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Research Paper

Some Remarks about the Hilbert Transform

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ABSTRACT:*In thispaperwepresentsomeremarks about the Hilbert transform on the real line and itsnumerical approximation, in connectionwithits application in signal processing.*
KEYWORDS: *Signal Processing, Hilbert Transform,Gaussian Rules*

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I. INTRODUCTION

The Hilbert transform has a variety of applications in many subjects of science and technology; the ones to be mentionedhere concern signal processing [1], [2], [3] and [4].More in general, in nonstationary signal analysis and processing, one needs to determine the instantaneous frequency of a real-valuedsignal $s(t)$. We want to write $s(t)$ in the form $\rho(t)\cos\theta(t)$, where $\rho(t)$ represents the instantaneous amplitude and $\theta(t)$the instantaneous phase. Then, the derivative of the phase $\theta(t)$ is defined as the instantaneous frequency of $s(t)$. This process is called signal demodulation. It is easy to understand that finding $\rho(t)$ and $\theta(t)$ satisfying $s(t) = \rho(t)\cos\theta(t)$ isequivalent to finding a function $v(t)$ such that$\rho(t)e^{i\theta(t)} = s(t) + \imath v(t)$.This means, a definition of the amplitude and phase of $s(t)$ corresponds to a definition of the imaginary part function $v(t)$. Generally, $v(t)$ depends on $s(t)$, we denote the relationship between $s(t)$ and $v(t)$ by writing $v(t) = Ps(t)$, and call P theimaginary part operator. There are infinite possibilities of P, which corresponds to different demodulation methods andproviding different definitions of instantaneous frequency and amplitude. If we choose P to be the Hilbert transform, thisdemodulation is the well-known analytic signal method, which was first proposed by D. Gabor [5] in 1946 and has beenwidely used ever since.Then, defining the Hilbert transform of a function f as the Cauchy principal value integral:

$$H(f;t) := \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(x)}{x-t}dx, \qquad \rightarrow (1)$$

the complex valued signal

$$z(t) = s(t) + \imath H(s;t) = \rho(t)\exp^{\imath\theta(t)}$$

is called its analytic signal and the instantaneous frequency of $s(t)$ is defined as

$$\omega(t) = \frac{1}{2\pi}\frac{d\theta(t)}{dt}, \qquad \theta(t) = \arctan\left(\frac{H(s;t)}{s(t)}\right).$$

In [6], [7] and [8] is proved that, given a function $s(t)$ with actual instantaneous phase $\theta(t)$ and amplitude $\rho(t)$, recovering $\theta(t)$ and $\rho(t)$ from $s(t)$ by the Hilbert transform is guaranteed if and only if it satisfies the following Bedrosian identity

$$H(\rho\cos\theta\,;t) = \rho(t)\sin\theta(t).$$

D. Vakman in [9] and [10] proved that the Hilbert transform is necessary for the definition of the instantaneous frequency, since it is the only operator that does not violate certain fundamental conditions of demodulation. Briefly, we recall these conditions that can be found in [11]:

i. H is linear and continuous;

ii. H maps $\exp^{\imath\omega(t)}$ to $-\imath\,\mathrm{sgn}(\omega)\exp^{\imath\omega(t)}$ for any $\omega \in \mathbb{R}$.

Therefore, the importance of the Hilbert transform coming from its many applications, justifies some interest in its numerical evaluation. For this reason, we want at first to recall some of the results obtained by the authors in the last years to compute numerically the finite and the infinite Hilbert transform based on Gaussian rules and product rules, and then, in this paper, we show new numerical schemes with the aim of finding efficient and stable procedures in order to compute effectively the Hilbert transform.In the case of the finite

Hilbert transformthere are other possible strategies using, for instance, splinesor wavelets approximation (see e.g. [12], [13]). Also in the case of the Hilberttransform defined on the real line there are several papers in the literature for example which use the sinc approximation (see for instance [14]). However we will not discuss about these cases.

The paper is organized as follow. In Section II we recall some methods for the finite Hilbert transform; Section III is devoted to present results about numerical methods for solving the Hilbert transform and in Section IV we present some efficient quadrature rules to this end. Finally, Section V is devoted to present some numerical simulations of the proposed methods that yield satisfactory numerical results in comparison with the MatLab routine "hilbert".

