minimal sequence and A be a central* set in (ℕ,+). Then there exists a sum subsystem$\langle x_n \rangle_{n=1}^{\infty}$of$\langle y_n \rangle_{n=1}^{\infty}$such thatFS($\langle x_n \rangle_{n=1}^{\infty}$)∪ FP ($\langle x_n \rangle_{n=1}^{\infty}$)⊆A.

*Proof.*[2, Theorem 2.4].

A similar result in this direction in the case of dyadic rational numbers has been proved by Bergelson, Hindman and Leader.

**Theorem 1.4.** There exists a finite partition $\mathbb{D}\setminus\{0\}=\cup_{i=1}^{r} A_i$ such that there do not exist i∈{1,2, . . . , r} and a sequence $\langle x_n \rangle_{n=1}^{\infty}$ withFS($\langle x_n \rangle_{n=1}^{\infty}$)∪ FP ($\langle x_n \rangle_{n=1}^{\infty}$)⊆$A_i$.
Proof.[3, Theorem 5.9].

In [3], the authors also presented the following conjecture and question.

**Conjecture 1.5.**There exists a finite partition $\mathbb{Q}\setminus\{0\}=\cup_{i=1}^{r} A_i$ such that there do not exist i∈{1,2, . . . , r} and a sequence $\langle x_n \rangle_{n=1}^{\infty}$ withFS($\langle x_n \rangle_{n=1}^{\infty}$)∪ FP ($\langle x_n \rangle_{n=1}^{\infty}$)⊆$A_i$.

**Problem 1.6.** Does there exists a finite partition $\mathbb{R}\setminus\{0\}=\cup_{i=1}^{r} A_i$ such that there do not exist i∈{1,2, . . . , r} and a sequence $\langle x_n \rangle_{n=1}^{\infty}$ withFS($\langle x_n \rangle_{n=1}^{\infty}$)∪ FP ($\langle x_n \rangle_{n=1}^{\infty}$)⊆$A_i$ ?

In the section 2, we shall first work on some combined algebraic properties near 0 in the ring of quaternions, denoted byℍ. The ring being non-abelian, is a division ring having an idempotent 0. In section 3, for any suitable dense sub semigroup S of ((0, ∞), +), our aim is to establish partition regularity among two matrices using additive and multiplicative structure of βS, Stone-Čech compactification of S.

## II. COMBINED ALGEBRAIC AND MULTIPLICATIVE PROPERTIES NEARAN IDEMPOTENT IN RELATION WITH QUATERNION RINGS

In the following discussion our aim is to extend Theorem 1.2 and Theorem 1.3 for dense subsemigroups (ℍ, +) in the appropriate context.

**Definition 2.1.** IfSis a dense subsemigroup of(ℍ,+), we define$0^+(S)$ ={p∈$\beta S_d$: ( r >0)($B_d(r)$∈ p)}.

It is proved in [7], that 0+(S) is a compact right topological subsemigroup of ($\beta S_d$, +) which is disjoint fromK($\beta S_d$) and hence gives some new informationwhich are not available from K($\beta S_d$). Being compact right topological semigroup 0+(S) contains minimal idempotents of 0+(S). A subsetAofSis said to be IP*-setnear 0 if it belongs to every idempotent of 0+(S) and a subset C of S is said to be central* set near 0 if it belongs to every minimal idempotent of 0+(S). In [5] the authors applied the algebraic structure of 0+(S) on their investigation of image partition regularity near 0 of finite and infinite matrices. Article [6] used algebraic structure of 0+(R) to investigate image partition regularity of matrices with real entries from R.

**Definition 2.2.**LetSbe a dense subsemigroup of(ℍ,+). A subsetAofSis said to be an IP setnear 0 if there exists a sequence $\langle x_n \rangle_{n=1}^{\infty}$ with $\sum_{n=1}^{\infty} x_n$ converges such that FS($\langle x_n \rangle_{n=1}^{\infty}$)⊆A. We call a subset D of S is an IP* set near 0 if for every subset C of S which is IP set near 0, C ∩ D is IP set near 0.