II. NUMERICAL SCHEMES FOR THE FINITE HILBERT TRANSFORM

The finite Hilbert transform $H(wf)$ of the function wf is defined by the integral in the Cauchy principal value sense

$$H(wf;t) := \int_{-1}^{1} \frac{f(x)}{x-t} w(x)dx = \lim_{\epsilon\to 0^+} \int_{|x-t|\ge\epsilon} \frac{f(x)}{x-t} w(x)dx, \qquad \to (2)$$

where w is a nonnegative weight function on $I := [-1,1]$ such that $0 < \int_{-1}^{1} w(t)dt < \infty$, and $x \in A := I - \{\text{singularities of } w\}$.

In the literature, essentially two kinds of quadrature rules of interpolatory type have been proposed to compute (2), according to whether among the nodes of quadrature one includes the point x or not.The former are the so-called "Gaussian rules" for the finite Hilbert transform, the latter are called "product rules". These rules have been extensively studied (cf. the literature cited in [15], [16], [17]). Throughout these papers there is an underlying theme of the instability of the computation and of the divergence of the rules. Here we present some our results about the numerical computation of (2).

Let $\{p_m(v)\}_{m\in\mathbb{N}}$ be a sequence of orthogonal polynomials on I associated with the weight function v. We denote the zeros of $p_m(v)$ by $x_{m,k} = x_{m,k}(v)$, $k = 1, \dots, m$. Let $L_m(v; f)$ be the Lagrange interpolating polynomial of f at the knots $x_{m,k}$. By replacing f by $L_m(v; f)$ in (2), we obtain the following interpolatory product rule for the evaluation of $H(wf; t)$

$H(wf;t) = H_m(v; wf; t) + E_m(v; wf; t)$, $\qquad \to (3)$

where

$$H_m(v; wf; t) = \sum_{k=1}^{m} A_{m,k}(v; w; t) f(x_{m,k}),$$

$$A_{m,k}(v; w; t) = H(wl_{m,k}(v); t), \qquad k = 1, \dots, m,$$

$l_{m,k}(v)$, $k = 1, \dots, m$ being the fundamental Lagrange polynomials and the error functional is denoted by $E_m(v; wf; t)$. An algorithm for the stable computation of the coefficients $A_{m,k}(v; w)$ has been proposed in [17], whatever the weight w may be. Irrespective of w, the quadrature nodes $x_{m,k}$ are always chosen to be the zeros of the Chebyshew polynomials of the first kind. Numerical computation of some finite Hilbert transform has been done by using this algorithm. Further, the uniform convergence of the product formula (3) when t tends to ± 1 or to the interior singularities of w at a prescribed rate has been proved under the assumptions that w is a generalized Ditzian-Totik weight and the quadrature knots are the zeros of the orthogonal polynomials associated with a suitable Jacobi weight. From now, let

$$v^{\alpha,\beta}(x) = v(x) = (1-x)^{\alpha}(1+x)^{\beta}, \qquad \alpha, \beta \succ 1, \qquad \to (4)$$

and

$$w(x) = (1-x)^{\gamma} \log^{\Gamma}\left(\frac{e}{1-x}\right) (1+x)^{\delta} \log^{\Delta}\left(\frac{e}{1+x}\right) \prod_{i=1}^{s} |t_i - x|_i^{\gamma} \log_i^{\Gamma}\left(\frac{e}{|t_i - x|}\right), \qquad \to (5)$$

where $\gamma, \delta, \gamma_i \succ 1$, $\Gamma, \Delta, \Gamma_i \ge 0$, $i = 1, \dots, s$ and $-1 < t_1 < \dots < t_s < 1$. We can state the following theorems.

Theorem 1

Let v and w be defined by (4) and (5) respectively, and $t \in [t_s + \epsilon, 1 - cm^{-2}]$ with $0 < \epsilon < 1 - t_s$. If the exponent γ of w satisfied $-1 < \gamma \le 0$, by choosing the exponent of v such that $2\gamma - \frac{1}{2} < \alpha < -\frac{1}{2}$, then, for $f \in Lip_M \lambda$,

$$|E_m(v; wf; t)| \le C \begin{cases} m^{-2\lambda-2\gamma} \log^{\Omega} m, & \text{if } -\gamma < \lambda \le -2\gamma, \\ m^{-\lambda} \log^{\Omega} m, & \text{if } -2\gamma < \lambda \le 1, \end{cases} \qquad \to (6)$$