From [10, Theorem 3.2], it follows that for a dense subsemigroup S of (ℍ, +) a subset A of S is an IP set near0 if only if there exists some idempotent p ∈$0^+$(S) with A ∈ p. Further it can be easily observed that a subset D of S is an IP* set near 0 if and only if it belongs to every idempotent of $0^+$(S).
Given c ∈H\ {0} and p ∈ $\beta\mathbb{H}_d$, the product c·p and p·c are defined in ($\beta\mathbb{H}_d$, ·). One has A ⊆ℍ is a member of c·p if and only if $c^{-1}A$ ={x ∈ℍ:c·x∈ A} is amember of p and similarly for c·p.

**Lemma 2.3.** Let S be a dense subsemigroup of (ℍ,+) such that S∩ℍis a sub-semigroup of (ℍ\ {0},·). If A is an IP set near 0 in S then sA is also an IP set near 0 for every s∈S∩$B_d(1)$\ {0}. Further if A is a an IP* set near 0 in (S,+) then both $s^{-1}A$ and $As^{-1}$ are IP set near 0 for every s∈S∩$B_d(1)$\ {0}.

*Proof.*SinceAis an IP set near0then by [7, Theorem 3.1] there exists a sequence$\langle x_n \rangle_{n=1}^{\infty}$ in S with the property that $\sum_{n=1}^{\infty} x_n$ converges and FS( $\langle x_n \rangle_{n=1}^{\infty}$ )⊆A. This implies that $\sum_{n=1}^{\infty}(s.x_n)$ is also convergent and FS($\langle s x_n \rangle_{n=1}^{\infty}$)⊆sA. This proves that sA is also IP set near 0. Similarly, we can prove that $As^{-1}$ is also

IP set near 0 for every s ∈ S $\cap B_d(1)\setminus\{0\}$. For the second let A be a an IP* set near 0 and s ∈ S $\cap B_d(1)\setminus\{0\}$. To prove that $s^{-1}A$ is a an IP* set near 0 it is sufficient to show that if B is any IP set near 0 then $B \cap s^{-1}A \neq \emptyset$. Since B is an IP set near 0, sB is also an IP set near 0 by the first part of the proof, so that $A \cap sB \neq \emptyset$. Choose t ∈sB∩ A and k ∈ B such that t = sk. Therefore k ∈ $s^{-1}A$so that $B \cap s^{-1}A \neq \emptyset$ .

Given A⊆S and s ∈ S, $s^{-1}A$ ={t ∈ S : st∈ A} and -s + A ={t ∈ S : s+ t∈A}. In case of product we must keep in mind the order of elements asthe product is noncommutative here.

**Definition 2.4.**Let$\langle x_n\rangle_{n=1}^{\infty}$be a sequence in the ring(ℍ,+,·), and letk∈ℕ.Then FP($\langle x_n\rangle_{n=1}^{k}$) is the set of all products of terms of $\langle x_n\rangle_{n=1}^{k}$ in any order with no repetitions. Similarly FP($\langle x_n\rangle_{n=1}^{\infty}$) is the set of all products of terms of $\langle x_n\rangle_{n=1}^{\infty}$ in any order with no repetitions.

**Theorem 2.5.** Let S be a dense subsemigroup of (ℍ,+), such that S$\cap B_d(1)\setminus\{0\}$ is a subsemigroup of ($B_d(1)\setminus\{0\}$,·). Also let $\langle x_n\rangle_{n=1}^{\infty}$ be a sequence in S such that$\sum_{n=1}^{\infty} x_n$converges to0and A be a IP* set near 0 in S. Then there exists a sumsubsystem$\langle y_n\rangle_{n=1}^{\infty}$ of $\langle x_n\rangle_{n=1}^{\infty}$ such that

$$FS(\langle y_n\rangle_{n=1}^{\infty}) \cup FP(\langle y_n\rangle_{n=1}^{\infty}) \subseteq A.$$

Proof. Since $\sum_{n=1}^{\infty} x_n$ converges to 0, from [7, Theorem 3.1] it follows that we can find some idempotent $p \in 0^+(S)$ for which $FS(\langle x_n\rangle_{n=1}^{\infty}) \in p$. In fact,$T = \cap_{m=1}^{\infty} cl_{\beta S_d} FS(\langle y_n\rangle_{n=m}^{\infty}) \subseteq 0^+(S)$ and $p \in T$. Again, since A is an IP* set near 0 in S, by the above Lemma 2.3 for every $s \in S\cap B_d(1)\setminus\{0\}$, both $s^{-1}A, As^{-1} \in p$. Let A* = {s∈A : −s + A∈p}. Then by [9, Lemma 4.14],A*∈p. We can choose$y_1 \in A^* \cap FS(\langle x_n\rangle_{n=1}^{\infty})$. Inductively let m ∈Nand$\langle y_i\rangle_{i=1}^{m}$,$\langle H_i\rangle_{i=1}^{m}$ in$P_f$(N)bechosen with the following properties:

(a) i∈{1, 2, . . . , m −1}Max $H_i$<Min $H_{i+1}$;

(b) If $y_i = \sum_{t\in H_i} x_t$ then $\sum_{t\in H_m} x_t \in A^*$ and $FS(\langle y_i\rangle_{i=1}^{m}) \subseteq A$.

We observe that $\left\{\sum_{t\in H} x_t : H \in P_f(N),\ minH > maxH_m\right\} \in p$. Let B $= \left\{\sum_{t\in H} x_t : H \in P_f(N),\ minH >\right.$ $maxHm$, let $E1=FS(yii=1m)$ and $E2=AP(yii=1m)$. Now consider

$$D = B \cap A^* \bigcap_{s\in E_1}(-s + A^*) \cap \bigcap_{s\in E_2}(s^{-1}A^*) \cap \bigcap_{s\in E_2}(A^* s^{-1})$$

Then $D \in p$. Now choose $y_{m+1} \in D$ and $H_{m+1} \in P_f(N)$ such that $minH_{m+1} > maxH_m$. Putting $y_{m+1} = \sum_{t\in H_{m+1}} x_t$ shows that the induction can be continued, and this proves the theorem.

## III. AN APPLICATION OF ADDITIVE AND MULTIPLICATIVE STRUCTURE OF βS

We shall like to produce an alternative proof of the above Theorem 3.1 using additive and multiplicative structure of βS. We need the following notion.

**Theorem 3.1.** Let u, v∈ℕ. Let M be a finite image partition regular matrix over ℕof orderu × v, and letNbe an infinite image partition regular near 0 matrixover a dense subsemigroup S of ((0,∞),+). Then

$$\begin{pmatrix} M & O \\ O & N \end{pmatrix}$$

is image partition regular near 0 over S.

**Definition 3.2.**LetSbe a subsemigroup of((0,∞),+)and letAbe a matrix, finite or infinite with entries fromℚ . Then I(A) = {p∈ $0^+$ : for every P∈p, there exists $\vec{x}$ with entries from S such that all entries of $A\vec{x}$are in P }.

The following lemma can be easily proved as [8, Lemma 2.5].

**Lemma 3.3.** Let A be a matrix, finite or infinite with entries from ℚ.

(a) The set I(A) is compact and I(A)≠ ∅if and only if A is image partition regular near 0.

(b) If A is finite image partition regular matrix, then I(A) is a sub-semigroup of ($0^+$, +).

Next, we shall investigate the multiplicative structure of I(A). In the following Lemma 3.4, we shall see that if A is an image partition regular near 0, then I(A) is a left ideal of (0+, ·). It is also a two-sided ideal of (0+, ·), provided A is a finite image partition regular near 0.

**Lemma 3.4.** Let A be a matrix, finite or infinite with entries from ℚ.

(a) If A is an image partition regular near 0, then I(A) is a left ideal of ($0^+$,·).

(b) If A is a finite image partition regular near 0, then I(A) is a two-sided ideal of $(0^+,\cdot)$.

Proof. (a). Let Abe au×vimage partition regular matrix, whereu,v∈ℕ∪{ω}.Let p ∈ $0^+$ and q ∈ I(A). Also let U ∈ p · q. Then{x ∈S : $x^{-1}$U∈ q} ∈ p. Choose z ∈ {x ∈S : $x^{-1}$U∈ q}. Then $z^{-1}$U ∈ q. So there exists $\vec{x}$ with entries from S such that $y_j$∈ $z^{-1}$U for 0 ≤ j < u where $\vec{y} = A\vec{x}$,

$\vec{y} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \end{pmatrix}$ and $\vec{x} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{pmatrix}$; $\vec{y}$ and $\vec{x}$ are u×1 and v×1 matrices respectively.