where $\Omega = 2 + \max\{\Gamma, \Delta, \Gamma_i, i = 1, \dots, s\}$ and C is a positive constant independent of m and f. Furthermore by choosing α such that $-\frac{1}{2} \le \alpha < 2\gamma + \frac{3}{2}$, if $-1 < \gamma \le -\frac{1}{2}$ or $-\frac{1}{2} \le \alpha < \frac{1}{2}$, if $-\frac{1}{2} < \gamma \le 0$, then, instead of (6), the following estimate holds

$$|E_m(v;wf;t)| \le Cm^{-\lambda-2\gamma+\alpha+\frac{1}{2}}, \qquad -\lambda-2\gamma+\alpha+\frac{1}{2} < \lambda \le 1. \qquad \to (7) \quad ■$$

Theorem 2

Let v and w be defined by (4) and (5) respectively, $t \in [t_p-\epsilon, t_p+\epsilon]$, $0<\epsilon<min\{t_p-t_{p-1}, t_{p+1}-t_p\}$ and $p \in \{1,\dots,s\}$ with $t_0=-1$ and $t_{s+1}=1$. If the exponent γ_p of w satisfies $-1<\gamma_p\le 0$, and $f \in Lip_M\lambda$, then for every t such that $|t-t_p|>m^{-1}$

$$|E_m(v;wf;t)| \le Cm^{-\gamma_p-\lambda}\, log^{\Omega}\, m, \qquad -\gamma_p<\lambda\le -1 \qquad \to (8)$$

where $\Omega = 2+max\{\Gamma,\Delta,\Gamma_i, I=1,\dots,s\}$ and C is a positive constant independent of m and f. ■

The proofs of the Theorems 1-2 can be found in [17].

Another possible method to compute (2) starts from

$$H(wf;t) = f(t)\int_{-1}^{1}\frac{w(x)}{x-t}dt + \int_{-1}^{1}\frac{f(x)-f(t)}{x-t}w(x)dx\,; \qquad \to (9)$$

hence, assuming the $H(w;t)$ can be computed, we approximate the second integral on the right-had side of (9) by a classical quadrature rule Q_m on m knots. This method has been frequently used (see [18] and the literature cited therein). Obviously, from a theoretical point of view, this method turns out to be convergent if the function f is sufficiently smooth, for instance, when $f \in C^1(I)$ and Q_m is the quadrature Gaussian rule with respect to the weight w. An attempt in order to eliminate the numerical problems and to assure the convergence with f not strongly smooth has been proposed, but the convergence results are proved only when t is a fixed point of $(-1,1)$.

In [16], starting from (9), the authors have proposed a quadrature formula Q_m based on a special set of knots, all of which are sufficiently far from the singularity t. This procedure was introduced in [15] where $w=v^{\alpha,\beta}$ and the quadrature knots are suitable $m-1$ zeros of the mth Jacobi polynomial $p_m(v^{\alpha,\beta})$. An approximation method to compute $H(wf)$ using still the classical Jacobi zeros as knots even if $w \ne v^{\alpha,\beta}$ has been proposed in [16]. Let us consider the formula of interpolatory type Q_{m-1} constructed by replacing the function in the integral with the Lagrange polynomial interpolating it at the points $\{x_{m,k}, k=1,\dots,m, k\ne c\}$, where $c=c(t,m)$ is defined by

$$|t-x_{c(t,m)}| = min\{|t-x_{m,k}|, k=1,\dots,m\}.$$

Now, starting from (9) and using the previous formula Q_{m-1}, we arrive at the formula

$$H(wf;t) = H_m^{\alpha,\beta}(f;w;t) + E_m^{\alpha,\beta}(f;w;t) \qquad \to (10)$$

where

$$H_m^{\alpha,\beta} = f(t)H(w;t) + Q_{m-1}\left(\frac{f-f(t)}{.-t}\right)$$
$$= f(t)H(w;t) + \sum_{k=1,k\ne c} c_{m,k}\frac{f(x_m,k)-f(t)}{x_{m,k}-t},$$

$$c_{m,k} = \int_{-1}^{1} l_{m,k}(v;x)w(x)dx - \frac{\lambda_{m,k}P_{m-1}(x_{m,k})}{\lambda_{m,c}P_{m-1}(x_{m,c})}\int_{-1}^{1} l_{m,c}(v;x)w(x)dx\,, \qquad k=1,\dots,m, \quad k\ne c$$

$\lambda_{m,k}$, $k=1,\dots,m$, being the Christoffel constants with respect to $v^{\alpha,\beta}$.