Now$y_i \in z^{-1}U$ for 0 ≤ i < u implies that $zy_i \in U$ for 0 ≤ i < u. Let $\vec{x'}$= z$\vec{x}$ and $\vec{y'}$= z$\vec{y}$. Then $\vec{y'} = A\vec{x'}$. So there exists $\vec{x'}$ with entries from S such that all entries of $A\vec{x'}$ are in U. Therefore $p.q \in I(A)$ is a left ideal of $(0^+, .)$.

(b). Let A be a u × v matrix, where u, v ∈ N. By previous lemma, I(A) is a left ideal. We now show that I(A) is a right ideal of $(0^+,\cdot)$. Let p ∈ βS and q ∈ I(A). Now let U ∈ q · p. Then {x ∈ S: $x^{-1}$U ∈ p} ∈ q. So there exists$\vec{x}$with entries in S such that $y_i \in \{x \in S: x^{-1}U \in p\}$ for 0 ≤ i< u, where $\vec{y} = A\vec{x}$, $\vec{y} = \begin{pmatrix} y_0 \\ \vdots \\ y_{u-1} \end{pmatrix}$ and

$\vec{x} = \begin{pmatrix} x_0 \\ \vdots \\ x_{v-1} \end{pmatrix}$; $\vec{y}$ and $\vec{x}$ are u×1 and v×1 matrices respectively. Now for 0 ≤ i < u, $y_i \in \{x \in S: x^{-1}U \in p\}$. Hence $y_i^{-1}U \in p$ for 0 ≤ i < u. This implies $\bigcap_{i=0}^{u-1} y_i^{-1}U \in p$. So $\bigcap_{i=0}^{u-1} y_i^{-1}U \neq \emptyset$. Let $z \in \bigcap_{i=0}^{u-1} y_i^{-1}U$. Therefore, $z \in y_i^{-1}U$ for all $i \in \{0,1,2,\cdots,u-1\}$. Hence $y_i z \in U$ for 0 ≤ i < u. Let $\vec{x'}$= $\vec{x}z$ and $\vec{y'}$= $\vec{y}$z. Then $\vec{y'} = A\vec{x'}$. So there exists $\vec{x'}$ with entries from S such that all entries of $A\vec{x'}$ are in U. Thus $q.p \in I(A)$. Therefore I(A) is also a right ideal of ($\beta$ℕ,·). Hence I(A) is a two-sided ideal of $(0^+,\cdot)$.

Alternative proof of Theorem 2.2.7.Let $r \in \mathbb{N}$ be given and $\epsilon > 0$. Let $\mathbb{Q} = \bigcup_{i=1}^{r} E_i$. Suppose that A be a $u \times v$ matrix where $u, v \in \mathbb{N}$. Also let $A = \begin{pmatrix} M & O \\ O & N \end{pmatrix}$. Now by previous lemma 3.4, I(M) is a two sided ideal of $(0^+,\cdot)$. So $K(0+,\cdot) \subseteq I(A)$. Also by lemma 3.4, I(M) is a left ideal of $(0^+,\cdot)$. Therefore, $K(0+,\cdot) \cap I(N) \neq \emptyset$. Hence, $I(M) \cap I(N) \neq \emptyset$. Now choose $p \in I(M) \cap I(N)$. Since $\mathbb{Q} = \bigcup_{i=1}^{r} E_i$, there exist $k \in \{1,2,\cdots,r\}$ such that $E_k \in p$. Thus,by definition of I(M) and I(N), there exist $\vec{x} \in S^v$ and $\vec{y} \in S^\omega$ such that $M\vec{x} \in E_k^u$ and $N\vec{y} \in E_k^\omega$. Take $\vec{z} = \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix}$. Then A$\vec{z} = \begin{pmatrix} M\vec{x} \\ N\vec{y} \end{pmatrix}$.So, A$\vec{z} \in E_k^\omega$. Therefore, $A = \begin{pmatrix} M & O \\ O & N \end{pmatrix}$ is image partition regular near 0.

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