In [16] it has been proved that the quadrature rule (10) is equivalent to the product rule (3) from a convergent point of view, i.e. in the same assumptions of the Theorems 1-2, the bounds (6),(7) and (8) are true for $E_m^{\alpha,\beta}(f;w;t)$, too. However, the formula (3) involves some computational efforts; indeed, calculation of the generalized functions of the second kind is required; whereas, the computational effort of (10) is the same of an interpolation quadrature rule to compute an ordinary integral.

III. NUMERICAL EVALUATION OF THE HILBERT TRANSFORM ON THE REAL LINE

We have been interested in developing a simple numerical integration method for weighted Hilbert transform $H(wf)$ defined by

$$H(wf;t) := \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(x)}{x-t}w(x)dx\,, \qquad t\in\mathbb{R}$$

where the divergence at $t=x$ is allowed for by taking the Cauchy principal value integral. Even if H is a bounded operator in the $L_p(\mathbb{R})$ spaces, $1<p<\infty$, it is an unbounded operator in the space of continuous function on $\mathbb{R}$ equipped with the uniform norm. Nevertheless, if f belongs to the set

$$W_0^{\infty} := \left\{g \in C_{LOC}^{0}(\mathbb{R}) : \lim_{|x|\to\infty} g(x)\,exp^{-x^2/2} = 0\right\}$$

and satisfy a Dini type condition by the Ditzian-Totik modulus of continuity, then $H(wf)$ is bounded on $\mathbb{R}$[19].

We have proposed an algorithm to compute $H(wf)$ assuming that the function f has good integration properties at the limits of the integration interval; these assumptions are the same ones to assure the boundedness of $H(wf)$. The proposed procedure is of interpolatory type and it uses as quadrature nodes the zeros of orthogonal polynomials with respect to the Hermite weight function w on $\mathbb{R}$.

Let $\{p_m(w)\}$ be a sequence of orthogonal Hermite polynomials associated with the weight function $w(x) = exp(-x^2)$. Let $L_m(w;f)$ be the lagrange interpolating polynomial of f at the zeros $x_{m,k}$ of $p_m(w)$. If we approximate the function f in $H(wf)$ by $L_m(w;f)$, than we obtain a formula of the type

$$H(wf;x) = H_m(w;f;x) + R_m(w;f;x),$$

with

$$H_m(w;f;x) = \sum_{k=1}^{m} A_{m,k}(w;x) f(x_{m,k}),$$

$$A_{m,k}(w;x) = H(wl_{m,k}(w);x), \quad k = 1,\dots,m,$$

where $l_{m,k}(w)$ are the fundamental Lagrange polynomials and $R_m(w;f;t)$ is the error functional. The coefficients $A_{m,k}(w;x)$ can be computed by

$$A_{m,k}(w) = \frac{1}{\pi}\lambda_{m,k}\sum_{j=0}^{m-1} p_j(w;x_{m,k}) q_j(w), \qquad k = 1,2,\dots,m,$$

where $\lambda_{m,k} = \lambda_{m,k}(w)$, $k = 1,2,\dots,m$, $m \in \mathbb{N}$ are the Christoffel constants with respect to the weight w and the functions $q_j(w)$ are the functions of the second kind. For the amplification coefficient of the rule $H_m(w;f)$ we can state the following theorem.

Theorem 3

For any $t \in \mathbb{R}$, we have

$$\sum_{k=1}^{m} |A_{m,k}(w;x)| \le C \log m, \qquad m \ge 1,$$

with some constant independent of m.■

In order to give a convergence result for the quadrature procedure $H_m(w;t)$ we need some notations. We set

$$E_m(f)_{\sqrt{w},\infty} := \inf_{P\in\backslash\mathbb{P}_m} \left\|(f-P)\sqrt{w}\right\|_\infty,$$

for any function $f \in C^0_{\sqrt{w}} := \{f\ continuous\ on\ \mathbb{R}\ and\ lim_{|x|\to\infty} f(x)\sqrt{w} = 0\}$, and where$\mathbb{P}_m$denotes the set of the polynomials of degree at most m. Denoting by $\omega^r(f,\delta)_{\sqrt{w},\infty}$ the rth Ditzian-Totik weighted modulus of smoothness, we can state the following result.

Theorem 4

Assume that $F\in W^\infty_0$ satisfies the condition

$$\int_0^1 \frac{\omega^r(f,\delta)_{\sqrt{w},\infty}}{\delta} d\delta < \infty.$$

Then,

$$\max_{t\in\mathbb{R}} |R_m(w;f;t)| \le C\left\{\log m\, E_{m-1}(f)_{\sqrt{w},\infty} + \int_0^{\frac{1}{\sqrt{w}}} \frac{\omega^r(f,\delta)_{\sqrt{w},\infty}}{\delta} d\delta\right\},$$

for some constant independent of f and m.■

The proof of the Theorems3 and 4 can be found in [20].

Starting from the identity

$$H(wf;t) = f(t)H(w;t) + \int_{-\infty}^{\infty} \frac{f(x)-f(t)}{x-t} w(x)dx,$$

a completely different approach for the numerical approximation of the Hilbert transform has been proposed in [21]. This procedure corresponds to evaluate exactly the integral $H(w;t)$ and to approximate the integral $\int_{-\infty}^{\infty} \frac{f(x)-f(t)}{x-t} w(x)dx$ by an ordinary Gaussian rule. As pointed out in [21] the convergence and stability of this quadrature rule is not assured and it depends on the distance of the singularity t by the points$x_{m,k}, k = 1,\dots,m$. In [21] it is also proved that this problem is overcome by choosing m in a suitable space $\overline{N}$.

More recently, in [22] the authors prove the convergence and the stability properties of the following quadrature rule

$$\overline{H}_m(wf;t) = f(t)H(w;t) + \sum_{k=1,k\neq c} \lambda_{m,k}\left\{1 - \frac{p_{m-1}(w;x_{m,k})}{p_{m-1}(w;x_{m,c})}\right\} \frac{f(x_{m,k}) - f(t)}{x_{m,c} - t},$$

where $\lambda_{m,k}$ and $x_{m,k}$, for $k = 1, \dots, m$ are the Christoffel constants and the knots of the mth orthogonal polynomial $p_m(w)$, respectively and $x_{m,c}$ is the closest knot near the singularity t.

IV. NEW SCHEMES FOR COMPUTING THE HILBERT TRANSFORM

In this Section we came back to consider the Hilbert transform defined in (1), where the weight function $w(x) = 1$. For a sufficiently large constant a, we consider the central interval with respect to the singularity t, i.e. the interval $[t-a, t+a]$; then, we can write the Hilbert transform Hf in (1) as

$$H(f;t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(x)}{x-t}dx = \frac{1}{\pi}\{\int_{-\infty}^{t-a} + \int_{t-a}^{t+a} + \int_{t+a}^{\infty}\}\frac{f(x)}{x-t}dx.$$

Now, operating a change of variable for the interval $[t-a, t+a] \to [-1,1]$ in place of Hf we consider an approximation by a suitable quadrature rule of the finite Hilbert transform

$$H^a f(t) = \frac{1}{\pi}\int_{t-a}^{t+a}\frac{f(x)}{x-t}dx = \frac{1}{\pi}\int_{-1}^{1}\frac{f(az+t)}{z}dz. \qquad \to (11)$$

Denoting with $w_{2m,i}^L$ and $x_{2m,i}^L$, for $i = 1, \dots, 2m$ the weights and the nodes of the $2m$ Gauss-Legendre (GL) quadrature rule, respectively, we compute (11) by the Gauss-Legendre (GL) rule

$$H^a f(t) \approx H_m^a f(t) = \frac{1}{\pi}\sum_{i=1}^{m}\frac{w_{2m,i}^L}{x_{2m,i}^L}\left[f\left(ax_{2m,i}^L + t\right) - f\left(-ax_{2m,i}^L + t\right)\right].$$

We remark that the error order of the proposed scheme is the same of the corresponding ordinary Gauss-Legendre quadrature rule (see [23] for more detail). Therefore, for obtaining a convergent result we need to bound the two integrals we drop, i.e.$\{\int_{-\infty}^{t-a} + \int_{t+a}^{\infty}\}\frac{f(x)}{x-t}dx$.

It is easy to show that

$$\left|\int_{t+a}^{\infty}\frac{f(x)}{x-t}dx\right| \le \frac{1}{a}\int_{t+a}^{\infty}|f(x)|dx,$$

and this quantity tends to 0 for every a sufficiently large, due to the integrability property of the function f on $\mathbb{R}$.

The second scheme consists in approximating the integral with a Gauss-Hermite quadrature rule

$$Hf(t) = \int_{-\infty}^{\infty}\frac{f(y+t)\,exp(y^2)}{y}exp(-y^2)\,dy$$

$$H_m f(t) = \frac{1}{\pi}\sum_{i=1}^{m}\frac{w_{2m,i}^H}{x_{2m,i}^H}\left[f\left(x_{2m,i}^H + t\right) - f\left(-x_{2m,i}^H + t\right)\right]exp({x_{2m,i}^H}^2),$$

where $w_{2m,i}^H$ and $x_{2m,i}^H$, for $i = 1, \dots, 2m$ denote weights and nodes of the $2m$ Gauss-Hermite rule, respectively.

Also in this case the error is the same of the ordinary Gauss-Hermite quadrature rule.

V. NUMERICAL TESTS

In this section experiments for two functions are conducted. These functionsare chosen for our experiments since their Hilbert transforms are analytically known.In Figs. 1,2 and 3,4 is shown the comparison of our methods with the "hilbert" routine of MatLab and the exact solution for these two test functions

$$f_1(t) = \frac{\sin t}{1+t^4}, \qquad f_2(t) = \frac{1}{1+t^2},$$

for which the Hilbert transform is given by

$$\mathrm{H}(\mathrm{f}_1;\mathrm{t}) = \frac{1}{1+\mathrm{t}^4}\left[\exp^{-\frac{1}{\sqrt{2}}}\cos\frac{1}{\sqrt{2}} + \exp^{-\frac{1}{\sqrt{2}}}\sin\frac{1}{\sqrt{2}}\mathrm{t}^2 - \cos\mathrm{t}\right],$$

$$\mathrm{H}(\mathrm{f}_2;\mathrm{t}) = \frac{\mathrm{t}}{1+\mathrm{t}^2}.$$

The exact Hilbert transform is graphically displayed with a bluesolid line, while the same transforms computed by MatLab method and by our proposed schemes (taking m=64 and for the first scheme $a = 12$) are shown with red dotted line, green solid line and violet dotted line, respectively. All the numerical computations are done in double precision arithmetics. One has no difficulty in seeing that our proposed methods give much finer numerical computation for all these test functions and they confirm that our proposed methods provides much better performance, especially in the neighborhoods of the boundary of the chosen interval. This circumstance can be shown also in Tables 1-2 where we report the exact value of the Hilbert transform evaluated in some points of the interval $[-6,0]$ and the correspondent value obtained with the two proposed methods and the MatLab routine"hilbert". We remark that in the paper the results are reported for $a = 12$, but these are still valid for a convenient choice of the parameter a.

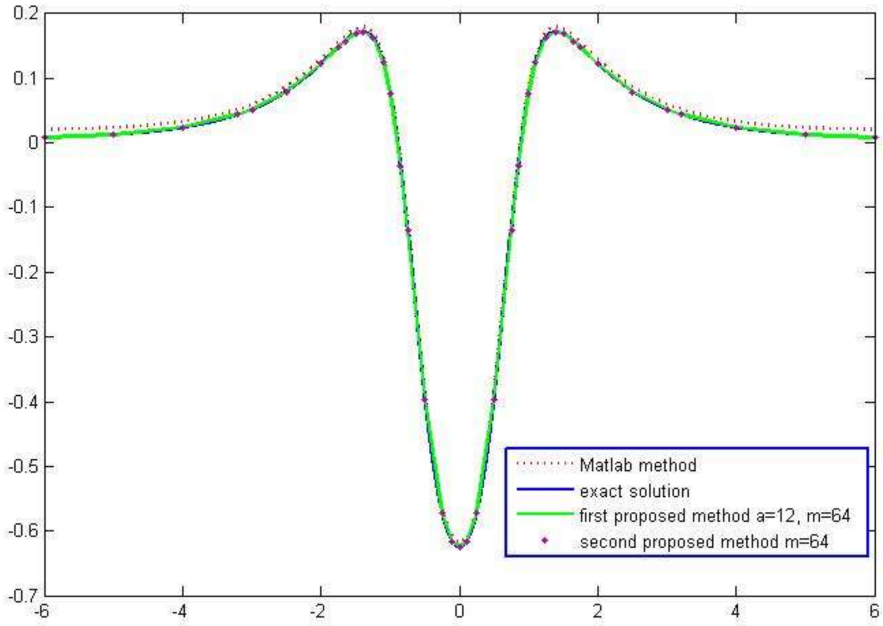

Figure 1: Comparison of the solution Hf_1 evaluated by the proposed methods, MatLab method and the exact solution evaluated by Mathematica Software

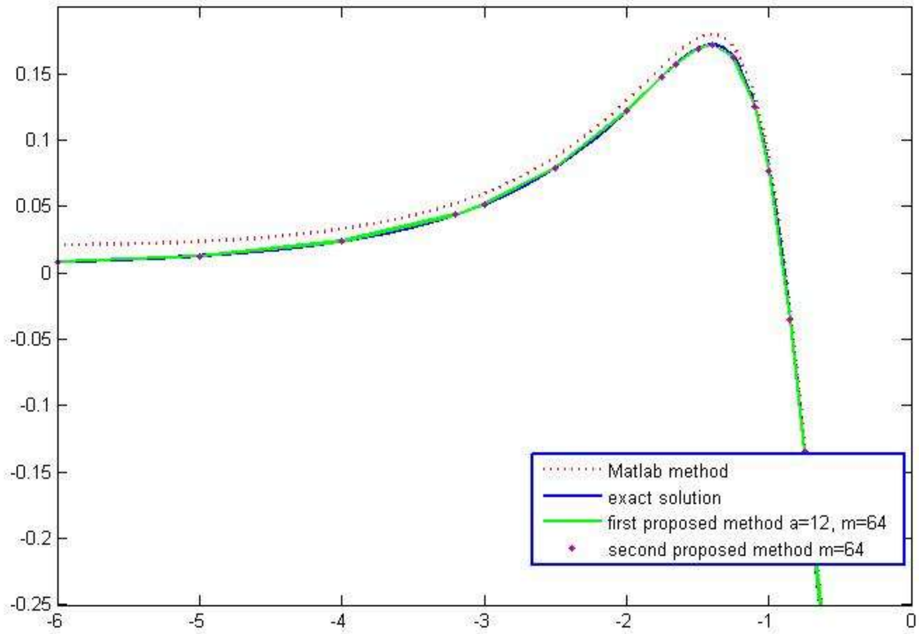

Figure 2: Comparison of the solution Hf_1 evaluated by the proposed methods, MatLab method and the exact solution evaluated by Mathematica Software, a detail

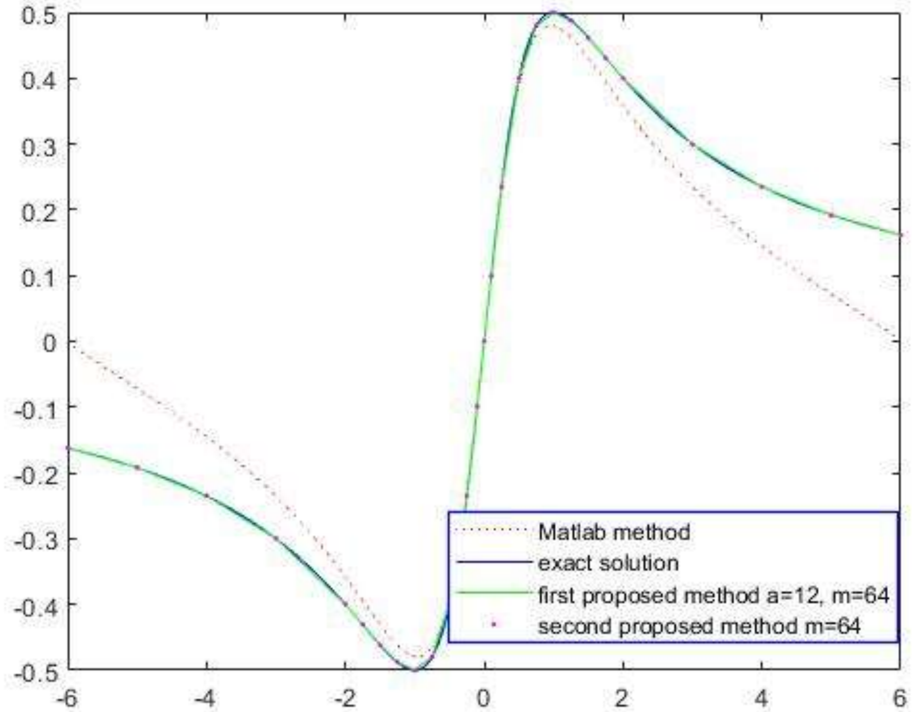

Figure 3: Comparison of the solution Hf_2 evaluated by the proposed methods, MatLab method and the exact solution evaluated by Mathematica Software

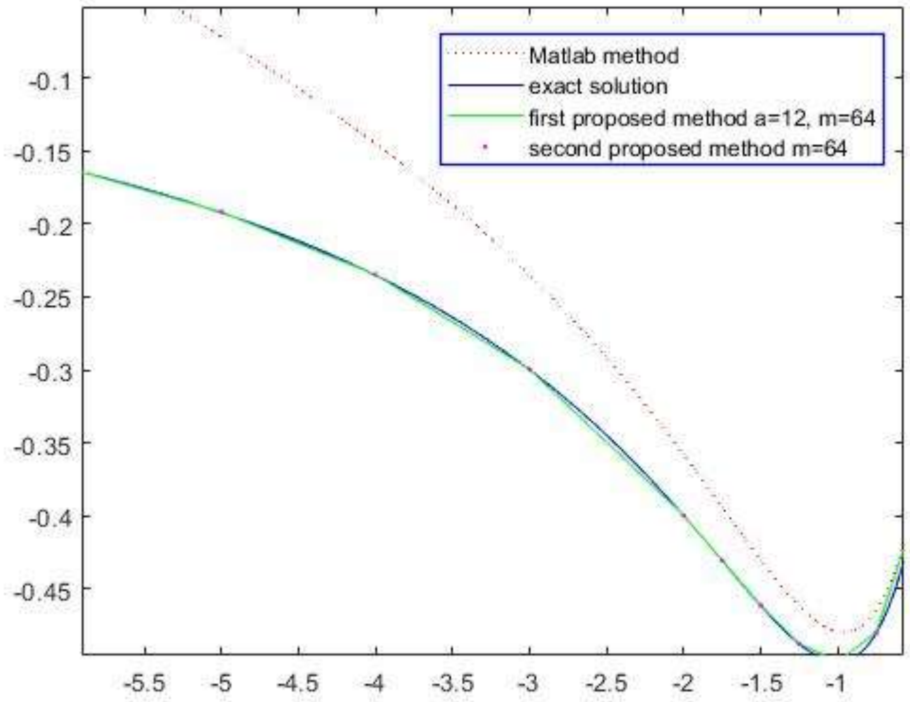

Figure 4: Comparison of the solution Hf_2 evaluated by the proposed methods, MatLab method and the exact solution evaluated by Mathematica Software, a detail

t	$\mathbf{Hf_1(t)}$	$\mathbf{H_m^a f_1(t)}$	$\mathbf{H_m f_1(t)}$	**MatLab**
-6	0.0084	0.0084	0.0084	0.0208
-5	0.0129	0.0129	0.0129	0.0237
-4	0.0239	0.0239	0.0239	0.0332
-3	0.0518	0.0518	0.0518	0.0601
-2	0.1218	0.1218	0.1218	0.1296
-1	0.0774	0.0774	0.0774	0.0849
0	-0.6251	-0.6251	-0.6251	-0.6177

Table 1test function $f_1(t)$: Comparison for various values of t, for $m = 64$, $a = 12$

t	$\mathbf{Hf_2(t)}$	$\mathbf{H_m^a f_2(t)}$	$\mathbf{H_m f_2(t)}$	**MatLab**
-6	-0.1621	-0.1616	-0.1613	-0.4852d-4
-5	-0.1923	-0.1918	-0.1916	-0.0719
-4	-0.2352	-0.2348	-0.2348	-0.4150
-3	-0.2999	-0.2996	-0.2996	-0.2352
-2	-0.4	-0.3996	-0.3997	-0.3580
-1	-0.4999	-0.4998	-0.4998	-0.4793
0	0	0	0	-0.2839d-16

Table 2 test function $f_2(t)$: Comparison for various values of t, for $m = 64$, $a = 12$

VI. CONCLUSION

In this paper the authors point out that very simple Gaussian quadrature rules can be used to approximate the Hilbert transform, recalling the good convergence properties of such rules. Moreover, they show that the proposed formulas work better than the most common built-in function “hilbert” designed for the computation of Hf in the software tool MatLab,often used by the scientific community in the engineering applications.

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